MA360 **Handout 8a (week of March 13)**
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- **<u>Problem 1, p 80 via LT Method</u>**

$\ddot{x}(t) - 5\dot{x}(t) = t - 2, x(0) = 0, \dot{x}(0) = 2$

Prior to applying the LT technique (in answer to student question) note that when solving this by UC method, one must choose a *quadratic* particular (steady-state) solution. Why? because the LHS (left hand side) involves *only derivatives* of the unknown function $x(t)$. (No term involves just $x(t)$.) So, for example, the first derivative term would involve a linear power of t.[1])

Since the auxiliary equation is: $r^2 - 5r = r(r - 5) = 0$

$\Rightarrow x_{tr}(t) = c_1 e^{0t} + c_2 e^{5t} = c_1 + c_2 e^{5t}$

Choose $x_{ss}(t) = At^2 + Bt + C \Rightarrow \dot{x}_{ss}(t) = 2At + B \Rightarrow \ddot{x}_{ss}(t) = 2A$

Inserting into ODE: $2A - 5(2At + B) = t - 2 \Rightarrow \begin{matrix} t^1: & -10A = 1 \Rightarrow A = -\frac{1}{10} \\ t^0: & 2A - 5B = -\frac{1}{5} - 5B = -2 \Rightarrow B = \frac{9}{25} \end{matrix}$

So: $x(t) = x_{tr}(t) + x_{ss}(t) = c_1 + c_2 e^{5t} - \frac{1}{10}t^2 + \frac{9}{25}t, \dot{x}(t) = 5c_2 e^{5t} - \frac{1}{5}t + \frac{9}{25}$

Inserting initial conditions: $x(0) = 0 = c_1 + c_2, \dot{x}(0) = 2 = 5c_2 + \frac{9}{25} \Rightarrow c_2 = \frac{41}{125}, c_1 = -\frac{41}{125}$

Hence: $x(t) = -\frac{41}{125} + \frac{9}{25}t - \frac{1}{10}t^2 + \frac{41}{125}e^{5t}$

Note that there are two different strategies (of roughly equal # of steps) when applying the LT method: 1) Combining all terms on RHS and using partial fractions, 2) Keeping all fractions separate on RHS and using THM12.

First, the LT of the ODE is:

$$s^2Y(s) - sx(0) - \dot{x}(0) - 5(sY(s) - x(0)) = s^2Y(s) - 5sY(s) - 2 = \frac{1}{s^2} - \frac{2}{2} \Rightarrow s(s-5)Y(s) = \frac{1}{s^2} - \frac{2}{s} + 2$$

Method 1: Combine all terms on RHS into one fraction, and use Partial Fractions

$$s(s-5)Y(s) = \frac{1-2s+2s^2}{s^2} \Rightarrow Y(s) = \frac{1-2s+2s^2}{s^3(s-5)} = \frac{A_1}{s} + \frac{A_2}{s^2} + \frac{A_3}{s^3} + \frac{B}{(s-5)}$$

[1] This can be generalized by saying that if the LHS has a lowest-order derivative of order k, and $g(t)$ on the RHS is an n degree polynomial, then we must choose $x_{ss}(t)$ to be an $(n + k)$ – degree polynomial.

So: $2s^2 - 2s + 1 = A_1 s^2(s-5) + A_2 s(s-5) + A_3(s-5) + Bs^3$

$(s=0) \Rightarrow 1 = -5A_3 \Rightarrow A_3 = -\frac{1}{5}$

$(s=5) \Rightarrow 41 = 125B \Rightarrow B = \frac{41}{125}$

$(s=1) \Rightarrow 1 = -4A_1 - 4A_2 + \frac{4}{5} + \frac{41}{125} = -4(A_1 + A_2) + \frac{141}{125} \Rightarrow -\frac{16}{125} = -4(A_1 + A_2) \Rightarrow \frac{4}{125} = A_1 + A_2$

