

The Fourier Transform and its Inverse

$$F[g(t)] = G(\omega) = \int_{-\infty}^{\infty} g(t)e^{-i\omega t} dt$$

$$F^{-1}[G(\omega)] = g(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} G(\omega)e^{i\omega t} d\omega$$

Selected Proofs of the Properties:

1. Proof of Property 2 (Scale Change)

$$F[g(kt)] = \int_{-\infty}^{\infty} g(kt)e^{-i\omega t} dt, \Rightarrow u = kt, du = kdt \Rightarrow dt = \frac{1}{k} du, t = \frac{1}{k} u$$

$$\Rightarrow F[g(kt)] = \int_{u(-\infty)}^{u(\infty)} g(u)e^{-i(\frac{\omega}{k})u} (\frac{du}{k}) = \frac{1}{k} \int_{u(-\infty)}^{u(\infty)} g(u)e^{-i(\frac{\omega}{k})u} du$$

- Case 1: $k > 0 \Rightarrow u(\pm\infty) = k(\pm\infty) = \pm\infty \therefore F[g(kt)] = \frac{1}{k} \int_{-\infty}^{\infty} g(u)e^{-i(\frac{\omega}{k})u} du$

- Case 2: $k < 0 \Rightarrow u(\pm\infty) = k(\pm\infty) = \mp\infty$

$$\therefore F[g(kt)] = \frac{1}{k} \int_{\infty}^{-\infty} g(u)e^{-i(\frac{\omega}{k})u} du = -\frac{1}{k} \int_{-\infty}^{\infty} g(u)e^{-i(\frac{\omega}{k})u} du$$

Combining both cases¹:

$$\therefore F[g(kt)] = \begin{cases} \frac{1}{k} & k > 0 \\ -\frac{1}{k} & k < 0 \end{cases} \int_{-\infty}^{\infty} g(u)e^{-i(\frac{\omega}{k})u} du = \frac{1}{|k|} \int_{-\infty}^{\infty} g(u)e^{-i(\frac{\omega}{k})u} du = \frac{1}{|k|} G(\frac{\omega}{k})$$

2. Proof of Property 3 (Time Delay)

$$F[g(t-a)] = \int_{-\infty}^{\infty} g(t-a)e^{-i\omega t} dt \Rightarrow u = t-a \Rightarrow du = dt, t = u+a$$

$$F[g(t-a)] = \int_{u(-\infty)}^{u(\infty)} g(u)e^{-i\omega(u+a)} dt = e^{-i\omega a} \int_{-\infty}^{\infty} g(u)e^{-i\omega u} du = e^{-i\omega a} G(\omega)$$

3. Proof of Property 4 (Modulation)

¹ By definition of absolute value: $|x| = \{x, \text{ if } x > 0 \text{ or } -x \text{ if } x < 0\}$

$$F[e^{i\alpha} g(t)] = \int_{-\infty}^{\infty} e^{i\alpha} g(t) e^{-i\alpha t} dt = \int_{-\infty}^{\infty} g(t) e^{-i(\omega-\alpha)t} dt = G(\omega - \alpha)$$

4. Proof of Property 5 (Time Differentiation)

By Induction:

Base Case ($k = 1$)

$$F\left[\frac{d}{dt} g(t)\right] = \int_{-\infty}^{\infty} \frac{d}{dt} (g(t)) e^{-i\alpha t} dt \Rightarrow U = g(t), dV = \frac{d}{dt} e^{-i\alpha t} dt = -i\omega e^{-i\alpha t}$$

$$\Rightarrow F\left[\frac{d}{dt} g(t)\right] = \lim_{b \rightarrow \infty} g(t) e^{-i\alpha t} \Big|_{-b}^b - (-i\omega) \int_{-\infty}^{\infty} g(t) e^{-i\alpha t} dt$$

$$= i\omega \int_{-\infty}^{\infty} g(t) e^{-i\alpha t} dt = (i\omega)^1 G(\omega)$$

(Note: Assume the function g and its derivatives vanish at infinity, hence:

$$0 \leq \operatorname{Re} g(t) e^{-i\alpha t} \leq |g(t)|$$

so by the Sandwich Thm.: $\lim_{b \rightarrow \infty} \operatorname{Re}(g(t) e^{-i\alpha t}) = 0$, since: $\lim_{t \rightarrow \infty} |g(t)| = 0$)

Induction Step:

Assume $F\left[\frac{d^k}{dt^k} g(t)\right] = (i\omega)^k G(\omega)$, so prove: $F\left[\frac{d^{k+1}}{dt^{k+1}} g(t)\right] = (i\omega)^{k+1} G(\omega)$

Proof: Let $h(t) = \frac{d^k}{dt^k} g(t)$, $\Rightarrow \frac{d^{k+1}}{dt^{k+1}} g(t) = \frac{d}{dt} h(t)$

$$F\left[\frac{d}{dt} h(t)\right] = \int_{-\infty}^{\infty} \frac{d}{dt} (h(t)) e^{-i\alpha t} dt \Rightarrow U = h(t), dV = \frac{d}{dt} e^{-i\alpha t} dt = -i\omega e^{-i\alpha t}$$

$$\Rightarrow F\left[\frac{d}{dt} h(t)\right] = \lim_{b \rightarrow \infty} h(t) e^{-i\alpha t} \Big|_{-b}^b - (-i\omega) \int_{-\infty}^{\infty} h(t) e^{-i\alpha t} dt$$

$$= i\omega \int_{-\infty}^{\infty} h(t) e^{-i\alpha t} dt = (i\omega)^1 F[h(t)] = (i\omega) F\left[\frac{d^k}{dt^k} g(t)\right] = (i\omega) [(i\omega)^k G(\omega)]$$

$$= (i\omega)^{k+1} G(\omega)$$

5. Proof of Property 6 (Frequency Differentiation)

This can be proven by Mathematical Induction, just as in 4. above. For a virtually identical proof, see page 7, *Handout 1b*.

6. Proof of Property 7 (Integration)

Using Property 5: ($n = 1$): $F\left[\frac{d}{dt} g(t)\right] = (i\omega)G(\omega) = (i\omega)F[g(t)]$

Re-name:

$$\frac{d}{dt} g(t) = f(t) \Rightarrow g(t) = \int_0^t f(u) du \Rightarrow F[f(t)] = (i\omega)F\left[\int_0^t f(u) du\right] \Rightarrow F\left[\int_0^t f(u) du\right] = \frac{1}{i\omega} F[f(t)]$$

7. Proof of Property 8 (Symmetry)

$$F[g(t)] = G(\omega) \Rightarrow F^{-1}[G(\omega)] = g(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} G(\omega) e^{i\omega t} d\omega$$

$$\text{Moreover: } F[G(t)] = \int_{-\infty}^{\infty} G(t) e^{-i\omega t} dt = 2\pi \cdot \frac{1}{2\pi} \int_{-\infty}^{\infty} G(t) e^{i\omega(-t)} dt$$

Since t is any dummy variable :

$$\therefore F[G(t)] = 2\pi \cdot \frac{1}{2\pi} \int_{-\infty}^{\infty} G(t) e^{i\omega(-t)} dt = 2\pi \cdot \frac{1}{2\pi} \int_{-\infty}^{\infty} G(t) e^{i\omega(-\omega)} dt = 2\pi g(-\omega)$$

Proving Property9 (Reversal) is obviously quite trivial (the sign of the argument simply switches). As for Property 10, see page 4, *Handout 3b*, as the procedure is virtually identical.

Lemma1: $F[1] = 2\pi\delta(\omega)$

Proof: $F[\delta(t)] = 1 \xrightarrow{\text{Pr}\#8} F[1] = 2\pi\delta(-\omega)$

However, since:

$$\begin{aligned} F^{-1}[1] = \delta(t) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} 1 \cdot e^{i\omega t} d\omega = \frac{1}{2\pi} \lim_{b \rightarrow \infty} \left\{ \int_{-b}^b (\cos \omega t + i \sin \omega t) d\omega \right\} \\ &= \frac{1}{2\pi} \lim_{b \rightarrow \infty} \left\{ 2 \int_0^b \cos \omega t d\omega + 0 \right\} = \frac{1}{\pi} \int_0^{\infty} \cos \omega t d\omega \end{aligned}$$

(cosine is even, sine is odd, hence the integral expressions on the second line)

$$\text{So: } \delta(-t) = \frac{1}{\pi} \int_0^{\infty} \cos \omega(-t) d\omega = \frac{1}{\pi} \int_0^{\infty} \cos \omega t d\omega = \delta(t) \Rightarrow F[1] = 2\pi\delta(-\omega) = 2\pi\delta(\omega)$$

