MA360 **Handout 12 (week of April 24)**
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Additional Hints (Vc, Assignment 3)

Note that $s_1(t) = \begin{Bmatrix} 5 & 0 \le t \le 8 \\ 0 & t < 0, 8 < t \end{Bmatrix}$

So:

$$(s_1 * s_2)(t) = \int_{-\infty}^{\infty} s_1(u)s_2(t-u)du = 5\int_0^8 s_2(t-u)du = 20\int_0^8 [U(t-u) - U(t-u-8)]du$$

$$= 20\int_0^8 U(t-u)du - 20\int_0^8 U(t-u-8)du$$

Note: The step functions are labeled by *U* to distinguish them from the dummy variable *u*.

Now finding an expression for the integrals (as functions of *t*) is easy. Consider the following *w*-substitution: $w(u) = t - u$, $dw = -du$ (note that *t* behaves as a constant in the integration)

So (for example) $\int_0^8 U(t-u)du = -\int_{w(0)}^{w(u)} U(w)dw = -\int_t^{t-8} U(w)dw$

(where: $U(w) = 1$, for $w > 0$)

- **Remark: Connection between Fourier Transforms and Fourier Series**

Consider the Fourier Series of any periodic function *f*

$$f(t) = \sum_{n=0}^{\infty} \{a_n \cos(\tfrac{2n\pi}{p}) + b_n \sin(\tfrac{2n\pi}{p})\}$$

In the ‘continuum limit’ this becomes (by way of the Fourier Integral Theorem):

$f(t) \to \int_{-\infty}^{\infty} [a(\omega)\cos \omega t + b(\omega)\sin \omega t]d\omega$ where, instead of periodicity, *f* must obey the *Dirichlet Boundary conditions* which, among other things, entail that *f* vanish at ± ∞ [1].

The formulae for the continuous Fourier coefficients (now defined as functions with respect to the dummy variable of integration ω) are:

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[1] This is alluded to in Sheng, at the beginning of the discussion of Fourier Transforms, p 121, when he points out that function need only be defined in the *finite* interval:
$-\infty < t < \infty$.

$$a(\omega)=\int_{-\infty}^{\infty} f(t)\cos\omega t dt$$

$$b(\omega)=\int_{-\infty}^{\infty} f(t)\sin\omega t dt$$

by inspection, we see immediately that the Fourier transform is nothing more but the fomula for the coefficients of the continuum limit in the *complex* representation! I.e[2].:

$$F[f]=C(\omega)=a(\omega)\pm ib(\omega)=\int_{-\infty}^{\infty} f(t)(\cos\omega t\pm i\sin\omega t)dt=\int_{-\infty}^{\infty} f(t)e^{\pm i\omega t}dt$$

Review Problems (Final Exam)

Ia) Given: $f(t)=\left\{\begin{matrix} 1 & 0<t\le 1 \\ t & 1<t\le 2 \\ 2 & 2<t \end{matrix}\right\}$ Find the LT of f using the definition of the LT

$$\begin{aligned} L\{f(t)\} &= \int_0^{\infty} f(t)e^{-st}dt=\int_0^1 e^{-st}dt+\int_1^2 te^{-st}dt+\int_2^{\infty} 2e^{-st}dt \\ &= -\tfrac{1}{s}e^{-st}\Big|_0^1+\left\{-\tfrac{t}{s}e^{-st}\Big|_1^2-\left(-\tfrac{1}{s}\right)\int_1^2 e^{-st}dt\right\}+2e^{-st}\Big|_2^{\infty} \\ &= -\tfrac{1}{s}\left(e^{-s}-1\right)-\tfrac{2}{s}e^{-2s}+\tfrac{1}{s}e^{-s}-\tfrac{1}{s^2}e^{-st}\Big|_1^2-\tfrac{2}{s}\left\{\lim_{b\to\infty} e^{-sb}-e^{-2s}\right\} \\ &= -\tfrac{1}{s}e^{-s}+\tfrac{1}{s}-\tfrac{2}{s}e^{-2s}+\tfrac{1}{s}e^{-s}-\tfrac{1}{s^2}e^{-2s}+\tfrac{1}{s^2}e^{-s}-\tfrac{2}{s}\left\{0-e^{-2s}\right\} \\ &= -\tfrac{1}{s}e^{-s}+\tfrac{1}{s}-\tfrac{2}{s}e^{-2s}+\tfrac{1}{s}e^{-s}-\tfrac{1}{s^2}e^{-2s}+\tfrac{1}{s^2}e^{-s}+\tfrac{2}{s}e^{-2s} \\ &= \tfrac{1}{s}+\tfrac{1}{s^2}e^{-s}-\tfrac{1}{s^2}e^{-2s} \end{aligned}$$

