MA360 **Handout 3a (week of Jan 23)**
Sp2006 Kallfelz

Assignment: P 41 Sheng: 1,2,4,5(a), b), c)),6(b), c)), 7a)

First session of Class: Remaining problems from list of student questions that I didn't get to discussing

- Exercise 10, page 4 (Sheng)

From the graph: $f(t) = \begin{Bmatrix} 0 & 0 \le t < 1 \\ (2t-2) & 1 \le t < 2 \\ 0 & t > 2 \end{Bmatrix}$

Hence:

$$L\{f(t)\} = \int_0^\infty f(t)e^{-st}dt = \int_0^1 0 \cdot e^{-st}dt + \int_1^2 (2t-2)e^{-st}dt + \int_2^\infty 0 \cdot e^{-st}dt = 2\int_1^2 (t-1)e^{-st}dt$$
$$= 2\left\{\int_1^2 te^{-st}dt - \int_1^2 e^{-st}dt\right\} = 2\left\{-\tfrac{t}{s}e^{-st}\Big|_1^2 + \tfrac{1}{s}\int_1^2 e^{-st}dt - \int_1^2 e^{-st}dt\right\} = 2\left\{-\tfrac{2}{s}e^{-2s} + \tfrac{1}{s}e^{-s} - \tfrac{1}{s^2}e^{-st}\Big|_1^2 + \tfrac{1}{s}e^{-st}\Big|_1^2\right\}$$
$$= 2\left\{-\tfrac{2}{s}e^{-2s} + \tfrac{1}{s}e^{-s} - \tfrac{1}{s^2}e^{-2s} + \tfrac{1}{s^2}e^{-s} + \tfrac{1}{s}e^{-2s} - \tfrac{1}{s}e^{-s}\right\} = 2\left\{-\tfrac{2}{s}e^{-2s} - \tfrac{1}{s^2}e^{-2s} + \tfrac{1}{s^2}e^{-s} + \tfrac{1}{s}e^{-2s}\right\}$$
$$= 2\left\{-\tfrac{1}{s}e^{-2s} - \tfrac{1}{s^2}e^{-2s} + \tfrac{1}{s^2}e^{-s}\right\} = \tfrac{2}{s^2}\left\{-se^{-2s} - e^{-2s} + e^{-s}\right\} = \tfrac{2}{s^2}\left(e^{-s} - e^{-2s} - se^{-2s}\right)$$

(This could also be further simplified to: $\frac{2e^{-s}}{s^2}\left[1 - e^{-s}(1+s)\right]$)

- Exercise 1, page 17 (Sheng)

Given: $L[\sin \omega t] = \frac{\omega}{s^2+\omega^2}$, find $L[\cos \omega t]$

$$L\left[e^{i\omega t}\right] = L[\cos \omega t + i \sin \omega t] = L[\cos \omega t] + iL[\sin \omega t] = L[\cos \omega t] + i\tfrac{\omega}{s^2+\omega^2}$$

But also:

$$L\left[e^{i\omega t}\right] = \tfrac{1}{s-i\omega t} = \tfrac{1}{s-i\omega t} \cdot \tfrac{s+i\omega t}{s+i\omega t} = \tfrac{s+i\omega t}{s^2+\omega^2} = \tfrac{s}{s^2+\omega^2} + i\tfrac{\omega}{s^2+\omega^2}$$

Hence comparing the real parts: $L[\cos \omega t] = \frac{s}{s^2+\omega^2}$

- Exercise 2c, page 17 (Sheng)

Find: $L[t(\cosh 3t - 4)]$

$$L[2t(\cosh 3t-4)]=2L[t\cosh 3t]-8L[t]$$

$$L[t]=\frac{\Gamma(1)}{s^{1+1}}=\frac{1}{s^2}$$
$$L[\cosh 3t]=\frac{s}{s^2-3^2}$$

So, according to Thm5:
$$L[t\cosh 3t]=(-1)^1\frac{d}{ds}\left(\frac{s}{s^2-9}\right)=-\left[\frac{(s^2-9)-s(2s)}{(s^2-9)^2}\right]$$
$$-\left[\frac{-s^2-9}{(s^2-9)^2}\right]=\frac{s^2+9}{(s^2-9)^2}$$

So:
$$L[2t(\cosh 3t-4)]=2L[t\cosh 3t]-8L[t]=\frac{2(s^2+9)}{(s^2-9)^2}-\frac{8}{s^2}=\frac{2s^2(s^2+9)-8(s^2-9)^2}{s^2(s^2-9)^2}$$
$$=\frac{2s^4+18s^2-8s^4+144s^2-648}{s^2(s^2-9)^2}=\frac{-6s^4+162s^2-648}{s^2(s^2-9)^2}=\frac{-6(s^4-27s^2+108)}{s^2(s^2-9)^2}$$

- Exercise 2e, page 17 (Sheng)

Find : $L[(t-3)^2\cos 4t]$

$$L\left[(t-3)^2\cos 4t\right]=L\left[(t^2-6t+9)\cos 4t\right]=L\left[t^2\cos 4t\right]-6L[t\cos 4t]+9L[\cos 4t]$$

Method 1:

Since: $L[\cos 4t]=\frac{s}{s^2+16}$

Then, using Thm5: $L[t\cos 4t]=-\frac{d}{ds}\frac{s}{s^2+16}=-\left[\frac{(s^2+16)-s(2s)}{(s^2+16)^2}\right]=-\left[\frac{-s^2+16}{(s^2+16)^2}\right]=\frac{s^2-16}{(s^2+16)^2}$