$(s=2) \Rightarrow 5 = -12A_1 - 6A_2 + \frac{3}{5} + \frac{328}{125} \Rightarrow 5 = -6(2A_1 + A_2) + \frac{403}{125} \Rightarrow \frac{111}{125} = -6A_1 - 3A_2$

Solving for A_1, A_2 :

$$\begin{matrix} \frac{12}{125} = 3A_1 + 3A_2 \\ \frac{111}{125} = -6A_1 - 3A_2 \end{matrix} \Rightarrow \frac{123}{125} = -3A_1 \Rightarrow A_1 = -\frac{41}{125}, A_2 + A_1 = \frac{4}{125} \Rightarrow A_2 = \frac{45}{125} = \frac{9}{25}$$

Hence:

$$Y(s) = \frac{1-2s+2s^2}{s^3(s-5)} = -\frac{41_1}{125s} + \frac{9}{25s^2} - \frac{1}{5s^3} + \frac{41}{125(s-5)}$$

$$= -\frac{41}{125} L^{-1}\{t^0\} + \frac{9}{25} L^{-1}\{t\} - \frac{1}{5} L^{-1}\{\tfrac{1}{2} t^2\} + \frac{41}{125} L^{-1}\{e^{5t}\}$$

So: $x(t) = L^{-1}\{Y(s)\} = -\frac{41}{125} + \frac{9}{25} t - \frac{1}{10} t^2 + \frac{41}{125} e^{5t}$

Method 2: Keep terms on RHS separate, and use THM12 (when necessary)

$$s(s-5)Y(s) = \frac{1}{s^2} - \frac{2}{s} + 2 \Rightarrow Y(s) = \frac{1}{s^3(s-5)} - \frac{2}{s^2(s-5)} + \frac{2}{s(s-5)} \equiv \alpha(s) + \beta(s) + \gamma(s)$$

Now, the last fraction (denoted $\gamma(s)$) has a denominator term which is the product of two linear irreducibles. Hence one can use the Heaviside Cover method:

$$\frac{2}{s(s-5)} = \frac{A}{s} + \frac{B}{(s-5)} \Rightarrow (s=0) \to A = -\frac{2}{5}, (s=5) \to B = \frac{2}{5}$$

$$L^{-1}\{\gamma(s)\} = L^{-1}\left\{-\frac{2}{5s} + \frac{2/5}{(s-5)}\right\} = -\frac{2}{5} + \frac{2}{5} e^{5t}$$

The second fraction (denoted $\beta(s)$) is a candidate for THM12, since factoring the denominator terms one readily recognizes simple inverse LT functions:

$$\beta(s) = \frac{-2}{s^2(s-5)} = -2\left\{\frac{1}{s^2} \cdot \frac{1}{(s-5)}\right\} = -2\{L\{t\} \cdot L\{e^{5t}\}\} = -2L\{(t * e^{5t})\}$$

where: $\left(t * e^{5t}\right) = \int_0^t (t-u)e^{5u}\,du = \int_0^t u e^{5(t-u)}\,du$

Selecting the second representation of the convolution yields a slightly simpler integral (since t behaves as a constant):

$$\int_0^t ue^{5(t-u)}du = e^{5t}\int_0^t ue^{-5u}du \quad \text{Let } w = -5u\text{, hence: } e^{5t}\int_0^t ue^{-5u}du = \tfrac{e^{5t}}{25}\int_0^{-5t} we^{w}dw$$

Integrating by parts:

$$\tfrac{e^{5t}}{25}\int_0^{-5t} we^{w}dw = \tfrac{e^{5t}}{25}\left\{ we^{w}\Big|_0^{-5t} - \int_0^{-5t} e^{w}dw \right\} = \tfrac{e^{5t}}{25}\left\{ -5te^{-5t} - e^{w}\Big|_0^{-5t} \right\}$$

$$= \tfrac{e^{5t}}{25}\left\{ -5te^{-5t} - e^{-5t} + 1 \right\} = \tfrac{1}{25}\left(-5t - 1 + e^{5t} \right)$$