b.) Find $L\{f(t)\}$ using the step function method

$$\begin{aligned} f(t) &= u(t)-u(t-1)+t(u(t-1)-u(t-2))+2u(t-1) \\ &= u(t)+(t-1)u(t-1)-(t-2)u(t-2) \\ \therefore L\{f(t)\} &= L\{u(t)\}+L\{u(t-1)(t-1)\}-L\{u(t-2)(t-2)\} \\ &= \tfrac{1}{s}+e^{-s}L\{t\}-e^{-2s}L\{t\}=\tfrac{1}{s}+\tfrac{e^{-s}}{s^2}-\tfrac{e^{-2s}}{s^2}=\tfrac{1}{s}+\tfrac{1}{s^2}e^{-s}-\tfrac{1}{s^2}e^{-2s} \end{aligned}$$

c.) Calculate $f'(t)$

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[2] The +/- is a convention

$$f'(t) = \delta(t) + u(t-1) + (t-1)\delta(t-1) - u(t-2) - (t-2)\delta(t-2)$$

d.) Calculate $L\{ f'(t)\}$ from c.)

$$\begin{aligned}
&L\{f'(t)\} = L\{\delta(t)\} + L\{u(t-1)\} + L\{t\delta(t-1)\} - L\{\delta(t-1)\} - L\{u(t-2)\} \\
&- L\{t\delta(t-2)\} + 2L\{\delta(t-2)\} \\
&= 1 + \tfrac{e^{-s}}{s} + (-1)^1 \tfrac{d}{ds} L\{\delta(t-1)\} - e^{-s} - \tfrac{e^{-2s}}{s} - (-1)^1 \tfrac{d}{ds} L\{\delta(t-2)\} + 2e^{-2s} \\
&= 1 + \tfrac{e^{-s}}{s} - \tfrac{d}{ds} e^{-s} - e^{-s} - \tfrac{e^{-2s}}{s} + \tfrac{d}{ds} e^{-2s} + 2e^{-2s} \\
&= 1 + \tfrac{e^{-s}}{s} + e^{-s} - e^{-s} - \tfrac{e^{-2s}}{s} - 2e^{-2s} + 2e^{-2ss} = 1 + \tfrac{e^{-s}}{s} - \tfrac{e^{-2s}}{s}
\end{aligned}$$

e.) Verify your answer in d.) using THM2

- According to THM2: $L\{f'(t)\} = sL\{f(t)\} - f(0)$

In this case, $f(0) = 0$, so : $L\{f'(t)\} = sL\{f(t)\}$

Now: $L\{f'(t)\} = 1 + \frac{e^{-s}}{s} - \frac{e^{-2s}}{s}, L\{f(t)\} = \frac{1}{s} + \frac{e^{-s}}{s^2} - \frac{e^{-2s}}{s^2}$

So clearly: $L\{f'(t)\} = sL\{f(t)\}$

f.) Verify THM10 for $f(t)$

$\lim_{t\to 0^+} f(t) = 1$

$$\begin{aligned}
&\lim_{s\to\infty} sF(s) = \lim_{s\to\infty} sL\{f(t)\} = \lim_{s\to\infty} L\{f'(t)\} = \lim_{s\to\infty} \left\{1 + \tfrac{e^{-s}}{s} - \tfrac{e^{-2s}}{s}\right\} \\
&= 1 + \lim_{s\to\infty} \tfrac{e^{-s}}{s} - \lim_{s\to\infty} \tfrac{e^{-2s}}{s^2} = 1 + \lim_{s\to\infty} \tfrac{1}{se^s} - \lim_{s\to\infty} \tfrac{1}{s^2e^{2s}} = 1 + 0 + 0 = 1 \\
&\therefore \lim_{t\to 0^+} f(t) = \lim_{s\to\infty} sF(s)
\end{aligned}$$

