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Philosophy of Physics Colloquium

Feb. 26, 2007

Geometric-Algebraic Approaches to Quantum Physics

Clifford Algebra

- Using results of Hamilton and Grassmann, W. K Clifford developed his algebra (1878-1882) to characterize rotations and spin, (generalized to any n –dimensional vector space V).
- A systematic collection of *directed* line segments (vectors), areas (bivectors), volumes (trivectors),..., n -dimensional hypervolumes (n -vectors or n -blades) $n = \dim V$.



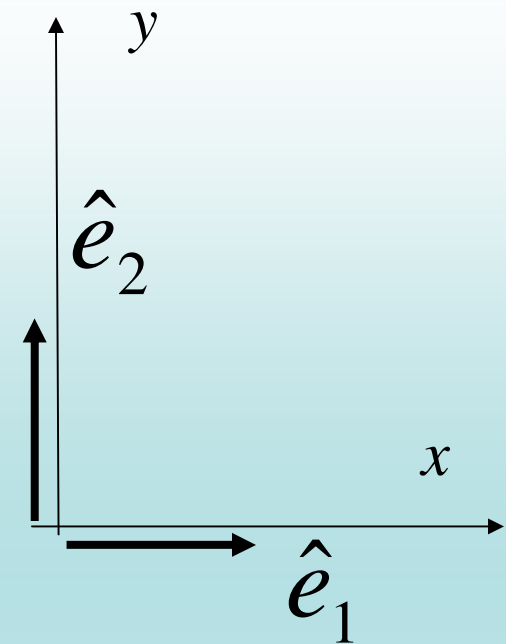
Toy Example: $CL(\mathbf{R}^2)$

$$\hat{e}_1^2 = \hat{e}_2^2 = 1, \quad \hat{e}_1 \bullet \hat{e}_2 = \hat{e}_2 \bullet \hat{e}_1 = 0$$

- Form Clifford Product:

$$\hat{e}_1 \hat{e}_2 = \hat{e}_2 \bullet \hat{e}_1 + \hat{e}_1 \wedge \hat{e}_2 = \hat{e}_1 \wedge \hat{e}_2 = -\hat{e}_2 \wedge \hat{e}_1 = -\hat{e}_2 \hat{e}_1$$

$$\begin{aligned} (\hat{e}_1 \hat{e}_2)^2 &= (\hat{e}_1 \hat{e}_2)(\hat{e}_1 \hat{e}_2) = \\ \hat{e}_1 (\hat{e}_2 \hat{e}_1) \hat{e}_2 &= -\hat{e}_1 (\hat{e}_1 \hat{e}_2) \hat{e}_2 = \\ -(\hat{e}_1 \hat{e}_1)(\hat{e}_2 \hat{e}_2) &= -(\hat{e}_1^2)(\hat{e}_2^2) = -1 \end{aligned}$$



Toy Example: $CL(\mathbf{R}^2)$ (cont.)

- Hence, the multivector $\hat{e}_1\hat{e}_2$ is algebraically isomorphic to $i = \sqrt{-1}$
- Moreover: $(\hat{e}_1\hat{e}_2)\hat{e}_1 = -\hat{e}_2$ and $(\hat{e}_1\hat{e}_2)\hat{e}_2 = \hat{e}_1$
- In matrix form:

$$\hat{e}_1\hat{e}_2 \equiv \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \hat{e}_1 \equiv \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \hat{e}_2 \equiv \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

- $\therefore CL(\mathbf{R}^2) \cong \mathbf{C} \otimes \mathbf{C} \cong \mathbf{H}$

