MA360 Handout 1a (week of Jan 9)
Sp2006 Kallfelz

Some Review of Integration Techniques and Series

No doubt you know how to use a Table of Integrals, like the list of integration formulas found in A6,A7 in Sheng. Nevertheless, for the sake of review (Calc. II), it's useful to see a few specific techniques applied (namely, integration by parts, trigonometric substitutions, etc.) I'll also briefly review here some other topics from Calc II relevant to this course: namely, the improper integral and Taylor Series methods. The review here will be short, and entirely example-driven. If you'd like a more extensive review, please let me know and I can send you some useful links. The key point to realize in this section, however, is to refresh yourself a bit on the *concepts* and *techniques*. But as you also learned about short-cut procedures in Calc II (when you got familiar enough with integration techniques that you could just refer to a Table, without having to re-derive each formula from scratch), so also in this course you will learn about various short-cut procedures for cranking out the Laplace Transforms (LTs). However, in the first batch of exercises (p.5, Sheng), we'll go about finding the (LTs) by brute force (just using the definition of the LT and its simple property like linearity.) Hence this review section here might be of use.

- Example 1: Trigonometric Substitutions (Differentiating and Integrating)

Underlying idea: Suppose you were asked to differentiate[1] an expression like:

$$\frac{d}{dx}\sin^{-1}\left(u(x)\right) = \frac{d}{dx}\arcsin(u(x)) \qquad \text{(Eqn II.1)}$$

One way to think about this is to consider what we mean by the arcsin, i.e., it's the inverse function of the sine, expressed therefore as an angular quantity (measured in radian units): θ = arcsin(u(x)). Therefore[2]: sinθ = sin(arcsin(u(x))) = $u(x)$.

Hence (Eqn II.1) amounts to asking us to find: $\frac{d}{dx}\theta$. We can solve this implicitly: θ = arcsin(u(x)) so: sinθ = $u(x)$.

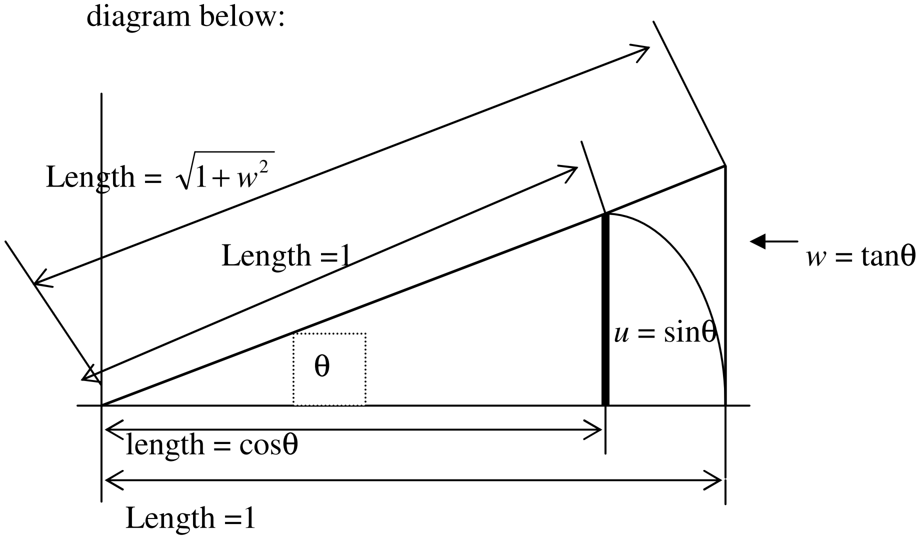
Differentiating both sides: $$\frac{d}{dx}\sin\theta = \cos\theta\frac{d\theta}{dx} = u'(x)$$

[1] I start with a differentiation exercise before talking about the integral. Integration, after all, is the inverse operation of differentiation.

[2] Since sin and arcsine are inverse functions, i.e. if $f(u)$ = sin(u) then $f^{-1}(u)$ = arcsin(u). By the definition of inverse functions: $f(f^{-1}(u)) = u = f^{-1}(f(\text{u}))$. Therefore, obviously, sin(arcsin(u)) = $f(f^{-1}(u)) = u$.