And:
$$L\left[t^2\cos 4t\right]=(-1)^2\frac{d^2}{ds^2}\frac{s}{s^2+16}=\frac{d}{ds}\left[\frac{-s^2+16}{(s^2+16)^2}\right]=\frac{(s^2+16)^2(-2s)-(-s^2+16)\left[2(s^2+16)2s\right]}{(s^2+16)^4}$$
$$=(s^2+16)\left[\frac{-2s(s^2+16)+(s^2-16)4s}{(s^2+16)^4}\right]=\frac{-2s^3-32s+4s^3-64s}{(s^2+16)^3}=\frac{2s(s^2-48)}{(s^2+16)^3}$$

So:
$$L\left[(t-3)^2\cos 4t\right]=L\left[t^2\cos 4t\right]-6L[t\cos 4t]+9L[\cos 4t]$$
$$=\frac{2s(s^2-48)}{(s^2+16)^3}-6\frac{(s^2-16)}{(s^2+16)^2}+9\frac{s}{(s^2+16)}=\frac{2s^3-96s-6(s^2-16)(s^2+16)+9s(s^2+16)^2}{(s^2+16)^3}$$
$$=\frac{2s^3-96s-6s^4+1536+9s^5+288s^3+2304s}{(s^2+16)^3}=\frac{9s^5-6s^4+290s^3+2208s+1536}{(s^2+16)^3}$$

Method 2:

$$L[\cos 4t] = L\left[\tfrac{1}{2}\left(e^{i4t} + e^{-i4t}\right)\right] = \tfrac{1}{2}\left\{L\left[e^{i4t}\right] + L\left[e^{-i4t}\right]\right\} = \tfrac{1}{2}\left\{\tfrac{1}{s-4i} + \tfrac{1}{s+4i}\right\}$$
$$= \tfrac{1}{2}\left\{\tfrac{s+4i+s-4i}{s^2+16}\right\} = \tfrac{1}{2}\left[\tfrac{2s}{s^2+16}\right] = \tfrac{s}{s^2+16}$$

So using Thm5:

$$L[t\cos 4t] = -\tfrac{1}{2}\tfrac{d}{ds}\left[\tfrac{1}{s-4i} + \tfrac{1}{s+4i}\right] = -\tfrac{1}{2}\left[\tfrac{-1}{(s-4i)^2} + \tfrac{-1}{(s+4i)^2}\right] = -\tfrac{1}{2}\left[\tfrac{-(s+4i)^2-(s-4i)^2}{(s^2+16)^2}\right] = -\tfrac{1}{2}\left[\tfrac{-s^2-8is+16-s^2+8is+16}{}\right]$$
$$= -\tfrac{1}{2}\left[\tfrac{-2s^2+32}{(s^2+16)^2}\right] = \tfrac{s^2-16}{(s^2+16)^2}$$

and:

$$L\left[t^2\cos 4t\right] = (-1)^2\tfrac{d^2}{ds^2}\tfrac{s}{s^2+16} = \tfrac{d}{ds}\left[\tfrac{-s^2+16}{(s^2+16)^2}\right] = \tfrac{(s^2+16)^2(-2s)-(-s^2+16)[2(s^2+16)2s]}{(s^2+16)^4}$$
$$= (s^2+16)\left[\tfrac{-2s(s^2+16)+(s^2-16)4s}{(s^2+16)^4}\right] = \tfrac{-2s^3-32s+4s^3-64s}{(s^2+16)^3} = \tfrac{2s(s^2-48)}{(s^2+16)^3}$$

$$L\left[(t-3)^2\cos 4t\right] = L\left[t^2\cos 4t\right] - 6L[t\cos 4t] + 9L[\cos 4t]$$

So: $= \tfrac{2s(s^2-48)}{(s^2+16)^3} - 6\tfrac{(s^2-16)}{(s^2+16)^2} + 9\tfrac{s}{(s^2+16)} = \tfrac{2s^3-96s-6(s^2-16)(s^2+16)+9s(s^2+16)^2}{(s^2+16)^3}$

$$= \tfrac{2s^3-96s-6s^4+1536+9s^5+288s^3+2304s}{(s^2+16)^3} = \tfrac{9s^5-6s^4+290s^3+2208s+1536}{(s^2+16)^3}$$

(**Note:** You don't really save any work in Method2, but I showed it anyway as a way to present an alternate characterization of cos4t via Euler Thm.)

- Exercise 2f, page 17 (Sheng)

Find $L[t^2e^{4t}]$

Method 1 (using material covered up to this section)

According to Thm5: $L\left[t^2e^{4t}\right] = (-1)^2\tfrac{d^2}{ds^2}L\left[e^{4t}\right] = \tfrac{d^2}{ds^2}\left(\tfrac{1}{s-4}\right) = \tfrac{d}{ds}\left[-(s-4)^2\right]$
$= 2(s-4)^3 = \tfrac{2}{(s-4)^3}$

Method 2 (using material covered in the next section)

According to Thm7: $L\left[e^{4t}t^2\right] = F(s-4), (F(s) = L\left[t^2\right] = \tfrac{2!}{s^3})$
$\therefore L\left[e^{4t}t^2\right] = \tfrac{2}{(s-4)^3}$