II a.) Find the inverse LT of $\frac{s}{(s^2+9)(s^2+4)}$

$$\tfrac{s}{(s^2+9)(s^2+4)} = \tfrac{s}{(s^2+3^2)} \cdot \tfrac{1}{(s^2+2^2)} = L\{\cos 3t\} \cdot \tfrac{1}{2} L\{\sin 2t\} = \tfrac{1}{2} L\{\cos 3t\} L\{\sin 2t\}$$

$$= \tfrac{1}{2} L\{\cos 3t * \sin 2t\} = \tfrac{1}{2} L\left\{ \int_0^t \cos 3(t-u) \sin 2u du \right\}$$

$$\therefore L^{-1}\left\{\tfrac{s}{(s^2+9)(s^2+4)}\right\} = \tfrac{1}{2} \int_0^t \cos 3(t-u) \sin 2u du = \tfrac{1}{2} \int_0^t \tfrac{1}{2}\{\sin(3t-u) - \sin(3t-5u)\} du$$

$$= \tfrac{1}{4} \int_0^t \sin(3t-u) du - \tfrac{1}{4} \int_0^t \sin(3t-5u) du = \tfrac{1}{4} \cos(3t-u)\Big|_0^t - \tfrac{1}{4} \cdot \tfrac{1}{5} \cos(3t-5u)\Big|_0^t$$

$$= \tfrac{1}{4}\{\cos 2t - \cos 3t\} - \tfrac{1}{20}\{\cos(-2t) - \cos 3t\} = \tfrac{1}{4} \cos 2t - \tfrac{1}{4} \cos 3t - \tfrac{1}{20} \cos 2t + \tfrac{1}{20} \cos 3t$$

$$= \tfrac{1}{5} \cos 2t - \tfrac{1}{5} \cos 3t = \tfrac{1}{5}(\cos 2t - \cos 3t)$$

IIb.) Use IIa) to solve: $\dot{x}(t) + 9\int_0^t x(\omega) d\omega = u(t-2)\cos 2(t-2)$, $x(0) = 0$

$$L\left\{ \dot{x} + 9\int_0^t x(\omega) d\omega \right\} = L\{u(t-2)\cos 2(t-2)\}$$

$$sY(s) - x(0) + \tfrac{9}{s} Y(s) = e^{-2s} \cdot \tfrac{2}{s^2+4}$$

$$(s + \tfrac{9}{s}) Y(s) = e^{-2s} \tfrac{2}{s^2+4}$$

$$(s^2 + 9) Y(s) = e^{-2s} \cdot \tfrac{2s}{(s^2+4)}$$

$$\therefore Y(s) = e^{-2s} \cdot \tfrac{2s}{(s^2+4)(s^2+9)} \Rightarrow x(t) = L\left\{e^{-2s} \cdot 2 \cdot \tfrac{s}{(s^2+4)(s^2+9)}\right\} = 2L\left\{e^{-2s} \cdot \tfrac{s}{(s^2+4)(s^2+9)}\right\}$$

$$\Rightarrow= 2L\{x(t+2)\}, \Rightarrow F(s) = L\{x(t)\} = \tfrac{s}{(s^2+4)(s^2+9)} \Rightarrow x(t) = \tfrac{1}{5}(\cos 2t - \cos 3t)$$

$$\therefore L^{-1} Y(s) = 2x(t-2), t > 2 \Rightarrow x(t) = \tfrac{1}{5}(\cos 2(t-2) - \cos 3(t-2)), t > 2$$

III.) Consider: $\ddot{x}(t) + 5\dot{x}(t) + 6x(t) = \cos 2t, x(0) = 1 = -\dot{x}(0)$

III.a) Solve using the method of Undetermined Coefficients (UC)

Auxiliary Eqn.: $r^2 + 5r + 6 = 0 \Rightarrow (r+3)(r+2) = 0 \Rightarrow x_{tr} = c_1 e^{-3t} + c_2 e^{-2t}$

$$x_{ss}(t) = A\cos 2t + B\sin 2t$$

$$\Rightarrow \ddot{x}_{ss}(t) + 5\dot{x}_{ss}(t) + 6x_{ss}(t) = \left\{ \begin{matrix} (-4A\cos 2t - 4B\sin 2t) + 5(-2A\sin 2t + 2B\cos 2t) \\ + 6(A\cos 2t + B\sin 2t) \end{matrix} \right\} = \cos 2t$$