Hence, isolating $\frac{d}{dx}\theta$ above, we get: $$\frac{d\theta}{dx} = \frac{u'(x)}{\cos\theta}$$

So now the question becomes: how do we express $\cos\theta$ in terms of the function $u(x)$ alone? This is the underlying idea of the trigonometric substitution. Consider the diagram below:



The above is a segment of the unit circle. The small inscribed right triangle has a hypotenuse of length = 1. Therefore we see immediately using the Pythagorean Theorem that the length of the subtending side, parallel to the x-axis, = cosθ = $\sqrt{1-u^2}$, if we call u = sinθ. By the same token, note that the lager triangle, according to the Pythaogorean Theorem, has a hypotenuse of length $\sqrt{1+w^2}$ if its opposite side (tanθ) is denoted by w. So, for instance, if we wanted to find a derivative formula for arctan(w) then using the same procedure above, we'd eventually arrive at an expression, after differentiating implicitly: $\frac{d\theta}{dx} = \frac{w'(x)}{\sec^2\theta}$. But $\sec^2\theta = 1/\cos^2\theta = (\text{Hypotenuse/adjacent})^2 = 1+w^2$.
The point is, using the above construction procedure, we can *derive all* the differentiation formulae for the inverse trigonometric functions (i.e., arcsin, arccos, arctan, arcsec, etc.) So the underlying technique of Trigonometric Substitutions is based on the geometry of right triangles!

Completing the above example involving arcsin(u):

$$\frac{d\theta}{dx} = \frac{u'(x)}{\cos\theta} = \frac{u'(x)}{\sqrt{1-u^2}}$$ Therefore: $$\frac{d}{dx}\sin^{-1}(u(x)) = \frac{d}{dx}\arcsin(u(x)) = \frac{u'(x)}{\sqrt{1-u^2}}$$

Hence the anti-derivative becomes[3]: $$\int \frac{du}{\sqrt{1-u^2}} = \arcsin(u) + C$$

[3] Adopting the differential notation: $du = u'\,dx$

So, for instance, looking at the most general instance (Formula 4, page A6)

$$\int \frac{du}{\sqrt{a^2 - u^2}} = \arcsin(\tfrac{u}{a}) + C$$

…A result which is arrived at by substituting: $u = a\sin\theta$

<u>Proof:</u> $u = a\sin\theta$, so: $\sqrt{a^2 - u^2} = \sqrt{a^2 - a^2\sin^2\theta} = a\sqrt{1-\sin^2\theta} = a\cos\theta$
Also: $u = a\sin\theta$, so: $du = a\cos\theta d\theta$
And: $\frac{u}{a} = \sin\theta \Rightarrow \theta = \arcsin\left(\tfrac{u}{a}\right)$ (the symbol "⇒" means "so," or "therefore.")

So: $$\boxed{\int \frac{du}{\sqrt{a^2 - u^2}} = \int \frac{a\cos\theta d\theta}{a\cos\theta} = \int d\theta = \theta + C = \arcsin\left(\frac{u}{a}\right) + C}$$

By the same token, suppose we were asked to find: $\frac{d}{dx}\theta$, where: θ = arctan(w(x)). As in the previous example involving arcsin, we differentiate implicitly:

$\frac{d}{dx}(\tan\theta) = \sec^2\theta \frac{d\theta}{dx} = w'(x)$. In addition, as the remarks under the illustration

on page 3 show: $\frac{d\theta}{dx} = \frac{w'(x)}{\sec^2\theta} = \frac{w'(x)}{1+w^2}$.

