MA360 Handout 1b (week of Jan 9)
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Assignment: 1,4,6,7,10 (pg5-Sheng) /2.(a),(c),(f),(h) (pg 17) [Read pp1-16, Sheng, and Review techniques and examples in Handouts1a, Handouts 1b]

I.) The Laplace Transform – Underlying Motivation(s)

Consider an n-th order non-homogenerous differential equations, with constant coefficients, i.e., a differential equation of the form:

$$D^n y(t) = f(t) \qquad \text{(Eqn. I.1)}$$

where: a.) $D^n = \sum_{k=0}^{n} a_k \frac{d^k}{dt^k} = \left(a_n \frac{d^n}{dt^n} + a_{n-1} \frac{d^{n-1}}{dt^{n-1}} + \ldots + a_1 \frac{d}{dt} + a_0 \right)$

(i.e., D^n is the *n-th* order differential operator, with constant coefficients: $a_0, a_1, \ldots a_n$)

b.) $y(t)$ is the unknown function, with initial conditions[1]:

$$y^{(n-1)}(0) = c_{n-1}, \ldots, y'(0) = c_1, y(0) = c_0$$

c.) $f(t)$ is a differentiable function to n-th order

You may be used to solving (Eqn I.1) using methods like Undetermined Coefficients (UC) or the Variation of Paramaters (VP), etc. But what you may also recall is the usefulness of such methods have limited applicability—the algebra can get awfully messy in cases when the order $n > 2$ of (Eqn I.1).

As an example, consider the particular case when $n = 2$, in the case of the following ODE:

$$D^2 y(t) \equiv \left(\frac{d^2}{dt^2} + 1 \right) y(t) = y''(t) + y(t) = 1$$

with initial conditions: $y'(0) = 0$, $y(0) = 1$

In other words, the above example is rather simple. Translating in terms of the general notation of (Eqn. I.1) above, the coefficients are: $a_0 = 1$, $a_1 = 0$, $a_2 = 1$. Moreover, the initial constants in this example are: $c_0 = 1$, $c_1 = 0$, and $f(t) = 1$

Anyway, you realize using either UC or VP it's a straightforward to solve. In other words, the degree of $n = 2$ bounds the (algebraic) complexity of the UC or VP procedures to acceptable limits.

But what about something only slightly more complicated? Consider, for example the following 3rd order equation:

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[1] The notation $y^{(k)}(0)$ is shorthand for the k-th derivative of $y(t)$ at $t = 0$ (for $0 \le k \le n-1$, in the above case of initial conditions.)

$$D^3 y(t) \equiv \left(\frac{d^3}{dt^3} - 3\frac{d}{dt} + 2 \right) y(t) = y^{(3)}(t) - 3y'(t) + 2y(t) = 2e^{-t}$$

with initial conditions: $y(0) = 1$, $y'(0) = -2$, $y''(0) = 1$. I.e, the coefficients are: $a_0 = 2$, $a_1 = -3$, $a_2 = 0$, $a_3 = 1$. Moreover, the initial constants in this example are: $c_0 = 1$, $c_1 = -2$, $c_2 = 1$ and $f(t) = 2e^{-t}$. It's only slightly more complicated *in form* than that of the first example, but trying to solve it using methods like UC or VP proves much messier. Just the homogeneous part of the solution (i.e., the solution of $y^{(3)}(t) - 3y'(t) + 2y(t) = 0$) involves solving the auxillary equation: $r^3 - 3r + 2 = 0$, which involves some non-trivial algebraic maneuvers.

So the question becomes: Is there a general and systematic way of solving equations of the form (Eqn I.1) which can by-pass some of the algebraic messiness of the more particular procedures like UC and VP? The answer is yes! This turns out to be one of the most attractive features of the Laplace Transform, as we shall discover in this course.