By Lemma1 above and Property4, it can be immediately determined that:

$$F[e^{i\alpha}] = F[e^{i\alpha} \cdot 1] = G(\omega - \alpha) = 2\pi\delta(\omega - \alpha)$$

Lemma2: If $g(t)$ is even, so is $G(\omega)$ (I.e. $g(-t) = g(t) \Rightarrow G(-\omega) = G(\omega)$). Moreover,

$$G(\omega) = \int_0^{\infty} g(t) \cos \omega t dt, g(t) = F^{-1}[G(\omega)] = \frac{1}{\pi} \int_0^{\infty} G(\omega) \cos \omega t d\omega$$

Proof:

$$\begin{aligned} G(\omega) &= \int_{-\infty}^{\infty} g(t) e^{-i\omega t} dt = \lim_{b \rightarrow \infty} \int_{-b}^b g(t) (\cos \omega t + i \sin \omega t) dt = \\ &= \lim_{b \rightarrow \infty} \int_{-b}^b g(t) \cos \omega t dt + i \lim_{b \rightarrow \infty} \int_{-b}^b g(t) \sin \omega t dt \end{aligned}$$

Now, since cosine is an even function, and the product of two even functions is even, then:

$$\int_{-b}^b g(t) \cos \omega t dt = 2 \int_0^b g(t) \cos \omega t dt$$

Now, since sine is an odd function, and the product of an even function with an odd function is odd, then:

$$\int_{-b}^b g(t) \sin \omega t dt = 0$$

$$\text{So: } G(\omega) = \lim_{b \rightarrow \infty} 2 \int_0^b g(t) \cos \omega t dt + 0 = 2 \int_0^{\infty} g(t) \cos \omega t dt$$

$$\text{Moreover: } G(-\omega) = 2 \int_0^{\infty} g(t) \cos(-\omega t) dt = 2 \int_0^{\infty} g(t) \cos \omega t dt = G(\omega),$$

In addition, by the same reasoning, (since G was shown to be even):

$$\begin{aligned} g(t) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} G(\omega) e^{i\omega t} d\omega = \lim_{b \rightarrow \infty} \frac{1}{2\pi} \int_{-b}^b G(\omega) (\cos \omega t + i \sin \omega t) d\omega \\ &= \lim_{b \rightarrow \infty} \frac{1}{2\pi} \int_{-b}^b G(\omega) \cos \omega t d\omega + i \lim_{b \rightarrow \infty} \frac{1}{2\pi} \int_{-b}^b G(\omega) \sin \omega t d\omega \\ &= \lim_{b \rightarrow \infty} \frac{1}{\pi} \int_0^b G(\omega) \cos \omega t d\omega + 0 = \frac{1}{\pi} \int_0^{\infty} G(\omega) \cos \omega t d\omega \end{aligned}$$

Note: It can be similarly shown that if $g(-t) = -g(t) \Rightarrow G(-\omega) = -G(\omega)$, and that:

$$G(\omega) = i \int_0^{\infty} g(t) \sin \omega t dt = e^{i\frac{\pi}{2}} \int_0^{\infty} g(t) \sin \omega t dt$$

$$g(t) = F^{-1}[G(\omega)] = \frac{i}{\pi} \int_0^{\infty} G(\omega) \sin \omega t d\omega = \frac{e^{i\frac{\pi}{2}}}{\pi} \int_0^{\infty} G(\omega) \sin \omega t d\omega$$

- Example (Problem 4(e), Exercise 12, Sheng, p 133)

Given: $F[g(t)] = G(\omega) = \frac{5+i\omega}{12-\omega^2+i7\omega}$, find: $F[g(4-t)]$

$$F[g(4-t)] = F[g((-1)(t-4))] \xrightarrow{\text{Pr#2}} \frac{1}{|-1|} G^*(-\omega) = G^*(-\omega)$$

$$\text{where: } G^*(\omega) = F[g(t-4)] \xrightarrow{\text{Pr#3}} e^{-i4\omega} G(\omega)$$

$$\text{Hence: } F[g(4-t)] = G^*(-\omega) = e^{i4\omega} G(\omega) = e^{i4\omega} \frac{5-i\omega}{12-\omega^2-i7\omega}$$

- Example (Problem 4(f), Exercise 12, Sheng, p 133)