Therefore: $\frac{d}{dx}\tan^{-1}(w(x)) = \frac{d}{dx}\arctan(w(x)) = \frac{w'(x)}{1+w^2}$ (for any differentiable function w)

Hence the antiderivative becomes: $\int \frac{du}{1+u^2} = \arctan(u) + C$ (for any differntialbe function u)

So, for instance, looking at the most general instance (Formula 3, page A6)

$$\int \frac{du}{a^2 + u^2} = \tfrac{1}{a}\arctan(\tfrac{u}{a}) + C$$

…A result which is arrived at by substituting: $u = a\tan\theta$

<u>Proof:</u> $u = a\tan\theta$, so: $a^2 + u^2 = a^2 + a^2\tan^2\theta = a^2\left(1+\tan^2\theta\right) = a^2\sec^2\theta$
Also: $u = a\tan\theta$, so: $du = a\sec^2\theta d\theta$

And: $\frac{u}{a} = \tan\theta \Rightarrow \theta = \arctan\left(\frac{u}{a}\right)$ (the symbol "⇒" means "so," or "therefore.")

So: $$\int \frac{du}{a^2+u^2} = \int \frac{a\sec^2\theta d\theta}{a^2\sec^2\theta} = \frac{1}{a}\int d\theta = \frac{1}{a}\theta + C = \frac{1}{a}\arctan\left(\frac{u}{a}\right) + C$$

- Exercise: Repeat the procedure above for θ = arcsec(u(x)). (I.e. find differentiation and integration formulae)

In general, the underlying strategy of trigonometric substitutions is summarized in the below:

1. If your integrand involves an expression of the form: $\left(a^2-u^2\right)^{\frac{n}{2}} = \left(\sqrt{a^2-u^2}\right)^n$ for any integer n, then substitute: $u = a\sin\theta$, and your integral will be transformed into one involving powers of sine and cosine.
2. If your integrand involves an expression of the form: $\left(a^2+u^2\right)^{\frac{n}{2}} = \left(\sqrt{a^2+u^2}\right)^n$ for any integer n, then substitute: $u = a\tan\theta$, and your integral will be transformed into one involving powers of tangent and secant.
3. If your integrand involves an expression of the form: $\left(u^2-a^2\right)^{\frac{n}{2}} = \left(\sqrt{u^2-a^2}\right)^n$ for any integer n, then substitute: $u = a\sec\theta$, and your integral will be transformed into one involving powers of tangent and secant.

Note: The above rules suffice for *all* of the associated (6) trig functions (sine, cosine, tangent, secant, cosecant, cotangent), since with no loss of generality Rules 1., 2., 3. could have instead been specified in terms of $u = a\cos\theta$, $u = a\cot\theta$, $u = a\csc\theta$. What connects the latter three trig functions with those listed above (i.e., $a\sin\theta$, $u = a\tan\theta$, $u = a\sec\theta$) are the three *Pythagorean Identities:* for any x: $\sin^2 x + \cos^2 x = 1$ *iff* $\tan^2 x + 1 = \sec^2 x$ *iff* $\cot^2 x + 1 = \csc^2 x$

• Example2: Integration by parts

Integration by parts is a recursive procedure you're no doubt well-familiar with. It's a 'divide and conquer strategy,' i.e., it transforms a more complicated integral into a function and a resulting integral that's less complicated. It's seldom the case that you can utilize integration by parts in just one iteration. (In fact, there's a general method you may be familiar with, *tabular integration* which speeds up the procedure if you're faced with an integrand requiring several iterations using Integration by Parts. As with most techniques in advanced integration, however, Tabular Integration can't be applied across-the-board: it depends on the form of the intgrand.)

Integration by Parts (Indefinite Integrals) $\int u\,dv = uv - \int v\,du$

Integration by Parts (Definite Integrals) $\int_a^b u\,dv = (uv)\Big|_a^b - \int_a^b v\,du$

(**where:** $(uv)|_a^b$ is shorthand for: $u(b)v(b) - u(a)v(a)$)

For example, Formulae 24.-31. (A 7) are all derivable by Integration by Parts;

1.) Formula 24: $\int \arcsin u\,du = u \arcsin u + \sqrt{1-u^2} + C$

Proof: Let $U = \arcsin u$ $\quad dV = du$

Then: $dU = \dfrac{du}{\sqrt{1-u^2}}$ (recall procedure in page 2 for differentiating arsine, above) $\quad V = u$

So: $\int U\,dV = UV - \int V\,dU = u \arcsin u - \int \dfrac{u\,du}{\sqrt{1-u^2}}$ (A)

