MATH 360 **Name: KEY**
EXAM II
March 29, 2006

Directions: For maximum credit, please show all work in most reasonable detail. If you run out of room, you may pick up extra paper (supplied by instructor/proctor) and attach to this exam. No books or notes. Formula sheet provided. <u>NO CALCULATORS</u> Please choose FOUR from the following FIVE (worth 25 pts each.) If you do more, I will grade the best <u>four</u>. Bonus problem included. Good luck!

I.) Solve: $\ddot{x} - 4x = -3e^t, x(0) = 1, \dot{x}(0) = 5$

(a) (12) By the UC (Undetermined Coefficients) Method

$$r^2 - 4 = 0 \Rightarrow (r-2)(r+2) = 0 \Rightarrow r_{1,2} = \pm 2 \Rightarrow x_{tr}(t) = c_1 e^{2t} + c_2 e^{-2t}$$
$$x_{ss}(t) = Ae^t \Rightarrow \ddot{x}_{ss}(t) - 4x_{ss}(t) = -3Ae^t = -3e^t \Rightarrow A = 1$$
$$\therefore x(t) = x_{tr}(t) + x_{ss}(t) = c_1 e^{2t} + c_2 e^{-2t} + e^t, \dot{x}(t) = 2(c_1 e^{2t} - c_2 e^{2t}) + e^t$$
$$x(0) = 1 = c_1 + c_2 + 1 \Rightarrow c_1 + c_2 = 0$$
$$\dot{x}(0) = 5 = 2(c_1 - c_2) + 1 \Rightarrow c_1 - c_2 = 2 \Rightarrow c_1 = 1, c_2 = -1$$
$$\therefore x(t) = e^{2t} - e^{-2t} + e^t$$

b) (13) By the LT method (Note: For algebraic ease, combine all terms on right hand side into one fraction before isolating *Y*(*s*))

$$L\{\ddot{x} - 4x\} = s^2 Y - s - 5 - 4Y = L\{-3e^t\} = \tfrac{-3}{s-1} \Rightarrow (s^2 - 4)Y = \tfrac{-3}{s-1} + s + 5$$
$$\Rightarrow (s-2)(s+2)Y = \tfrac{s^2+4s-8}{s-1} \Rightarrow Y(s) = \tfrac{s(s+4)-8}{(s-1)(s+2)(s-2)} = \tfrac{A}{(s-1)} + \tfrac{B}{(s+2)} + \tfrac{C}{(s-2)}$$
$$\Rightarrow A = \tfrac{5-8}{3\cdot(-1)} = 1, B = \tfrac{-2\cdot 2-8}{(-3)\cdot(-4)} = -1, C = \tfrac{2\cdot 6-8}{1\cdot 4} = 1$$
$$\therefore Y(s) = \tfrac{1}{s-1} - \tfrac{1}{s+2} + \tfrac{1}{s-2} \Rightarrow x(t) = L^{-1}\left\{\tfrac{1}{s-1} - \tfrac{1}{s+2} + \tfrac{1}{s-2}\right\} = e^t - e^{-2t} + e^{2t}$$

II. a) (10) Use THM12 to find the inverse LT of: $\frac{s^2}{(s^2+1)(s^2+4)}$

$$\frac{s^2}{(s^2+1)(s^2+4)} = \frac{s}{s^2+1^2}\cdot\frac{s}{s^2+2^2} = L\{\cos t\}L\{\cos 2t\} = L\{\cos t * \cos 2t\}$$

where:

$$\cos t * \cos 2t = \int_0^t \cos(t-u)\cos 2u\,du = \tfrac{1}{2}\int_0^t [\cos(t-3u)+\cos(t+u)]du$$
$$= -\tfrac{1}{6}\sin(t-3u)\Big|_0^t + \tfrac{1}{2}\sin(t+u)\Big|_0^t = -\tfrac{1}{6}[-\sin 2t - \sin t] + \tfrac{1}{2}[\sin 2t - \sin t]$$
$$= \tfrac{2}{3}\sin 2t - \tfrac{1}{3}\sin t = \tfrac{1}{3}(2\sin 2t - \sin t)$$