$$\Rightarrow \cos 2t = (-4A + 10B + 6A)\cos 2t \Rightarrow 1 = 2A + 10B$$

$$\Rightarrow 0 = (-4B - 10A + 6B)\sin 2t \Rightarrow 0 = 2B - 10A \Rightarrow B = 5A \Rightarrow A = \tfrac{1}{52}, B = \tfrac{5}{52}$$

$$
\begin{aligned}
&x(t) = x_{tr}(t) + x_{ss}(t) = c_1 e^{-3t} + c_2 e^{-2t} + \tfrac{1}{52}(\cos 2t + 5\sin 2t) \\
&\dot{x}(t) = -3c_1 e^{-3t} - 2c_2 e^{-2t} + \tfrac{1}{52}(-2\sin 2t + 10\cos 2t) \\
&x(0) = c_1 + c_2 + \tfrac{1}{52} = 1, \dot{x}(0) = -3c_1 - 2c_2 + \tfrac{5}{26} = -1 \\
&\Rightarrow \begin{matrix} 2c_1 + 2c_2 = \tfrac{51}{26} \\ -3c_1 - 2c_2 = -\tfrac{31}{26} \end{matrix} \Rightarrow -c_1 = \tfrac{20}{26} \Rightarrow c_1 = -\tfrac{10}{13}, c_2 = \tfrac{51}{52} + \tfrac{40}{52} = \tfrac{91}{52}
\end{aligned}
$$

$$
\begin{aligned}
&\therefore x(t) = x_{tr}(t) + x_{ss}(t) = -\tfrac{10}{13} e^{-3t} + \tfrac{91}{52} e^{2t} + \tfrac{1}{52}\cos 2t + \tfrac{5}{52}\sin 2t \\
&= \tfrac{1}{52}\left(-40e^{-3t} + 91e^{2t} + \cos 2t + 5\sin 2t\right)
\end{aligned}
$$

III.b) Solve using LTs:

$$
\begin{aligned}
&L\{\ddot{x} + 5\dot{x} + 6x\} = (s^2 Y(s) - sx(0) - \dot{x}(0)) + 5(sY(s) - x(0)) + 6Y(s) = L\{\cos 2t\} \\
&\Rightarrow (s^2 + 5s + 6)Y(s) - s + 1 - 5 = \tfrac{s}{s^2+4} \Rightarrow (s+3)(s+2)Y(s) = \tfrac{s}{s^2+4} + s + 4 \\
&\Rightarrow Y(s) = \tfrac{s}{(s^2+4)(s+3)(s+2)} + \tfrac{s+4}{(s+3)(s+2)}
\end{aligned}
$$

i.)

$$
\begin{aligned}
&\tfrac{s}{(s^2+4)(s+3)(s+2)} = \tfrac{A_1 s + A_2}{s^2+4} + \tfrac{B}{s+3} + \tfrac{C}{s+2} \\
&\Rightarrow s = (A_1 s + A_2)(s+3)(s+2) + B(s^2+4)(s+2) + C(s^2+4)(s+3) \\
&(s = -2) \Rightarrow -2 = 8C \Rightarrow C = -\tfrac{1}{4} \\
&(s = -3) \Rightarrow -3 = -13B \Rightarrow B = \tfrac{3}{13} \\
&s^1 : 1 = 6A_1 + 5A_2 + \tfrac{12}{13} - 1 \Rightarrow 6A_1 + 5A_2 = \tfrac{14}{13} \\
&s^3 : 0 = A_1 + \tfrac{3}{13} - \tfrac{1}{4} \Rightarrow A_1 = \tfrac{1}{52} \Rightarrow A_2 = \tfrac{10}{52}
\end{aligned}
$$

$$
\therefore \tfrac{s}{(s^2+4)(s+3)(s+2)} = \frac{\tfrac{1}{52}(s+40)}{s^2+4} + \frac{\tfrac{3}{13}}{s+3} - \frac{\tfrac{1}{4}}{s+2} = \frac{1}{52}\left\{\tfrac{s+10}{s^2+4} + \tfrac{12}{s+3} - \tfrac{13}{s+2}\right\}
$$

ii.) $\tfrac{s+4}{(s+3)(s+2)} = \tfrac{A}{s+3} + \tfrac{B}{s+2} \Rightarrow A = -1, B = 2 \Rightarrow \tfrac{s+4}{(s+3)(s+2)} = -\tfrac{1}{s+3} + \tfrac{2}{s+2}$