The integral on the left hand side is easily evaluated using the following substitution:

$W = (1- u^2)$ $\quad$ so $dW = -2u\,du \Rightarrow u\,du = -\tfrac{1}{2}\,dW$

So according to the Power Rule:

$$\int \frac{u\,du}{\sqrt{1-u^2}} = -\tfrac{1}{2}\int W^{-\frac{1}{2}}dW = -\tfrac{1}{2}\left[\frac{W^{-\frac{1}{2}+1}}{-\frac{1}{2}+1}\right] = -W^{\frac{1}{2}} = -\sqrt{1-u^2} \qquad \text{(B)}$$

Hence substituting this result (B) into (A) above, and adding the undetermined constant, we get the desired result.

2.) Formula 30: $\int e^{au}(\sin bu)\,du = \dfrac{e^{au}}{(a^2+b^2)}[a \sin bu - b \cos bu] + C$

This involves a trick involving iterating twice and isolating your final answer algebraically, i.e., like an unknown in an algebra equation.

Let $U = e^{au}$ $\quad dV = \sin bu \quad \Rightarrow \quad dU = ae^{au}du \quad V = -\tfrac{1}{b}\cos bu$

$$\int U\,dV = \int e^{au} \sin bu\,du = UV - \int V\,dU = -\frac{e^{au}}{b}\cos bu + \tfrac{a}{b}\int e^{au}\cos bu\,du \qquad \text{(C)}$$

Now perform the same kind of substitution on right hand side integral:

Let $U = e^{au}$ $\qquad dV = \cos bu \quad \Rightarrow \quad dU = ae^{au}du \quad V = {}^{1}/_{b}\sin bu$

$$\int UdV = \int e^{au}\cos buu = UV - \int VdU = \frac{e^{au}}{b}\sin bu - \tfrac{a}{b}\int e^{au}\sin budu \qquad \text{(D)}$$

Substituting result (D) into (C) :

$$\int e^{au}\sin bu = -\frac{e^{au}}{b}\cos bu + \tfrac{a}{b}\left\{\frac{e^{au}}{b}\sin bu - \tfrac{a}{b}\int e^{au}\sin budu\right\}$$
$$= -\tfrac{e^{au}}{b}\cos bu + \tfrac{ae^{au}}{b^2}\sin bu - \tfrac{a^2}{b^2}\int e^{au}\sin budu$$

So the right hand side has an integral of the same expression as the left hand side. We can carry over the integral on the right hand side, therefore, and isolate it like an unknown in an algebraic equation:

$$\left(1 + \tfrac{a^2}{b^2}\right)\int e^{au}\sin budu = e^{au}\left(-\tfrac{1}{b}\cos bu + \tfrac{a}{b^2}\sin bu\right) = \frac{e^{au}}{b^2}(-b\cos bu + a\sin bu)$$

(rearranging terms on the right hand side in the above expression, by getting a common denominator)

So: $$\boxed{\int e^{au}\sin budu = \frac{e^{au}}{b^2}\frac{(-b\cos bu + a\sin bu)}{\left(1 + \tfrac{a^2}{b^2}\right)} = \frac{e^{au}(a\sin bu - b\cos bu)}{b^2 + a^2} + C}$$

There are some general 'rules of thumb' for choosing U and dV the right way, to simplify an integral. As I wrote, many important results listed in tables of integrals are constructed using Integration by Parts (including 24.-31. in A 7). (For more information, regarding more complicated integrals, you can visit my Calculus I,II,III link, under "**Recent Courses Taught"** in my homepage:
(www.glue.umd.edu/~wkallfel/wmkhome.html) for instance, where I have numerously detailed handouts in the Calc II folder.) In this course, in particular, two important rules of thumb to keep in mind (the first is the basis of Formula 28., A 7) include:

a.) If your integrand involves the expression: $e^{au}u^n du$, then choose: $dV = e^{au}du$ and $U = u^n$.
b.) If your integrand involves the expression: $(\ln bu)^m u^n du$, then choose: $dV = u^n du$ and $U = (\ln bu)^m du$.