b.) (15) Use your result obtained in a) and THM8 to solve:

$$\dot{x}(t) + \int_0^t x(\omega) = u(t+1)\cos(2(t+1)),\, x(0) = 0$$

$$L\left\{\dot{x} + \int_0^t x(\omega)d\omega\right\} = L\{u(t+1)\cos(2(t+1))\} \Rightarrow sY + \tfrac{1}{s}Y = e^s L\{\cos 2t\} = e^s\cdot\tfrac{s}{s^2+4}$$
$$\Rightarrow (s^2+1)Y = e^s\cdot\tfrac{s^2}{s^2+4} \Rightarrow Y(s) = e^s\cdot\tfrac{s^2}{(s^2+1)(s^2+4)} = e^s L\{\tfrac{1}{3}(2\sin 2t - \sin t)\}$$
$$= L\{\tfrac{1}{3}u(t+1)[2\sin 2(t+1) - \sin(t+1)]\} \Rightarrow x(t) = \tfrac{1}{3}[2\sin 2(t+1) - \sin(t+1)],\, t > -1$$

III) Consider: $\dot{x} - 2x + 2\int_0^t x(\omega)d\omega = \delta(t-1)$, $x(0) = 0$

a.) (5) Find the LT of this differential equation, and isolate *Y*(s) in such a manner that you have completed the square in the denominator term

$$L\{\dot{x} - 2x + 2\int_0^t x(\omega)d\omega\} = L\{\delta(t-1)\} \Rightarrow sY - 2Y + \tfrac{2}{s}Y = e^{-s}$$
$$\Rightarrow (s^2 - 2s + 2)Y = se^{-s} \Rightarrow Y(s) = \tfrac{se^{-s}}{(s^2-2s+2)} = \tfrac{se^{-s}}{s^2-2s+1+1}$$
$$\Rightarrow Y(s) = \tfrac{se^{-s}}{(s-1)^2+1}$$

b.) (10) Use THM7 and THM8 to solve for *x*(*t*). (Note: consider the following 'trick of 0' in the numerator term: *s* = (*s* – 1) + 1)

$$Y(s) = e^{-s}\tfrac{(s-1)+1}{(s-1)^2+1} = e^{-s}\tfrac{(s-1)}{(s-1)^2+1} + e^{-s}\tfrac{1}{(s-1)^2+1} = e^{-s}F(s-1) + e^{-s}G(s-1)$$

According to THM7, $\begin{array}{l} F(s) = \frac{s}{s^2+1} = L\{\cos t\} \Rightarrow F(s-1) = L\{e^t \cos t\} \\ G(s) = \frac{1}{s^2+1} = L\{\sin t\} \Rightarrow G(s-1) = L\{e^t \sin t\} \end{array}$

According to THM8:

$$Y(s) = e^{-s} L\{e^t (\cos t + \sin t)\} = L\{u(t-1)e^{(t-1)}[\cos(t-1) + \sin(t-1)]\}$$
$$\therefore x(t) = e^{(t-1)}[\cos(t-1) + \sin(t-1)], t > 1$$

c) (10) Method 2: Use Lemma1 (Handout 5b) solve the ODE Use the Pythagorean Theorem and the sine addition formulae to show your answer matches a)

$$Y(s) = \frac{se^{-s}}{(s-1)^2+1} = e^{-s} \frac{s}{(s-1)^2+1}$$

According to Lemma1: $L^{-1}\left\{\frac{s+a}{(s+b)^2+\omega^2}\right\} = \frac{k}{\omega} e^{-bt} \sin(\omega t + \phi)$ (where: $k = \sqrt{(a-b)^2 + \omega^2}, \phi = \arctan\left(\frac{\omega}{a-b}\right)$)

Here: a = 0, b = -1, ω= 1 $k = \sqrt{(0--1)^2 + 1^2} = \sqrt{2}, \phi = \arctan\left(\frac{1}{0--1}\right) = \arctan(1) = \frac{\pi}{4}$