Given: $F[g(t)] = G(\omega) = \frac{5+i\omega}{12-\omega^2+i7\omega}$, find: $F[g(t)\cos t]$

$$\begin{aligned} F[g(t)\cos t] &= F[g(t)\frac{1}{2}(e^{it} + e^{-it})] = \frac{1}{2}F[e^{it}g(t)] + \frac{1}{2}F[e^{-it}g(t)] \\ &= \frac{1}{2}(G(\omega-1)) + \frac{1}{2}G(\omega+1) = \frac{1}{2}\left\{\frac{5+i(\omega-1)}{12-(\omega-1)^2+i7(\omega-1)} + \frac{5+i(\omega+1)}{12-(\omega+1)^2+i7(\omega+1)}\right\} \\ &= \frac{5+i(\omega-1)}{2(11+2\omega-\omega^2)+i14(\omega-1)} + \frac{5+i(\omega+1)}{2(11-2\omega-\omega^2)+i14(\omega+1)} \end{aligned}$$

- Example (Problem 5(b), Exercise 12, Sheng p 133)

Method 1 (first principles)

$$g(t) = \begin{cases} A & 0 \leq t \leq \frac{a}{2} \\ -A & -\frac{a}{2} \leq t < 0 \end{cases}$$

$$\begin{aligned} \therefore F[g(t)] &= \int_{-\infty}^{\infty} g(t)e^{-i\omega t} dt = A \int_0^{\frac{a}{2}} e^{-i\omega t} dt - A \int_{-\frac{a}{2}}^0 e^{-i\omega t} dt = \frac{iA}{\omega} e^{-i\omega t} \Big|_0^{\frac{a}{2}} - \frac{iA}{\omega} e^{-i\omega t} \Big|_{-\frac{a}{2}}^0 \\ &= \frac{iA}{\omega} \left\{ e^{-i\omega \frac{a}{2}} - 1 - e^{-i\omega \alpha} + e^{-i\omega \frac{a}{2}} \right\} = \frac{A}{-i\omega} \left\{ 1 + 2e^{-i\omega \frac{a}{2}} - e^{-i\omega \alpha} \right\} = \frac{A}{i\omega} \left(1 - 2e^{-i\omega \frac{a}{2}} + e^{-i\omega \alpha} \right) \\ &= \frac{A}{i\omega} \left(1 - e^{-i\omega \frac{a}{2}} \right)^2 \end{aligned}$$

Method 2 (unit step function method)

$$g(t) = A(u(t) - u(t - \frac{a}{2})) - A(u(t - \frac{a}{2}) - u(t - a)) \Rightarrow g'(t) = A(\delta(t) - \delta(t - \frac{a}{2})) - A(\delta(t - \frac{a}{2}) - \delta(t - a))$$

$$\Rightarrow F[g'(t)] = (i\omega)^1 F[g(t)] = A\{F[\delta(t)] - 2F[\delta(t - \frac{a}{2})] + F[\delta(t - a)]\}$$

$$\Rightarrow F[g(t)] = \frac{A}{i\omega}\{F[\delta(t)] - 2F[\delta(t - \frac{a}{2})] + F[\delta(t - a)]\}$$

(Recall²: (Example 7-3-1, 7-3-3 Sheng): $F[\delta(t - \alpha)] = \int_{-\infty}^{\infty} \delta(t - \alpha)e^{-i\omega t} dt = e^{-i\omega\alpha}$)

$$\text{Hence: } F[g(t)] = \frac{A}{i\omega}\left[1 - 2e^{-i\omega\frac{a}{2}} + e^{-i\omega a}\right] = \frac{A}{i\omega}\left(1 - e^{-i\omega\frac{a}{2}}\right)^2$$

- Example (problems 7,8 Exercise 12)

$$F[g(t)\sin at] = F\left[g(t)\frac{1}{2i}(e^{iat} - e^{-iat})\right] = \frac{1}{2i}F[e^{iat}g(t)] - \frac{1}{2i}F[e^{-iat}g(t)]$$

$$\xrightarrow{\text{Pr\#4}} \frac{1}{2i}G(\omega - a) - \frac{1}{2i}G(\omega + a) = \frac{G(\omega - a) - G(\omega + a)}{2i}$$

$$\text{Hence: } F[e^{-bt}\sin atu(t)] = \frac{1}{2i}(G(\omega - a) + G(\omega + a))$$