Toy Example: $CL(\mathbf{R}^3)$

- Adopt abbreviation: $e_{i\dots k} = \hat{e}_i \dots \hat{e}_k$
- $CL(\mathbf{R}^3) = CL_{(0)} \oplus CL_{(1)} \oplus CL_{(2)} \oplus CL_{(3)}$
- Where:
 - $CL_{(0)} = \langle 1 \rangle \cong R$
 - $CL_{(1)} = \langle (e_1, e_2, e_3) \rangle$
 - $CL_{(2)} = \langle (e_{12}, e_{13}, e_{23}) \rangle$
 - $CL_{(3)} = \langle I \rangle = \langle e_{123} \rangle$
- Hence: $CL(\mathbf{R}^3) \cong \mathbf{C} \otimes (\mathbf{C} \otimes \mathbf{C}) \cong \mathbf{C} \otimes \mathbf{H} \cong \mathbf{Oct}$

For any $CL(V)$, $\dim V = N$

- $CL(V) = CL_{(0)} \oplus CL_{(1)} \oplus \dots \oplus CL_{(N)}$
- The closed subspaces $CL_{(k)}$ of grade k in $CL(V)$ have dimensionality $C(N,k) = \frac{N!}{[k!(N-k)!]}$
- (i.e, are spanned by $C(N,k)$ multivectors of grade k .)
- Hence the total number of Clifford basis elements generated by the Clifford product acting on the basis elements of the underlying vector space is:

$$2^N = \sum_{k=0}^N C(N, k)$$

Some General Definitions & Remarks

- For any $A, B \in CL(V)$ their (associative) Clifford product is defined by: $AB = A \bullet B + A \wedge B$

$A \bullet B$ is their (commutative and associative) *inner* product.

$A \wedge B$ is their anti-commutative ($A \wedge B = -B \wedge A$) and associative *exterior* (or Grassmann) product.

- CL is equipped with an adjoint \dagger and grade operator $\langle \rangle_r$ where $\langle \rangle_r$ is defined as isolating the r th grade of a Clifford element A . For any Clifford elements A, B :

$$\langle AB \rangle_r^\dagger = (-1)^{C(r,2)} \langle B^\dagger A^\dagger \rangle_r$$

$$\text{(where: } C(r, 2) = \frac{r!}{2!(r-2)!} = r(r-1)/2)$$

Some General Definitions & Remarks (cont.): The Unit Element I

- The unit pseudoscalar I should *not* be interpreted as a multiplicative identity: it is certainly *not* the case that for any $A \in CL(V)$, $AI = A = IA$.
- The unit pseudoscalar is adopted to define an element of dual grade A^* : for any pure Clifford element A_k , (where $0 \leq k < N$) $\langle A_k I \rangle$ (or $\langle A_k^* \rangle$) = $(N - k)$ and vice versa.
- Thus an inverse element A^{-1} can in principle still be constructed for every nonzero $A \in CL(V)$. So the linear equation $AX = B$ has the formal solution $X = A^{-1}B$ in $CL(V)$. “**Much of the power of geometric (Clifford) algebra lies in this property of invertibility.**” (Lasenby, et. al. (2000), 25)

David Hestenes (1984-6)

- Clifford algebras offer a reformulation of quantum theory associating any significant mathematical entity a semantic isomorphism with its algebraic and geometric meaning.
- On the other hand in the case of complex vector spaces (i.e. a Hilbert space in standard QM mathematical formalism) “[o]ne can distinguish three fundamentally different geometric roles tacitly assigned to the unit imaginary i , namely:

David Hestenes (1984-6)

- The generator of rotations in a plane :

$$\tilde{\varphi}(x, t) \mapsto e^{i\theta} \varphi(x, t)$$

- The generator of duality transformations :

$$\psi^{\uparrow}_B = (M_{AB} \psi^A)^*$$

- The indicator of an indefinite metric:

$$\psi^{\uparrow} \varphi = (\psi^A M_{AB} \psi^B)^*$$

David Hestenes (1984-6)(cont.)

“Confusion is difficult to avoid when i is required to perform more than one of these roles in a single system. Worse yet, in physics all three possibilities are sometimes realized in a single theory confounded with problems of physical interpretation...[t]he multiplicity of geometric interpretations shows that conventional mathematical formalisms are profoundly deficient in their tacit assumption that there is a unique ‘imaginary unit’. Therefore, in the interest of fidelity to geometric interpretation, the convention that complex numbers are scalars should be abandoned in favor of...a system in which each basic geometric distinction has a unique algebraic representation. Geometric [Clifford] Algebra has this property.” (1984, xii – xiii).