So: $\begin{array}{l} L^{-1}\left\{\frac{s}{(s-1)^2+1^2}\right\} = \sqrt{2}e^t \sin\left(t + \frac{\pi}{4}\right) = \sqrt{2}e^t \left(\sin t \cos\frac{\pi}{4} + \cos t \sin\frac{\pi}{4}\right) \\ = \sqrt{2}e^t \left(\frac{\sqrt{2}}{2}\sin t + \frac{\sqrt{2}}{2}\cos t\right) = e^t (\sin t + \cos t) \end{array}$

According to THM8:

$$Y(s) = \frac{se^{-s}}{(s-1)^2+1} = e^{-s} \frac{s}{(s-1)^2+1} = e^{-s} L\{e^t (\sin t + \cos t)\}$$
$$= L\{u(t-1)e^{(t-1)}[\sin(t-1) + \cos(t-1)]\}$$
$$\therefore x(t) = e^{(t-1)}[\sin(t-1) + \cos(t-1)], t > 1$$

IV.) Solve: $\ddot{x} - 3\dot{x} + 2x = 2e^{-t}, x(0) = 2, \dot{x}(0) = -1$

(a) (12) By the UC (Undetermined Coefficients) Method

Auxiliary Eqn.: $r^2 - 3r + 2 = (r-2)(r-1) = 0 \Rightarrow x_{tr}(t) = c_1 e^{2t} + c_2 e^t$

$g(t) = 2e^{-t} \Rightarrow x_{ss}(t) = Ae^{-t}, \dot{x}_{ss}(t) = -Ae^{-t}, \ddot{x}_{ss}(t) = x_{ss}(t) = Ae^{-t}$

$\ddot{x}_{ss}(t) - 3\dot{x}_{ss}(t) + 2x_{ss}(t) = 6Ae^{-t} = 2e^{-t} \Rightarrow A = \frac{1}{3}$

So:

$$x(t) = x_{tr}(t) + x_{ss}(t) = c_1 e^{2t} + c_2 e^t + \tfrac{1}{3} e^{-t}$$
$$x(0) = 2 = c_1 + c_2 + \tfrac{1}{3} \Rightarrow c_1 + c_2 = \tfrac{5}{3}$$
$$\dot{x}(t) = 2c_1 e^{2t} + c_2 e^t - \tfrac{1}{3} e^{-t}$$
$$\dot{x}(0) = -1 = 2c_1 + c_2 - \tfrac{1}{3} \Rightarrow 2c_1 + c_2 = -\tfrac{2}{3}$$
$$\therefore c_1 = -\tfrac{7}{3} \Rightarrow c_2 = 4$$

Hence: $x(t) = -\frac{7}{3} e^{2t} + 4e^t + \frac{1}{3} e^{-t}$

b) (13) By the LT method

$$L\{\ddot{x} - 3\dot{x} + 2x\} = L\{2e^{-t}\} \Rightarrow \left(s^2 Y(s) - 2s + 1\right) - 3\left(sY(s) - 2\right) + 2Y(s) = \tfrac{2}{s+1}$$
$$\Rightarrow \left(s^2 - 3s + 2\right)Y(s) = \tfrac{2}{s+1} - 7 + 2s \Rightarrow (s-2)(s-1)Y(s) = \tfrac{2s^2 - 5s - 5}{s+1}$$
$$\Rightarrow Y(s) = \tfrac{2s^2 - 5(s+1)}{(s+1)(s-2)(s-1)} = \tfrac{A}{(s+1)} + \tfrac{B}{(s-2)} + \tfrac{C}{(s-1)}$$

By Heaviside Cover Method:

$$(s = -1) \Rightarrow A = \tfrac{2}{6} = \tfrac{1}{3}$$
$$(s = 2) \Rightarrow B = \tfrac{-7}{3 \cdot 1} = -\tfrac{7}{3}$$
$$(s = 1) \Rightarrow C = \tfrac{-8}{2 \cdot (-1)} = 4$$
$$\therefore Y(s) = \tfrac{1/3}{(s+1)} - \tfrac{7/3}{(s-2)} + \tfrac{4}{(s-1)} \Rightarrow x(t) = L^{-1}\{Y(s)\} = \tfrac{1}{3} e^{-t} - \tfrac{7}{3} e^{2t} + 4e^t$$