$$\text{Where: } G(\omega) = F[e^{-bt}u(t)] = \int_{-\infty}^{\infty} e^{-bt}u(t)e^{-i\omega t} dt = \int_0^{\infty} e^{-(b+i\omega)t} dt = -\frac{1}{(b+i\omega)}e^{-(b+i\omega)t}\Big|_0^{\infty} = \frac{1}{b+i\omega}$$

$$\text{So: } F[e^{-bt}\sin atu(t)] = \frac{1}{2i}(G(\omega - a) + G(\omega + a)) = \frac{1}{2i}\left\{\frac{1}{b+i(\omega - a)} - \frac{1}{b+i(\omega + a)}\right\}$$

$$= \frac{1}{2i}\left\{\frac{(b+i(\omega + a)) - (b+i(\omega - a))}{b^2 + 2i\omega b - (\omega^2 - a^2)}\right\} = \frac{1}{2i} \cdot \frac{2ia}{a^2 + (b^2 + 2i\omega b - \omega^2)} = \frac{a}{a^2 + (b+i\omega)^2}$$

HINTS (Assignment III)

1a) From Handout 9b:

$$f(t) = t^2 = \frac{4}{3}\pi^2 + \sum_{n=1}^{\infty} \left[\frac{4}{n^2} \cos nt - \frac{4\pi}{n} \sin nt \right] = 4 \left\{ \frac{1}{3}\pi^2 + \sum_{n=1}^{\infty} \frac{1}{n} \left(\frac{1}{n} \cos nt - \pi \sin nt \right) \right\}$$

for $0 < t \leq 2\pi$. Evaluate $f(2\pi)$ and simplify

II) Do *not* square the Fourier Series of f and then try to integrate! (Much more tedious)

$$\text{Instead, note that if: } f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left\{ a_n \cos\left(\frac{2n\pi}{p}\right) + b_n \sin\left(\frac{2n\pi}{p}\right) \right\}$$

$$\text{Then: } \int_{-p/2}^{p/2} (f(t))^2 dt = \frac{a_0^2}{2} \int_{-p/2}^{p/2} dt + \sum_{n=1}^{\infty} a_n \int_{-p/2}^{p/2} f(t) \cos\left(\frac{2n\pi t}{p}\right) dt + \sum_{n=1}^{\infty} b_n \int_{-p/2}^{p/2} f(t) \sin\left(\frac{2n\pi t}{p}\right) dt$$

Note how the expressions for a_0 , b_n , a_n just ‘jump right out’.

² This is the ‘sifting property’ of $\delta(t - a)$ first introduced in page 48, Sheng.

IV) a) Use Lemma 2 in this Handout. Let $G(\alpha) = \int_0^{\infty} f(t) \cos \alpha t dt = 2(1 - \alpha), 0 < \alpha < 1$

Then according to Lemma 2: $f(t) = \frac{1}{\pi} \int_0^{\infty} G(\alpha) \cos \alpha t d\alpha = \frac{2}{\pi} \int_0^1 (1 - \alpha) \cos \alpha t d\alpha$

b) Take the limit $\alpha \rightarrow 0$ from your answer in a)

V) Consider the right hand expression: $\frac{1}{y^2 + b^2}$. It's an even function. Take the Fourier Transform of both sides of the integral equation. On the right hand side:

$$F\left[\frac{1}{y^2 + b^2}\right] = \int_{-\infty}^{\infty} \frac{e^{-i\alpha u}}{u^2 + b^2} du = 2 \int_0^{\infty} \frac{\cos \alpha u du}{u^2 + b^2} = \frac{2}{b} \int_0^{\infty} \frac{\cos b\omega v dv}{v^2 + 1}, (v = \frac{1}{b} u)$$

You may use the result (without proving) that: $\int_0^{\infty} \frac{\cos b\omega v dv}{v^2 + 1} = \frac{\pi}{2} e^{-b\omega}$

Now observe that the FT of the LHS is expressible as the FT of a convolution. Hence according to Property 10:

$F[LHS] = F[x(u)]F\left[\frac{1}{y^2 + a^2}\right]$. Now observe the (obvious) connection between $F\left[\frac{1}{y^2 + a^2}\right]$ and $F\left[\frac{1}{y^2 + b^2}\right]$ as depicted above. You can now isolate $F[x(u)]$. Take the inverse FT of that to isolate x . You should end up with the expression:

$$x(u) = \frac{1}{2\pi} \frac{a}{b} \int_{-\infty}^{\infty} e^{i\alpha u} e^{-(b-a)\omega} d\omega, \text{ which is straightforward to evaluate}$$