II. The Laplace Transform: Definition & Its Linearity

Defn.: $L\{f(t)\} = F(s) = \int_0^\infty f(t)e^{-ts}\,dt$

As page 6 (Sheng) shows, the Laplace Transform (LT) is a *linear* operator. (The basic reason is because the LT is defined as an integral. The integral (or antiderivative) operator, is linear in its functional argument (just like the derivative). So:

$$L\{af(t) + bg(t)\} = \int_0^\infty [af(t) + bg(t)]e^{-ts}\,dt = \int_0^\infty [af(t)e^{-st} + bg(t)e^{-st}]dt = a\int_0^\infty f(t)e^{-st}\,dt + b\int_0^\infty g(t)e^{-st}\,dt$$
$$= aL\{f(t)\} + bL\{g(t)\}$$

The exercises below are worked out in explicit detail from first principles, using the definition of LT and its linearity alone:

- <u>Example (Exercise 1, problem 2, Sheng, page4)</u>

Find: $L\{6\sin(8t)\}$

Since L is linear: $L\{6\sin(8t)\} = 6L\{\sin(8t)\}$

And: $L\{\sin(8t)\} = \int_0^\infty \sin(8t)e^{-ts}\,dt = \lim_{d\to\infty} \int_0^d \sin(8t)e^{-ts}\,dt$ (**Note1:** Formally, we treat this as an improper integral. For more details, See **Handout 1a** I sent, pages 6 (bottom) to page 8 (top).

$$So: \lim_{d\to\infty} \int_0^d \sin(8t)e^{-ts}\,dt = \lim_{d\to\infty} \left\{ \frac{e^{-st}}{-(s)^2+8^2}\left[-s\sin 8t - 8\cos(-st)\right] \right\}\Bigg|_0^d$$

(**Note 2:** Using Formula 31, **A 7**, with $a = -s$, $b = 8$. To see how to derive this result from scratch, using integration by parts, see page 5 –6, of **Handout 1a.)**

$$So: \lim_{d\to\infty} \left\{ \frac{e^{-st}}{-(s)^2+8^2}\left[-s\sin 8t - 8\cos(-st)\right] \right\}\Bigg|_0^d$$
$$= \lim_{d\to\infty} \left\{ \frac{e^{-sd}}{s^2+64}\left[-s\sin 8d - 8\cos sd\right] \right\} - \left\{ \frac{e^{0}}{s^2+64}\left[-s\sin 0 - 8\cos 0\right] \right\}$$

(**Note 3:** Plugging in the integration limits *d* (the upper limt) and 0 (the lower limit). Note also the simplification in the second step, cos(-A) = cosA.)

$$So: \lim_{d\to\infty} \left\{ \frac{e^{-sd}}{s^2+64}\left[-s\sin 8d - 8\cos sd\right] \right\} - \left\{ \frac{e^{0}}{s^2+64}\left[-s\sin 0 - 8\cos 0\right] \right\}$$
$$= \frac{1}{s^2+64}\left\{-s\lim_{d\to\infty}\left(e^{-sd}\sin 8d\right) - 8\lim_{d\to\infty}\left(e^{-sd}\cos sd\right)\right\} + \frac{8}{s^2+64}$$

(**Note 4:** Distributing the limit operation across the numerator term in the first expression. Keep in mind that the argument of the limit is *d* only (*s* behaves as a constant)).

$$So: \frac{1}{s^2+64}\left\{-s\lim_{d\to\infty}\left(e^{-sd}\sin 8d\right) - 8\lim_{d\to\infty}\left(e^{-sd}\cos sd\right)\right\} + \frac{8}{s^2+64}$$
$$= \frac{1}{s^2+64}\{0-0\} + \frac{8}{s^2+64} = \frac{8}{s^2+64}$$

(**Note 5:** Using property of limits: $\lim_{x\to\infty}(e^{-x}\sin x) = \lim_{x\to\infty}(e^{-x}\cos x) = 0$. This can be formally proved using the "Sandwich Theorem:" Since $-1 \le \sin x \le 1$, $-1 \le \cos x \le 1$, and $\lim_{x\to\infty}(-e^{-x}) = \lim_{x\to\infty}(e^{-x}) = 0$. Therefore: $\lim_{x\to\infty}(e^{-x}\sin x) = \lim_{x\to\infty}(e^{-x}\cos x) = 0$, because: $\lim_{x\to\infty}(-e^{-x}) \le \lim_{x\to\infty}(e^{-x}\sin x) \le \lim_{x\to\infty}(e^{-x})$, and $\lim_{x\to\infty}(-e^{-x}) \le \lim_{x\to\infty}(e^{-x}\cos x) \le \lim_{x\to\infty}(e^{-x})$. An informal way of thinking about this is to note that while e^{-x} goes to 0 "very fast," the sine and cosine terms just oscillate between –1, 1. So the e^{-x} term dominates in the expression $e^{-x}\sin x$ or $e^{-x}\cos x$. Similar reasoning holds for cases like the product of an exponential decay term and a power-term: I.e., $\lim_{x\to\infty}(x^n e^{-x}) = 0$, for *any n*. This can be proven using L'Hopital's Rule and logarithmic differentiation. Viewed informally the exponential term shrinks 'much faster' than the power term increases. So in the expression $x^n e^{-x}$, the exponential term dominates. For further details, see **Handout 1a,** page 8.)