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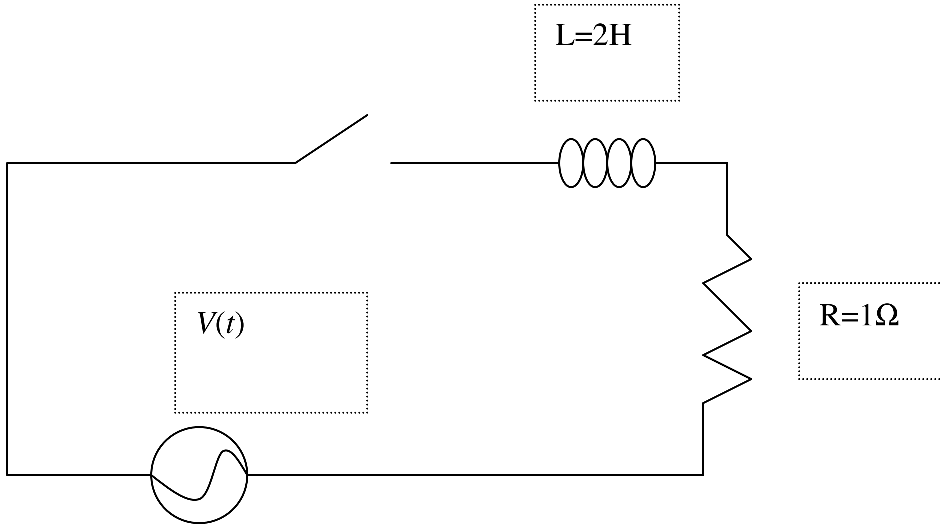
V) (25) A resistor of 1ohms is hooked up to a voltage source *V*(*t*) is series with an inductor of 2 henries, and a switch. Initially the current = 0. Find the current when:



…and $V(t) = 2\sin t$

$$V(s) = L\{2\sin t\} = \tfrac{2}{s^2+1^2} = 2\left(s+\tfrac{1}{2}\right)I(s) \Rightarrow \tfrac{1}{s^2+1^2} = \left(s+\tfrac{1}{2}\right)I(s)$$
$$\Rightarrow I(s) = \tfrac{1}{(s^2+1^2)(s+.5)}$$

<u>Method 1 (THM12)</u>:

$$I(s) = \tfrac{1}{s^2+1^2}\cdot\tfrac{1}{s+0.5} = L\{\sin t\}L\{e^{-0.5t}\} = L\{(\sin t * e^{-0.5t})\}$$

where: $\left(\sin t * e^{-0.5t}\right) = \int_0^t \sin(t-u)e^{-0.5u}\,du = \int_0^t \sin u e^{-0.5(t-u)}\,du$

(For algebraic simplicity, select the representation of the convolution on the right)

$$\int_0^t \sin u e^{-\frac{1}{2}(t-u)}\,du = e^{-\frac{1}{2}t}\int_0^t \sin u e^{\frac{1}{2}u}\,du = e^{-\frac{1}{2}t}\left\{\frac{e^{\frac{1}{2}u}}{(\frac{1}{2})^2+1^2}\left(\tfrac{1}{2}\sin u - \cos u\right)\Big|_0^t\right\}$$
$$= e^{-0.5t}\left\{\tfrac{4e^{0.5t}}{5}\left(\tfrac{1}{2}\sin t - \cos t\right) - \tfrac{4}{5}(-1)\right\} = \tfrac{4e^{-0.5t}}{5}\left\{e^{0.5t}\left(\tfrac{1}{2}\sin t - \cos t\right) + 1\right\}$$
$$= \tfrac{2}{5}\left[(\sin t - 2\cos t) + 2e^{-0.5t}\right]$$