Exercise: Find: $\int (\ln x)^3 x^4 dx$

- Example 3: Improper Integrals

Improper integrals are *definite* integrals, whose limit-points diverge (i.e., go to ∞) or the whose integrand diverges (i.e., has an essential singularity, i.e., a vertical asymptote) or both. Some examples include of course the Laplace Transform itself:

$$L\{f(t)\} = F(s) = \int_0^\infty f(t)e^{-ts}\,dt \text{ , etc.}$$

Formally, the divergences should be treated as *limits*: That is to say:

$$\int_0^\infty f(t)e^{-ts}\,dt \quad \text{is shorthand for:} \qquad \lim_{d\to\infty} \int_0^d f(t)e^{-ts}\,dt$$

a.) Consider, for example, the definite integral: $\int_0^1 \frac{dx}{\sqrt{1-x^2}}$. This integral is *improper*, because on the interval [0,1] of integration, the integrand has an essential singularity at Right endpoint: $x = 1$. Hence we need to rewrite the integral in terms of a limit:

$$\int_0^2 \frac{dx}{\sqrt{1-x^2}} = \lim_{c\to 1^-} \int_0^c \frac{dx}{\sqrt{1-x^2}}$$

(where the "–" superscript on 1 means it's a left-hand limit)

Hence, evaluating:

$$\int_0^2 \frac{dx}{\sqrt{1-x^2}} = \lim_{c\to 1^-} \int_0^c \frac{dx}{\sqrt{1-x^2}} = \lim_{c\to 1^-} [\arcsin x]\big|_0^c = (\lim_{c\to 1^-} \arcsin c) - \arcsin 0 = \tfrac{\pi}{2}$$

So even though the integrand has a vertical asymptote at its right endpoint, the area under it is still finite.

b.) Consider the integral: $\int_0^\infty t^3 e^{-st}\,dt = \lim_{d\to\infty} \int_0^d t^3 e^{-ts}\,dt$, i.e., the Laplace transform for $f(t) = t^3$. We could iterate Formula 28. (A-7) several times, to get the antiderivative, but an even quicker way involves Tabular Integration by Parts:

***U* and its derivatives**		***dV* and its antiderivatives (note: *s* behaves as a constant)**
t^3	(+)	e^{-st}
$3t^2$	(-)	$-s^{-1}e^{-st}$
$6t$	(+)	$s^{-2}e^{-st}$
6	(-)	$-s^{-3}e^{-st}$
0		$s^{-4}e^{-st}$

Hence, from the above Table, the antiderivative is:

$$\int_0^\infty t^3 e^{-st}dt = \lim_{d\to\infty}\int_0^d t^3 e^{-ts}dt = \lim_{d\to\infty}\left\{-t\frac{e^{-st}}{s} - 3t^2\frac{e^{-st}}{s^2} - 6t\frac{e^{-st}}{s^3} - 6\frac{e^{-st}}{s^4}\right\}\Bigg|_0^d$$

$$= \lim_{d\to\infty}\left\{-d\frac{e^{-sd}}{s} - 3d^2\frac{e^{-sd}}{s^2} - 6d\frac{e^{-sd}}{s^3} - 6\frac{e^{-sd}}{s^4}\right\} - \left\{-\frac{6}{s^4}\right\} = \frac{6}{s^4}$$

Note: the limits are all zero since e^{-st} shrinks must faster than t^n. In fact, you can show L'Hopital's Rule that: $\lim_{t\to\infty} t^n e^{-st} = 0$, for any integer n. Also be careful to note that when evaluating the antiderivative at the point 0, most terms vanish except for the very last term in {…} above, since $e^0 = 1$.

- Example 4: Taylor and McClaurin Series

For an in-depth look at Taylor's Theorem, including its proof and formulae for error bounds, see the first three handouts in the MA355link under "**Recent Courses Taught**" in my homepage:
(www.glue.umd.edu/~wkallfel/wmkhome.html)

Here I just prove an interesting result, i.e. Euler's Theorem, (which has much relevance in MA360) using series methods.