Hence: $L^{-1}\{\beta(s)\} = -2\{(t * e^{5t})\} = -\tfrac{2}{25}\left(-5t - 1 + e^{5t}\right) = \tfrac{1}{25}\left(10t - 2e^{5t} + 2\right)$

Similarly for the first fraction (denoted $\alpha(s)$) is another candidate for THM12:

$$\alpha(s) = \tfrac{1}{s^3(s-5)} = \left\{ \tfrac{1}{s^3} \cdot \tfrac{1}{(s-5)} \right\} = \left\{ L\left\{ \tfrac{1}{2}t^2 \right\} \cdot L\left\{ e^{5t} \right\} \right\} = \tfrac{1}{2} L\left\{ \left(t^2 * e^{5t} \right) \right\}$$

where: $\left(t^2 * e^{5t} \right) = \int_0^t (t-u)^2 e^{5u}du = \int_0^t u^2 e^{5(t-u)}du$

Again, selecting the second representation of the convolution yields a slightly simpler integral (since *t* behaves as a constant):

$$\int_0^t ue^{5(t-u)}du = e^{5t}\int_0^t u^2 e^{-5u}du \quad \text{Let } w = -5u\text{, hence: } e^{5t}\int_0^t u^2 e^{-5u}du = -\tfrac{e^{5t}}{125}\int_0^{-5t} w^2 e^{w}dw$$

Integrating by parts:

$$= -\tfrac{e^{5t}}{125}\int_0^{-5t} w^2 e^{w}dw = -\tfrac{e^{5t}}{125}\left\{ w^2 e^{w}\Big|_0^{-5t} - 2\int_0^{-5t} we^{w}dw \right\} = -\tfrac{e^{5t}}{125}\left\{ 25t^2 e^{-5t} - 2we^{w}\Big|_0^{-5t} + 2\int_0^{-5t} e^{w}dw \right\}$$

$$= -\tfrac{e^{5t}}{125}\left\{ 25t^2 e^{-5t} + 10te^{-5t} + 2e^{w}\Big|_0^{-5t} \right\} = -\tfrac{e^{5t}}{125}\left(25t^2 e^{-5t} + 10te^{-5t} + 2e^{-5t} - 2 \right) = -\tfrac{1}{5}t^2 - \tfrac{2}{25}t - \tfrac{2}{125} + \tfrac{2}{125}e^{5t}$$

Hence: $L^{-1}\{\alpha(s)\} = \tfrac{1}{2}\left\{ \left(t^2 * e^{5t} \right) \right\} = \tfrac{1}{2}\left\{ -\tfrac{1}{5}t^2 - \tfrac{2}{25}t - \tfrac{2}{125} + \tfrac{2}{125}e^{5t} \right\} = -\tfrac{1}{10}t^2 - \tfrac{1}{25}t - \tfrac{1}{125} + \tfrac{1}{125}e^{5t}$

So:

$$x(t) = L^{-1}\{Y(s)\} = L^{-1}\{\alpha(s)\} + L^{-1}\{\beta(s)\} + L^{-1}\{\gamma(s)\}$$
$$= \left\{ -\tfrac{1}{10}t^2 - \tfrac{1}{25}t - \tfrac{1}{125} + \tfrac{1}{125}e^{5t} \right\} + \tfrac{1}{25}\left\{ 10t + 2 - 2e^{5t} \right\} - \tfrac{2}{5} + \tfrac{2}{5}e^{5t}$$
$$= -\tfrac{1}{10}t^2 + \tfrac{9}{25}t - \tfrac{41}{125} + \tfrac{41}{125}e^{5t}$$

Of course, there is yet *another* method in which one could apply partial fractions to the above terms $\alpha(s)$, $\beta(s)$, but this would take just as much work (if not more) as Method1, where all the fractions are lumped together into one fraction, and subsequently applying partial fractions to it.