Hence: $i(t) = \tfrac{2}{5}\left[(\sin t - 2\cos t) + 2e^{-0.5t}\right]$

<u>Method 2 (Partial Fractions):</u>

$$I(s) = \frac{1}{(s^2+1)(s+0.5)} = \frac{A_1 s + A_2}{(s^2+1)} + \frac{B}{s+0.5} \Rightarrow 1 = (A_1 s + A_2)(s+0.5) + B(s^2+1)$$
$$(s = -0.5) \Rightarrow 1 = \tfrac{5}{4}B \Rightarrow B = \tfrac{4}{5}$$
$$\Rightarrow 1 = (A_1 s + A_2)(s+0.5) + \tfrac{4}{5}(s^2+1)$$

$$(s = 0) \Rightarrow 1 = 0.5A_2 + \tfrac{4}{5} \Rightarrow \tfrac{1}{2}A_2 = \tfrac{1}{5} \Rightarrow A_2 = \tfrac{2}{5}$$

Differentiating the above eqn.:

$$0 = \left(A_1 s + \tfrac{2}{5}\right) + A_1(s+0.5) + \tfrac{8}{5}s = \left(2A_1 + \tfrac{8}{5}\right)s + \left(0.5A_1 + \tfrac{2}{5}\right)$$
$$s^1 : 2A_1 + \tfrac{8}{5} = 0 \Rightarrow A_1 = -\tfrac{4}{5}$$
$$check : s^0 : 0.5A_1 + \tfrac{9}{10} = 0 = \tfrac{1}{2}\cdot\left(-\tfrac{4}{5}\right) + \tfrac{2}{5}\, OK$$

Hence:

$$I(s) = \frac{-\frac{4}{5}s + \frac{2}{5}}{s^2+1} + \frac{\frac{4}{5}}{s+0.5} = \frac{1}{5}\left\{\frac{-4s+2}{s^2+1} + \frac{4}{s+0.5}\right\} = \frac{2}{5}\left\{\frac{-2s+1}{s^2+1} + \frac{2}{s+0.5}\right\}$$
$$\Rightarrow i(t) = \frac{2}{5}\left\{L^{-1}\left\{\frac{-2s+1}{s^2+1^2}\right\} + L^{-1}\left\{\frac{2}{s+0.5}\right\}\right\}$$

Using Lemma1 (Handout 5b) on the first term:

$$L^{-1}\left\{\frac{s+a}{(s+b)^2+\omega^2}\right\} = \frac{k}{\omega}e^{-bt}\sin(\omega t + \phi) \qquad \text{(where: } k = \sqrt{(a-b)^2+\omega^2}, \phi = \arctan\left(\tfrac{\omega}{a-b}\right)\text{)}$$

Hence: $L^{-1}\left\{\frac{-2s+1}{(s+0)^2+1^2}\right\} = -2L^{-1}\left\{\frac{s-0.5}{(s-0)^2+1^2}\right\} = -2\sqrt{\tfrac{5}{4}}e^{0t}\sin(t+\phi) = -\sqrt{5}\sin(t+\phi)$

Where:
$$\phi = \tan^{-1}\left(\frac{\quad}{-0.5}\right) \Rightarrow \tan\phi = \tfrac{2}{-1} \Rightarrow O = 2, A = -1 \Rightarrow H = \sqrt{A^2+O^2} = \sqrt{5}$$
$$\therefore \cos\phi = \tfrac{A}{H} = -\tfrac{1}{\sqrt{5}}, \sin\phi = \tfrac{O}{H} = \tfrac{2}{\sqrt{5}}$$

So:
$$L^{-1}\left\{\frac{-2s+1}{(s+0)^2+1^2}\right\} = -2L^{-1}\left\{\frac{s-0.5}{(s-0)^2+1^2}\right\} = -\sqrt{5}(\sin t\cos\phi + \cos t\sin\phi)$$
$$= -\sqrt{5}\left(-\tfrac{1}{\sqrt{5}}\sin t + \tfrac{2}{\sqrt{5}}\cos t\right) = \sin t - 2\cos t$$

Hence:
$$i(t) = \frac{2}{5}\left\{L^{-1}\left\{\frac{-2s+1}{s^2+1^2}\right\} + L^{-1}\left\{\frac{2}{s+0.5}\right\}\right\} = \frac{2}{5}\left\{\sin t - 2\cos t + 2e^{-0.5t}\right\}$$