$$
\begin{aligned}
&\therefore Y(s) = \tfrac{1}{52}\left\{\tfrac{s+10}{s^2+4} + \tfrac{12}{s+3} - \tfrac{13}{s+2} - \tfrac{52}{s+3} + \tfrac{104}{s+2}\right\} = \tfrac{1}{52}\left\{\tfrac{s+10}{s^2+4} - \tfrac{40}{s+3} + \tfrac{91}{s+2}\right\} \\
&\Rightarrow x(t) = \tfrac{1}{52}\left\{L^{-1}\left(\tfrac{s+10}{s^2+4}\right) - 40L^{-1}\left(\tfrac{1}{s+3}\right) + 91L^{-1}\left(\tfrac{1}{s+2}\right)\right\} \\
&= \tfrac{1}{52}\left\{L^{-1}\left(\tfrac{s}{s^2+4}\right) + 5L^{-1}\left(\tfrac{2}{s^2+4}\right) - 40L^{-1}\left(\tfrac{1}{s+3}\right) + 91L^{-1}\left(\tfrac{1}{s+2}\right)\right\} \\
&= \tfrac{1}{52}\left\{\cos 2t + 5\sin 2t - 40e^{-3t} + 91e^{-2t}\right\}
\end{aligned}
$$

IV.) Consider: $f(t) = \begin{cases} t+1 & -1 \le t \le 0 \\ 1 & 0 < t \le 1 \end{cases}, p = 2$

a.) Find its Fourier Coefficients (real series representation)

$$a_0 = \tfrac{1}{2}\int_{-1}^{1} f(t)dt = \tfrac{1}{2}\left\{\int_{-1}^{0}(t+1)dt + \int_{0}^{1}dt\right\} = \tfrac{1}{2}\left\{\left(\tfrac{1}{2}t^2 + t\right)\Big|_{-1}^{0} + t\Big|_{0}^{1}\right\}$$
$$= \tfrac{1}{2}\left\{\left(0 - \tfrac{1}{2} + 1\right) + 1\right\} = \tfrac{3}{4}$$

$$a_n = \tfrac{2}{2}\int_{-1}^{1} f(t)\cos\left(\tfrac{2n\pi t}{2}\right)dt = \int_{-1}^{0}(t+1)\cos(n\pi t)dt + \int_{0}^{1}\cos(n\pi t)dt$$
$$= \int_{-1}^{0} t\cos n\pi t dt + \int_{-1}^{0}\cos n\pi t dt + \int_{0}^{1}\cos n\pi t dt = \left\{\tfrac{t}{n\pi}\sin n\pi t\Big|_{-1}^{0} - \tfrac{1}{n\pi}\int_{-1}^{0}\sin n\pi t dt\right\} + \int_{-1}^{1}\cos n\pi t dt$$
$$= \left\{0 + \tfrac{1}{(n\pi)^2}\cos n\pi t\Big|_{-1}^{0}\right\} + 2\int_{0}^{1}\cos n\pi t dt = \tfrac{1}{(n\pi)^2}\left(\cos 0 - \cos(-n\pi)\right) + \tfrac{2}{n\pi}\sin n\pi t\Big|_{0}^{1}$$
$$= \tfrac{1}{n^2\pi^2}\left(1 - \cos n\pi\right) + 0 = \tfrac{2}{(n\pi)^2}, n = 2j + 1$$

(i.e., when n is odd)

$$b_n = \tfrac{2}{2}\int_{-1}^{1} f(t)\sin\left(\tfrac{2n\pi t}{2}\right)dt = \int_{-1}^{0}(t+1)\sin(n\pi t)dt + \int_{0}^{1}\sin(n\pi t)dt$$
$$= \int_{-1}^{0} t\sin n\pi t dt + \int_{-1}^{0}\sin n\pi t dt + \int_{0}^{1}\sin n\pi t dt = \left\{-\tfrac{t}{n\pi}\cos n\pi t\Big|_{-1}^{0} + \tfrac{1}{n\pi}\int_{-1}^{0}\cos n\pi t dt\right\} + \int_{-1}^{1}\sin n\pi t dt$$
$$= \left\{(0 - \tfrac{1}{n\pi}\cos n\pi) + \tfrac{1}{(n\pi)^2}\sin n\pi t\Big|_{-1}^{0}\right\} + 0 = -\tfrac{1}{n\pi}\cos n\pi = \tfrac{(-1)^{n+1}}{n\pi}$$