- **Problem 3, p 81 via LT Method:**

$$\ddot{x}(t) - 3\dot{x}(t) + 2x(t) = 8t^2 + 12e^{-t}, x(0) = 0, \dot{x}(0) = 2$$

As a review, according to UC method:

Auxiliary equation: $r^2 - 3r + 2 = (r-2)(r-1) = 0 \Rightarrow x_{tr}(t) = c_1 e^{2t} + c_2 e^t$

Since

$$g(t) = 8t^2 + 12e^{-t} \Rightarrow x_{ss}(t) = At^2 + Bt + C + De^{-t}, \dot{x}_{ss}(t) = 2At + B - De^{-t}, \ddot{x}_{ss}(t) = 2A + De^{-t}$$

Hence inserting $x_{ss}(s)$ into the ODE:

$$\begin{aligned}
&\left(2A + De^{-t}\right) - 3\left(2At + B - De^{-t}\right) + 2\left(At^2 + Bt + C + De^{-t}\right) = 8t^2 + 12e^{-t} \\
&t^2: \quad 2A = 8 \Rightarrow A = 4 \\
&t^1: \quad -6A + 2B = 0 \Rightarrow B = 12 \\
&t^0: \quad 2A - 3B + 2C = 0 \Rightarrow C = 14 \\
&e^{-t}: \quad D + 3D + 2D = 12 \Rightarrow D = 2
\end{aligned}$$

$$\begin{aligned}
&x(t) = c_1 e^{2t} + c_2 e^t + 4t^2 + 12t + 14 + 2e^{-t}, \dot{x}(t) = 2c_1 e^{2t} + c_2 e^t + 8t + 12 - 2e^{-t} \\
&\Rightarrow x(0) = 0 = c_1 + c_2 + 16, \dot{x}(0) = 2 = 2c_1 + c_2 + 10 \\
&\therefore c_1 = 8, c_2 = -24
\end{aligned}$$

So: $\quad x(t) = 8e^{2t} - 24e^t + 4t^2 + 12t + 14 + 2e^{-t}$

Using LTs:

$$\begin{aligned}
&L\{\ddot{x}(t) - 3\dot{x}(t) + 2x(t)\} = \left(s^2 Y - s \cdot 0 - 2\right) - 3(sY - 0) + 2Y = L\left\{8t^2 + 12e^{-t}\right\} = \tfrac{16}{s^3} + \tfrac{12}{s+1} \\
&\Rightarrow \left(s^2 - 3s + 2\right)Y - 2 = \tfrac{16}{s^2} + \tfrac{12}{s+1} \Rightarrow Y(s) = \tfrac{16}{s^3(s-2)(s-1)} + \tfrac{12}{(s+1)(s-2)(s-1)} + \tfrac{2}{(s-2)(s-1)}
\end{aligned}$$

The second and third fractions, since their denominator terms consist in non repeating linear irreducibles, can be easily resolved using the Heaviside Cover method:

$$\frac{12}{(s+1)(s-2)(s-1)}=\frac{A}{s+1}+\frac{B}{s-2}+\frac{C}{s-1}\Rightarrow\begin{cases}s=-1: & A=\frac{12}{(-3)(-2)}=6\\ s=2: & B=\frac{12}{3}=4\\ s=1: & C=\frac{12}{-2}=-6\end{cases}$$

$$\frac{2}{(s-2)(s-1)}=\frac{A}{s-2}+\frac{B}{s-1}\Rightarrow\begin{cases}s=2: & A=2\\ s=1: & B=-2\end{cases}$$

Hence the inverse LTs of the second and third fractions are:

$$L^{-1}\left\{\left(\tfrac{2}{s+1}+\tfrac{4}{s-2}-\tfrac{6}{s-1}\right)+\left(\tfrac{2}{s-2}-\tfrac{2}{s-1}\right)\right\}=\left(2e^{-t}+4e^{2t}-6e^{t}\right)+\left(2e^{2t}-2e^{t}\right)=6e^{2t}-8e^{t}+2e^{-t}$$