Hence: $L\{\sin 8t\} = \dfrac{8}{s^2+64} \Rightarrow L\{6\sin 8t\} = 6L\{\sin 8t\} = \dfrac{48}{s^2+64}$

- <u>Example (Problem 3, Exercise 1, p 4, Sheng)</u>

Find $L\{2t - 3e^{-t}\}$

Because of linearity: $L\{2t - 3e^{-t}\} = 2L\{t\} - 3L\{e^{-t}\}$.

Step 1: $L\{t\} = \int_0^\infty te^{-ts}\,dt = \lim_{d\to\infty} \int_0^d te^{-ts}\,dt$ (**Note1:** We're treating this formally of course as an improper integral. As we shall soon see, however, due to various short-cut in the processes of obtaining results of LTs via the various theorems, much of these formalities can and will be by-passed. But for starters, in these examples, everything is explicitly laid out.)

The above integral can be done using Integration by Parts (for details, see pages 4,5,6 of **Handout 1a** or we can Formula 27., page A-7 (definite integral version). If you use the formula, however, make sure you perform the proper u – substitution correctly:

$$27. \int_a^b ue^u\,du = (u-1)e^u\Big|_a^b \Rightarrow u = -ts, du = -sdt, \Rightarrow dt = -\tfrac{1}{s}du, t = -\tfrac{1}{s}u$$

$$\therefore \int_0^d te^{-st}\,dt = \int_{u(0)}^{u(d)} \left(-\tfrac{1}{s}u\right)e^u\left(-\tfrac{1}{s}du\right) = \tfrac{1}{s^2}\int_0^{-sd} ue^u\,du = \tfrac{1}{s^2}(u-1)e^u\Big|_0^{-sd}$$

So:

$$L\{t\} = \lim_{d\to\infty} \int_0^d te^{-ts}\,dt = \lim_{d\to\infty} \left\{\tfrac{1}{s^2}(u-1)e^u\right\}\Big|_0^{-sd}$$

$$= \left\{\lim_{d\to\infty} \tfrac{1}{s^2}(-sd-1)e^{-sd}\right\} - \left\{\tfrac{1}{s^2}(0-1)e^0\right\} = \left\{-\tfrac{1}{s^2}\lim_{d\to\infty}(sd+1)e^{-sd}\right\} + \tfrac{1}{s^2} = \tfrac{1}{s^2}$$

(**Note2:** We're Using similar reasoning with the limits as discussed in Notes4,5 in the above example on page 3. Informally, the exponential term e^{-sd} dominates over $(sd + 1)$. Keep in mind, again, that the argument of the limit is d, so the s behaves like a constant.)

Step 2: $L\{e^{-t}\} = \int_0^\infty e^{-t}e^{-ts}\,dt = \lim_{d\to\infty}\int_0^d e^{-t}e^{-ts}\,dt = \lim_{d\to\infty}\int_0^d e^{-(1+s)t}\,dt$

This integral can be evaluated using a simple u-substitution. If, for instance, you use Formula 26. (A-7), definite integral version:

$$26. \int_a^b e^u\,du = e^u\Big|_a^b \Rightarrow u = -(1+s)t, du = -(1+s)dt, \Rightarrow dt = -\tfrac{1}{(s+1)}du, t = -\tfrac{1}{(s+1)}u$$

$$\therefore \int_0^d e^{-(1+s)t}\,dt = \int_{u(0)}^{u(d)} e^u\left(-\tfrac{1}{(s+1)}du\right) = -\tfrac{1}{(s+1)}\int_0^{-(s+1)d} e^u\,du = -\tfrac{1}{(s+1)}e^u\Big|_0^{-(s+1)d}$$

So:

$$L\{e^{-t}\} = \lim_{d\to\infty}\left\{-\tfrac{1}{(s+1)}e^u\right\}\Big|_0^{-(s+1)d}$$

$$= \left\{\lim_{d\to\infty} -\tfrac{1}{(s+1)}e^{-(s+1)d}\right\} - \left\{-\tfrac{1}{(s+1)}e^0\right\} = \left\{-\frac{1}{(s+1)}\lim_{d\to\infty} e^{-(s+1)d}\right\} + \frac{1}{(s+1)} = \frac{1}{(s+1)}$$