b.) Find its complex series coefficients:

$$C_k = \tfrac{1}{2}\int_{-1}^{1} f(t)e^{-ik\left(\frac{2\pi}{2}\right)t}dt = \tfrac{1}{2}\left\{\int_{-1}^{0}(t+1)e^{-ik\pi t}dt + \int_{0}^{1}e^{-ik\pi t}dt\right\}$$
$$= \tfrac{1}{2}\int_{-1}^{0} te^{-ik\pi t}dt + \tfrac{1}{2}\int_{-1}^{1} e^{-ik\pi t}dt = \tfrac{1}{2}\left\{\tfrac{it}{k\pi}e^{-ik\pi t}\Big|_{-1}^{0} - \tfrac{i}{k\pi}\int_{-1}^{0}e^{-ik\pi t}dt\right\} + \tfrac{i}{2k\pi}e^{-ik\pi t}\Big|_{-1}^{1}$$
$$= \tfrac{1}{2}\left\{0 + \tfrac{i}{k\pi}e^{ik\pi}\right\} + \tfrac{1}{2(k\pi)^2}e^{-ik\pi t}\Big|_{-1}^{0} + \tfrac{i}{2k\pi}\left(e^{-ik\pi} - e^{ik\pi}\right)$$
$$= \tfrac{i}{2k\pi}e^{ik\pi} + \tfrac{1}{2(k\pi)^2}\left(e^0 - e^{ik\pi}\right) + 0 = \tfrac{1}{2k\pi}\left[ie^{ik\pi} + \tfrac{1}{k\pi}\left(1 - e^{ik\pi}\right)\right]$$
$$= \tfrac{1}{2k\pi}\left[\tfrac{1}{k\pi} + e^{ik\pi}\left(i - \tfrac{1}{k\pi}\right)\right]$$

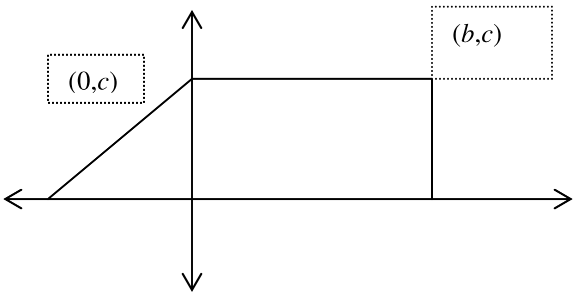
c.) Verify: $a_k = C_k + C_k^*$

$$a_k = C_k + C_k^* = \tfrac{1}{2k\pi}\left[\tfrac{1}{k\pi} + e^{ik\pi}\left(i - \tfrac{1}{k\pi}\right)\right] + \tfrac{1}{2k\pi}\left[\tfrac{1}{k\pi} + e^{-ik\pi}\left(-i - \tfrac{1}{k\pi}\right)\right]$$
$$= \tfrac{1}{(k\pi)^2} + \tfrac{i}{2k\pi}\left(e^{ik\pi} - e^{-ik\pi}\right) - \tfrac{1}{2(k\pi)^2}\left(e^{ik\pi} + e^{-ik\pi}\right)$$
$$= \tfrac{1}{(k\pi)^2} + \tfrac{i}{2k\pi}(0) - \tfrac{2}{2(k\pi)^2}(\cos k\pi) = \tfrac{1}{(k\pi)^2} - \tfrac{\cos k\pi}{(k\pi)^2} = \tfrac{2}{(k\pi)^2}, k = 2j+1$$

(i.e., when k is odd)

from part a.): $a_n = \frac{2}{(n\pi)^2}, n(odd)$, so we have agreement, since, as demonstrated here:
$a_k = C_k + C_k^* = \frac{2}{(k\pi)^2}, k(odd)$
(label of integer dummy index is clearly arbitrary)