Applying partial fractions to the first term:

$$\frac{16}{s^3(s-2)(s-1)}=\frac{A_1}{s}+\frac{A_2}{s^2}+\frac{A_3}{s^3}+\frac{B}{s-2}+\frac{C}{s-1}\Rightarrow 16=(A_1s^2+A_2s+A_3)(s-2)(s-1)+Bs^3(s-1)+Cs^3(s-2)$$

$s=0:\quad 16=2A_3\Rightarrow A_3=8$

$s=1:\quad 16=-C\Rightarrow C=-16$

$s=2:\quad 16=8B\Rightarrow B=2$

So: $\begin{matrix}s=-1: & 16=6A_1-6A_2+48+4-48\Rightarrow 2=A_1-A_2\\ s=-2: & 16A_1-24A_2+96+48-512\Rightarrow 16=2A_1-A_2\end{matrix}\Rightarrow A_1=14, A_2=12$

Hence the inverse LT of the first term is:

$$L^{-1}\left\{\tfrac{14}{s}+\tfrac{12}{s^2}+\tfrac{8}{s^3}+\tfrac{2}{s-2}-\tfrac{16}{s-1}\right\}=14+12t+4t^2+2e^{2t}-16e^{t}$$

Combining the two results:

$$x(t)=L^{-1}\{Y\}=\left(14+12t+4t^2+2e^{2t}-16e^{t}\right)+\left(6e^{2t}-8e^{t}+2e^{-t}\right)=14+12t+4t^2+8e^{2t}-24e^{t}+2e^{-t}$$

- Problem 7 (p 81 Sheng)

$$\ddot{x}(t)+x(t)=-9\cos 2t,\ x(0)=2,\ \dot{x}(0)=1$$

Again, for the sake of review, using the method of UC:

$$r^2+1=0\Rightarrow r_{1,2}=\pm i\Rightarrow x_{tr}(t)=c_1\cos t+c_2\sin t$$

$$x_{ss}(t) = A\cos 2t + B\sin 2t \Rightarrow \ddot{x}_{ss}(t) = -4x_{ss}(t) \Rightarrow \ddot{x}_{ss}(t) + x_{ss}(t) = -3x_{ss}(t)$$
$$= -3A\cos 2t - 3B\sin 2t = -9\cos 2t \Rightarrow A = 3, B = 0$$
$$\therefore x(t) = c_1\cos t + c_2\sin t + 3\cos 2t, \dot{x}(t) = -c_1\sin t + c_2\cos t - 6\sin 2t$$
$$x(0) = 2 = c_1 + 3 \Rightarrow c_1 = -1, \dot{x}(0) = 1 = c_2$$
$$\therefore x(t) = -\cos t + \sin t + 3\cos 2t$$

Using the method of LTs:

$$L\{\ddot{x} + x\} = \left(s^2 Y - sx(0) - \dot{x}(0)\right) + Y = L\{-9\cos 2t\} = \tfrac{-9s}{s^2+4}$$
$$\Rightarrow \left(s^2 + 1\right)Y = \tfrac{-9s}{s^2+4} + 2s + 1$$
$$\therefore Y(s) = \tfrac{-9s}{(s^2+1)(s^2+4)} + 2\tfrac{s}{(s^2+1)} + \tfrac{1}{(s^2+1)} = \tfrac{-9s}{(s^2+1)(s^2+4)} + L\{2\cos t + \sin t\}$$

(The last two terms are immediately obtainable, as shown above)

The first term, however, as in the case of Problem1, can be solved either using partial fractions or THM12:

Method1 (Partial Fractions)

$$\tfrac{-9s}{(s^2+1)(s^2+4)} = \tfrac{A_1 s + A_2}{s^2+1} + \tfrac{B_1 s + B_2}{s^2+4} \Rightarrow -9s = (A_1 s + A_2)(s+4) + (B_1 s + B_2)(s+1)$$
$$\Rightarrow -9s = (A_1 + B_1)s^3 + (A_2 + B_2)s^2 + (4A_1 + B_1)s + (4A_2 + B_2)$$
$$s^3: \quad 0 = A_1 + B_1 \Rightarrow A_1 = -B_1$$
$$s^2: \quad 0 = A_2 + B_2 \Rightarrow A_2 = -B_2$$
$$s^1: \quad -9 = 4A_1 + B_1 = 4A_1 - A_1 \Rightarrow A_1 = -3, B_1 = 3$$
$$s^0: \quad 0 = 4A_2 + B_2 = -4B_2 + B_2 = 0 \Rightarrow B_2 = A_2 = 0$$

So: $L^{-1}\left\{\tfrac{-9s}{(s^2+1)(s^2+4)}\right\} = L^{-1}\left\{\tfrac{-3s}{s^2+1} + \tfrac{3s}{s^2+4}\right\} = -3\cos t + 3\cos 2t$

Hence: $x(t) = L^{-1}\{Y(s)\} = (-3\cos t + 3\cos 2t) + (2\cos t + \sin t) = -\cos t + \sin t + 3\cos 2t$

Method2 (THM12)

$$-\tfrac{9s}{(s^2+1)(s^2+4)} = -9\left(\tfrac{s}{s^2+1}\cdot\tfrac{1}{s^2+4}\right) = -9L\{\cos t\}L\{\tfrac{1}{2}\sin 2t\} = L\{-9(\cos t * \tfrac{1}{2}\sin 2t)\} = -\tfrac{9}{2}L\{\cos t * \sin 2t\}$$

where: $\cos t * \sin 2t = \int_0^t \cos(t-u)\sin 2u\,du = \int_0^t \cos u \sin 2(t-u)\,du$

Picking the first version (though both integrals involve roughly the same amount of work):

$$\int_0^t \cos(t-u)\sin 2u\,du = \tfrac{1}{2}\int_0^t [\sin(t+u)-\sin(t-3u)]du = \tfrac{1}{2}\int_0^t \sin(t+u)du - \tfrac{1}{2}\int_0^t \sin(t-3u)du$$

$$= \tfrac{1}{2}\int_{U_1(0)=t}^{U_1(t)=2t} \sin U_1 dU_1 + \tfrac{1}{6}\int_{U_2(0)=t}^{U_2(t)=-2t} \sin U_2 dU_2 = -\tfrac{1}{2}\cos U_1\Big|_t^{2t} - \tfrac{1}{6}\cos U_2\Big|_t^{-2t} = -\tfrac{1}{2}(\cos 2t - \cos t) - \tfrac{1}{6}(\cos 2t - \cos t)$$

$$= \tfrac{1}{2}\cos t - \tfrac{1}{2}\cos 2t - \tfrac{1}{6}\cos 2t + \tfrac{1}{6}\cos t = \tfrac{2}{3}\cos t - \tfrac{2}{3}\cos 2t = \tfrac{2}{3}(\cos t - \cos 2t)$$

Note: In the two integrals, the first and second U-substitutions are: $U_1(u) = (t + u)$, $U_2(u) = (t - 3u)$. Also note how the property cos(-A) = cosA was used in evaluating the second integral at the upper limit $-2t$.

So: $L^{-1}\left\{\frac{-9s}{(s^2+1)(s^2+4)}\right\} = -\frac{9}{2}\cdot\frac{2}{3}(\cos t - \cos 2t) = -3\cos t + 3\cos 2t$

Hence: $x(t) = L^{-1}\{Y(s)\} = (-3\cos t + 3\cos 2t) + (2\cos t + \sin t) = -\cos t + \sin t + 3\cos 2t$