Hence combining results in Step 1, and Step 2:

$L\{2t - 3e^{-t}\} = 2L\{t\} - 3L\{e^{-t}\} = \frac{2}{s^2} - \frac{3}{(s+1)} = \frac{2s+2-3s^2}{s^2(s+1)}$ (getting a common denominator)

- Example (Problem 6, Exercise 1, Sheng, page 4)

Find $L\{4e^{-3t+2}\}$

By linearity: $L\{4e^{-3t+2}\} = 4L\{e^{-3t+2}\}$

$$\text{So } L\{e^{-3t+2}\} = \int_0^\infty e^{-3t}e^2e^{-ts}\,dt = \lim_{d\to\infty}\int_0^d e^{-3t}e^2e^{-ts}\,dt = e^2\lim_{d\to\infty}\int_0^d e^{-(3+s)t}\,dt$$

(**Note 1:** e^2 is obviously a constant, so we can pull it out of the integral. By Linearity of LT, in fact, it could have bee pulled out in the first step: $4L\{e^{-3t+2}\} = 4e^2L\{e^{-3t}\}$. Moreover, notice that what we're left with is almost identical to the integral in **Step 2** in the previous example of page 4. From scratch, this integral can be evaluated using the u – substitution : $u(t) = -(3 + s)t$. And Formula 26. A-7. Borrowing from the result derived in **Step 2:**

$$26.\int_a^b e^u\,du = e^u\Big|_a^b \Rightarrow u = -(3+s)t, du = -(3+s)dt, \Rightarrow dt = -\tfrac{1}{(s+3)}du, t = -\tfrac{1}{(s+3)}u$$

$$\therefore \int_0^d e^{-(3+s)t}\,dt = \int_{u(0)}^{u(d)} e^u\left(-\tfrac{1}{(s+3)}du\right) = -\tfrac{1}{(s+3)}\int_0^{-(s+3)d} e^u\,du = -\tfrac{1}{(s+3)}e^u\Big|_0^{-(s+3)d}$$

So:

$$L\{e^{-3t}\} = \lim_{d\to\infty}\left\{-\tfrac{1}{(s+3)}e^u\right\}\Big|_0^{-(s+3)d}$$

$$= \left\{\lim_{d\to\infty} -\tfrac{1}{(s+3)}e^{-(s+3)d}\right\} - \left\{-\tfrac{1}{(s+3)}e^0\right\} = \left\{-\tfrac{1}{(s+3)}\lim_{d\to\infty} e^{-(s+1)d}\right\} + \tfrac{1}{(s+3)} = \tfrac{1}{(s\ +3)}$$

Hence: $L\{4e^{-3t+2}\} = 4e^2L\{e^{-3t}\} = \frac{4e^2}{(s+3)}$

It is no accident that the two results are so similar. We can by-pass the rigor in doing everything from the ground up each and every time, when evaluating LTs by way of the ***Shifting Theorems*** (Sheng, p. 22, 25)

- Example: (Problem 9, Sheng, page 4)

Here we have a graph of a piece-wise continuous function: $f(t) = \begin{Bmatrix} 2t & 0 \le t \le \pi \\ 2\pi & t > \pi \end{Bmatrix}$

Hence: $L\{f\} = \int_0^\infty f(t)e^{-ts}dt = \int_0^\pi 2te^{-ts}dt + \int_\pi^\infty 2\pi e^{-ts}dt = 2\int_0^\pi te^{-ts}dt + 2\pi\int_\pi^\infty e^{-ts}dt$

The two integrals (on the last expression to the right) are easy to evaluate, based on all the previous steps performed in the previous examples. They both involve simple *u*-substitutions. In the first integral (using the definite integral version of formula 27):

$$27.\int_a^b ue^u du = (u-1)e^u\Big|_a^b \Rightarrow u = -ts, du = -sdt, \Rightarrow dt = -\tfrac{1}{s}du, t = -\tfrac{1}{s}u$$

$$\therefore \int_0^\pi te^{-st}dt = \int_{u(0)}^{u(\pi)} \left(-\tfrac{1}{s}u\right)e^u\left(-\tfrac{1}{s}du\right) = \tfrac{1}{s^2}\int_0^{-s\pi} ue^u du = \tfrac{1}{s^2}(u-1)e^u\Big|_0^{-s\pi} =$$

$$\left\{\tfrac{1}{s^2}(-s\pi-1)e^{-s\pi}\right\} - \left\{\tfrac{1}{s^2}(0-1)e^0\right\} = \tfrac{-(s\pi+1)e^{-s\pi}+1}{s^2}$$