The functions e^x, sin$x$, cos$x$ all have the following interval of convergence: (-∞, ∞). That is to say, the associated Taylor polynomial (of degree n, centered at x_0) :

$P_n(x) = \sum_{k=0}^{n}\frac{f^{(k)}(x_0)}{k!}(x-x_0)^k$ converges (i.e., $\lim_{n\to\infty}P_n(x)$ exists, and remainder term $\lim_{n\to\infty}R_n(x) = 0$) for *any* x_0 lying in interval (-∞, ∞), for the above functions. Hence, for any x_0 lying in interval (-∞, ∞), the (infinite) Taylor is formally defined as:

$$P_f(x) = \lim_{n\to\infty} P_n(x) = \sum_{k=0}^{\infty}\frac{f^{(k)}(x_0)}{k!}(x-x_0)^k$$

Moreover: $\quad P_f(x) = \sum_{k=0}^{\infty}\frac{f^{(k)}(x_0)}{k!}(x-x_0)^k = f(x) \qquad$ for $f(x) = e^x$, sin$x$, cos$x$

Choosing $x_0 = 0$, we produce the McClaurin Series: $M(x) = \sum_{k=0}^{\infty}\frac{f^{(k)}(0)}{k!}x^k$. The McClaurin Series for e^x, sin$x$, cos$x$ are[4]:

[4] For details of the derivation, consult the first few handouts in the MA355 folder

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \tfrac{x^2}{2!} + \tfrac{x^3}{3!} + \tfrac{x^4}{4!} + \ldots + \tfrac{x^n}{n!} + \ldots$$

$$\sin x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!} = x - \tfrac{x^3}{3!} + \tfrac{x^5}{5!} - \tfrac{x^7}{7!} + \ldots + (-1)^n \tfrac{x^{(2n+1)}}{(2n+1)!} + \ldots$$

$$\cos x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!} = 1 - \tfrac{x^2}{2!} + \tfrac{x^4}{4!} - \tfrac{x^6}{6!} + \ldots + (-1)^n \tfrac{x^{(2n)}}{(2n)!} + \ldots$$

We will use the above McClaurin series to verify Euler's Theorem: $e^{i\theta} = \cos\theta + i\sin\theta$.

Observe:

$$e^{i\theta} = \sum_{k=0}^{\infty} \frac{(i\theta)^k}{k!} = 1 + (i\theta) + \tfrac{(i\theta)^2}{2!} + \tfrac{(i\theta)^3}{3!} + \tfrac{(i\theta)^4}{4!} + \ldots + \tfrac{(i\theta)^n}{n!} + \ldots$$
$$= \left\{1 - \tfrac{\theta^2}{2!} + \tfrac{\theta^4}{4!} - \tfrac{\theta^6}{6!} + \ldots\right\} + \left\{i\theta - \tfrac{i\theta^3}{3!} + \tfrac{i\theta^5}{5!} - \tfrac{i\theta^7}{7!} + \ldots\right\}$$

using property: $i^2 = -1$, $i^3 = -i$, $i^4 = 1$, $i^5 = i, \ldots, i^n = i^{n(\mathrm{mod}4)}$ and rearranging terms in the above series. (Note: the superscript n(mod4) stands for the remainder of n when n is divided by 4. For example: 23(mod4)=3.)

But factoring out the i-term in the above series:

$$e^{i\theta} = \sum_{k=0}^{\infty} \frac{(i\theta)^k}{k!} = \left\{1 - \tfrac{\theta^2}{2!} + \tfrac{\theta^4}{4!} - \tfrac{\theta^6}{6} + \ldots\right\} + i\left\{\theta - \tfrac{\theta^3}{3!} + \tfrac{\theta^5}{5!} - \tfrac{\theta^7}{7!} + \ldots\right\}$$

By inspection we see the first terms in {…} is the McClaurin series for $\cos\theta$, whereas the terms in the second expression i{…} is the McClaurin series for $\sin\theta$. Hence:

$$e^{i\theta} = \sum_{k=0}^{\infty} \frac{(i\theta)^k}{k!} = \left\{\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} \theta^{2k}\right\} + i\left\{\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} \theta^{2k+1}\right\} = \cos\theta + i\sin\theta$$