V.) Given $f(t) = \begin{Bmatrix} c\left(\frac{t}{a}+1\right) & -a \le t < 0 \\ c & 0 \le t \le b \end{Bmatrix}$



a.) Find its FT using the definition of FT:

$$F[f(t)] = \int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt = c\int_{-a}^{0}\left(\tfrac{t}{a}+1\right)e^{-i\omega t}dt + c\int_{0}^{b} e^{-i\omega t}dt$$
$$= \tfrac{c}{a}\int_{-a}^{0} te^{-i\omega t}dt + c\int_{-a}^{b} e^{-i\omega t}dt = \tfrac{c}{a}\left\{\tfrac{it}{\omega}e^{-i\omega t}\Big|_{-a}^{0} - \tfrac{i}{\omega}\int_{-a}^{0} e^{-i\omega t}dt\right\} + \tfrac{ic}{\omega}e^{-i\omega t}\Big|_{-a}^{b}$$
$$= \tfrac{c}{a}\left\{\left(0 + \tfrac{ia}{\omega}e^{i\omega a}\right) + \tfrac{1}{\omega^2}e^{-i\omega t}\Big|_{-a}^{0}\right\} + \tfrac{ic}{\omega}\left(e^{-i\omega b} - e^{i\omega a}\right)$$
$$= \tfrac{ic}{\omega}e^{i\omega a} + \tfrac{c}{a\omega^2}\left(1 - e^{i\omega a}\right) + \tfrac{ic}{\omega}e^{-i\omega b} - \tfrac{ic}{\omega}e^{i\omega a}$$
$$= \tfrac{c}{a\omega^2}\left[1 - e^{i\omega a} + ia\omega e^{-i\omega b}\right]$$

b.) Find the FT using the (unit) step function method

$$f(t) = c\left(\tfrac{t}{a}+1\right)\left(u(t+a)-u(t)\right)+c\left(u(t)-u(t-b)\right)$$

$$= \tfrac{c}{a}tu(t+a)-\tfrac{c}{a}tu(t)+cu(t+a)-cu(t)+cu(t)-cu(t-b)$$

$$= \tfrac{c}{a}tu(t+a)-\tfrac{c}{a}tu(t)+cu(t+a)-cu(t-b)$$

$$f'(t) = \tfrac{c}{a}u(t+a)+\tfrac{c}{a}t\delta(t+a)-\tfrac{c}{a}u(t)-\tfrac{c}{a}t\delta(t)+c\delta(t+a)-c\delta(t-b)$$

$$F\left[f'(t)\right] = (i\omega)^1 F\left[f(t)\right] = \tfrac{c}{a}F\left[u(t+a)\right]+\tfrac{c}{a}F\left[t\delta(t+a)\right]-\tfrac{c}{a}F\left[u(t)\right]-\tfrac{c}{a}F\left[t\delta(t)\right]$$

$$+cF\left[\delta(t+a)\right]-cF\left[\delta(t-b)\right]$$

$$= \tfrac{c}{a}\cdot\tfrac{1}{(i\omega)^1}F\left[\delta(t+a)\right]+\tfrac{c}{a}i\tfrac{d}{d\omega}F\left[\delta(t+a)\right]-\tfrac{c}{a}\left[\tfrac{1}{i\omega}+\pi\delta(\omega)\right]-\tfrac{c}{a}\cdot i\tfrac{d}{d\omega}F\left[\delta(t)\right]$$

$$+ce^{i\omega a}-ce^{-i\omega b}$$

$$= -\tfrac{ic}{a\omega}e^{i\omega a}+\tfrac{ic}{a}\cdot\tfrac{d}{d\omega}e^{i\omega a}+\tfrac{ic}{a\omega}-\tfrac{c\pi}{a}\delta(\omega)-i\tfrac{c}{a}\tfrac{d}{d\omega}[1]+c\left(e^{i\omega a}-e^{-i\omega b}\right)$$

$$= -\tfrac{ic}{a\omega}e^{i\omega a}-ce^{i\omega a}+\tfrac{ic}{a\omega}-\tfrac{c\pi}{a}\delta(\omega)+ce^{i\omega a}-ce^{-i\omega b}$$

$$= -\tfrac{ic}{a\omega}e^{i\omega a}+\tfrac{ic}{a\omega}-\tfrac{c\pi}{a}\delta(\omega)-ce^{-i\omega b}$$

$$\Rightarrow F\left[f(t)\right] = \tfrac{1}{i\omega}\left[-\tfrac{ic}{a\omega}e^{i\omega a}+\tfrac{ic}{a\omega}-ce^{-i\omega b}\right]+\kappa\delta(\omega)$$

$$= \tfrac{c}{a\omega^2}\left[1-e^{i\omega a}+ia\omega e^{-i\omega b}\right]+\kappa\delta(\omega)$$

Note: the extra $\delta(\omega)$ term arises due to the $F[u(t)]$ ‘pathology’ (Example 7-3-9, Sheng).