The second integral is just a simple improper integral using a *u*-substitution in formula 26):

$$26.\int_a^b e^u du = e^u\Big|_a^b \Rightarrow u = -st, du = -sdt, \Rightarrow dt = -\tfrac{1}{s}du, t = -\tfrac{1}{s}u$$

$$\therefore \int_\pi^\infty e^{-st}dt = \lim_{d\to\infty}\int_{u(\pi)}^{u(d)} e^u\left(-\tfrac{1}{s}du\right) = -\tfrac{1}{s}\lim_{d\to\infty}\int_{-s\pi}^{-sd} e^u du = -\tfrac{1}{s}\lim_{d\to\infty} e^u\Big|_{-s\pi}^{-sd}$$

$$= -\tfrac{1}{s}\left\{\lim_{d\to\infty} e^{-sd} - e^{-s\pi}\right\} = \tfrac{e^{-s\pi}}{s}$$

Hence:

$$L\{f\} = 2\int_0^\pi te^{-ts}dt + 2\pi\int_\pi^\infty e^{-ts}dt = 2\left[\tfrac{-(s\pi+1)e^{-s\pi}+1}{s^2}\right] + 2\pi\left[\tfrac{e^{-s\pi}}{s}\right] = \tfrac{-2s\pi e^{-s\pi}-2e^{-\pi s}+2+2s\pi e^{-\pi s}}{s^2}$$

$$= \tfrac{-2e^{-\pi s}+2}{s^2} = \tfrac{2}{s^2}\left(1-e^{-\pi s}\right)$$

(Getting a common denominator and canceling like terms)

III. Some Shortcuts in Calculating the LT

Recall, central results derived in Example 1-1-2 Example 1-2-4 (pages2, 9, Sheng):

$$L(e^{-at}) = \tfrac{1}{s-(-a)} \qquad L(t^n) = \tfrac{1}{s}L\left(nt^{n-1}\right)$$

As shown in pp 6-9. The latter recursion is a result of Theorem 2 (page 8), i.e. the Derivative Formula:

$$L\{f'(t)\} = sL\{f(t)\} - f(0)$$

As shown in page 8, the proof of Thm 2 involves a simple Integration by Parts: Let $U = e^{-st}$, $dV = f^{/}(t)dt$. For more details on the method of Integration by Parts, see pages 4-6, **Handout 1a**.)

In addition, there is the result of Theorem 5 (page 14):

$$L\{t^n f(t)\} = (-1)^n D^n{}_s L\{f(t)\}$$

(D_s is the derivative operator, with respect to variable s. Keep in mind that $L\{f(t)\}$ is a function of s.)

Sheng proves Theorem 5 in the special case for $n = 1$ (page 14). We can use that proof to prove the general result using Mathematical Induction:

Step1 (Base) ($n = 1$) $L\{tf(t)\} = (-1)D_s L\{f(t)\}$
(Proved in page 14 using the Leibnitz Rule)

Induction: Suppose $L\{t^k f(t)\} = (-1)^k D^k{}_s L\{f(t)\}$ for some k. Prove that:

$$L\{t^{k+1} f(t)\} = (-1)^{k+1} D^{k+1}{}_s L\{f(t)\}$$

Proof: $L\{t^{k+1} f(t)\} = L\{tt^k f(t)\}$. Redefine: $F(t) = t^k f(t)$. Then according to the Base Case: $L\{tF(t)\} = (-1)D_s L\{F(t)\}$

However, according to the induction assumption:

$$L\{F(t)\} = L\{t^k f(t)\} = (-1)^k D^k{}_s L\{f(t)\}$$

Hence: $L\{tF(t)\} = L\{t^{k+1} f(t)\} = (-1)D_s[(-1)^k D^k{}_s L\{f(t)\}] = (-1)^{k+1} D^{k+1}{}_s L\{f(t)\}$

Therefore $L\{t^n f(t)\} = (-1)^n D^n{}_s L\{f(t)\}$ for any integer n

We can apply the above results in parts of Problem 2, Exercise 2 (page 17)

- Example 2(a)

Find $L\{4t(\sin 4t - e^{-2t})\}$

By Linearity: $4L\{t\sin 4t\} - L\{e^{-2t}\}$

Using Theorem 5: $L\{t^n f(t)\} = (-1)^n D^n{}_s L\{f(t)\}$ (where $n = 1$):

$$L\{t\sin 4t\} = (-1)D_s L\{\sin 4t\}$$

To find $L\{\sin 4t\}$, use Euler's Theorem (See page 9, Handout1a for a derivation involving series methods):

Aside: Page 7 (Sheng) demonstrates how one may derive LTs for sine and cosine using Euler's Theorem and the linearity of the LT. Alternatively, we may also derive the same results by first isolating algebraically sine and cosine from Euler's Formulae, and then Applying the LT:

$$e^{i\theta} = \cos\theta + i\sin\theta$$
$$e^{-i\theta} = \cos\theta - i\sin\theta$$
$$\therefore \sin\theta = \tfrac{1}{2i}\left(e^{i\theta} - e^{-i\theta}\right), \cos\theta = \tfrac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right)$$

Hence:

$$L(\sin\theta t) = L\left(\tfrac{e^{i\theta t} - e^{-i\theta t}}{2i}\right) = \tfrac{1}{2i}L\left(e^{i\theta t}\right) - \tfrac{1}{2i}L\left(e^{-i\theta t}\right) = \tfrac{1}{2i}\tfrac{1}{s-i\theta} - \tfrac{1}{2i}\tfrac{1}{s+i\theta}$$
$$= \tfrac{1}{2i}\left[\tfrac{s+i\theta-(s-i\theta)}{s^2+\theta^2}\right] = \tfrac{1}{2i}\left[\tfrac{2i\theta}{s^2+\theta^2}\right] = \tfrac{\theta}{s^2+\theta^2}$$

$$L(\cos\theta t) = L\left(\tfrac{e^{i\theta t} + e^{-i\theta t}}{2}\right) = \tfrac{1}{2}L\left(e^{i\theta t}\right) + \tfrac{1}{2}L\left(e^{-i\theta t}\right) = \tfrac{1}{2}\tfrac{1}{s-i\theta} + \tfrac{1}{2i}\tfrac{1}{s+i\theta}$$
$$= \tfrac{1}{2}\left[\tfrac{s+i\theta+(s-i\theta)}{s^2+\theta^2}\right] = \tfrac{1}{2}\left[\tfrac{2s}{s^2+\theta^2}\right] = \tfrac{s}{s^2+\theta^2}$$

…which are the desired results. I mention this alternative approach because it's useful to refresh your memory on the algebra of complex numbers as much as you can.

Hence: $$(-1)D_s L(\sin 4t) = -\tfrac{d}{ds}\tfrac{4}{s^2+16} = -\left[-(8s)\left(s^2+16\right)^{-2}\right] = \tfrac{8s}{\left(s^2+16\right)^2}$$

(Using the Chain Rule and the Power rule in differentiating. You can also use the Quotient rule.)

Also, using the result derived in Example 1-2-2:

$$L\left(e^{-2t}\right) = \tfrac{1}{s-(-2)} = \tfrac{1}{s+2}$$

And applying Theorem 5 (I.e., the formula: $L\{t^n f(t)\} = (-1)^n D^n{}_s L\{f(t)\}$, when $n = 1$

$$L\left(te^{-2t}\right) = (-1)\tfrac{d}{ds}L\left(e^{-2t}\right) = -\tfrac{d}{ds}\tfrac{1}{s+2} = (s+2)^{-2} = \tfrac{1}{(s+2)^2}$$

Hence:
$$L\left\{4t\left(\sin 4t - e^{-2t}\right)\right\} = 4L(t\sin 4t) - 4L\left(te^{-2t}\right) = 4\left\{\tfrac{8s}{\left(s^2+16\right)^2} - \tfrac{1}{(s+2)^2}\right\}$$
$$= 4\left[\tfrac{8s(s+2)^2 - \left(s^2+16\right)^2}{\left(s^2+16\right)^2(s+2)^2}\right] = 4\left[\tfrac{8s^3+32s^2+32s-s^4-32s^2-256}{\left(s^2+16\right)^2(s+2)^2}\right] = \tfrac{-4\left(s^4-8s^3-32s+256\right)}{\left(s^2+16\right)^2(s+2)^2}$$