David Hestenes (1984-6)(cont.)

- Physicists generally regard the σ^k [Pauli spin matrices] as three components of a single vector, instead of an orthonormal frame of three vectors...Consequently, they write:

$$\vec{\sigma} \cdot \vec{v} = \sum_{k=1}^3 \sigma_k v_k$$

...and to facilitate manipulation they employ the identity:

$$(\vec{\sigma} \cdot \vec{v})(\vec{\sigma} \cdot \vec{w}) = \vec{v} \cdot \vec{w} + i\vec{\sigma} \cdot (\vec{v} \times \vec{w})$$

... a good example of the redundancy in the language of physics which complicates the manipulations and obscures the meanings unnecessarily.” (1986, 323)

David Hestenes (1984-6)(cont.)

- The redundancy in the above identity is due to its ‘overlapping geometric content’: The (vector) dot and cross products of course comprise the binary operations of standard (Gibbs’) vector algebra in \mathbf{R}^3 , while the Pauli spin matrices acting as the ‘vector coefficients’ belong to the spinor algebra \mathbf{C}^2 .
- The geometric contents of \mathbf{R}^3 and \mathbf{C}^2 can be unified, however, when one instead considers σ^k as generators of a Clifford algebra, thereby “eliminat[ing] all redundancy incorporating both languages into a single coherent language.”

David Hestenes (1984-6)(cont.)

- To see this, simply write for any 3-vector:

$$\vec{v} = \sum_{k=1}^3 v_k \sigma_k$$

- Then the above identity with its (otherwise geometrically overlapping content) now simply is represented as: .

$$\vec{v} \vec{w} = \vec{v} \bullet \vec{w} + \vec{v} \wedge \vec{w}$$

- This is just precisely the definition of the Clifford product of two 3-vectors.

David Hestenes (1984-6)(cont.)

- Using the same reasoning as demonstrated above in the case of the Pauli spin algebra, Hestenes (1986) likewise shows how the 4×4 matrix algebra \mathbf{C}_4 of Dirac spinors γ_μ is algebraically isomorphic to the Clifford algebra CL_4 , or the (grade 4) Clifford algebra generated by 4-dimensional vectors with complex coefficients.
- This Clifford algebra is projectively isomorphic to the Minkowski spacetime algebra $\mathbf{R}_{1,3}$, or the Clifford algebra generated by four linearly independent rotation matrices in the Minkowski spacetime $\mathbf{R}_{1,3}$.

David Hestenes (1984-6)(cont.)

- “The relation of the Dirac theory [of spinors] to classical electrodynamics is not well understood...[with the projective extension into $\mathbf{R}_{1,3}$. however] **it is more intimate than originally thought....**”
- “This intimate relation between ...the Dirac theory and trajectories of the classical theory [shown in the $\mathbf{R}_{1,3}$ reformulation] provides a much more detailed correspondence between the classical and quantum theories than the conventional approach using expectation values and Ehrenfest’s theorem...”

David Hestenes (1984-6)(cont.)

- “The basic idea... [to] provide a general geometrical approach to the interpretation of the Dirac theory [goes] as follows...
- Any solution $\psi = \psi(x)$ of the Dirac Equation of form:

$$\psi = (\rho e^{i\beta})^{1/2} R$$

[where: ρ is a probability density

R is a spinor representation of a Lorentz transformation Λ ,

β is an arbitrary phase factor]

David Hestenes (1984-6)(cont.)

- [The above solution] determines a field of orthonormal frames $e_\mu = e_\mu(x)$...at each spacetime point there's a streamline $x = x(\tau)$ with tangent $v = v(x(\tau))$.
- [Then] $e_\mu = e_\mu(x(\tau))$ is to be regarded as a 'comoving frame', on the streamline, where e_1, e_2 rotate about the 'spin axis' e_3 ." (332-333)

David Hestenes (1984-6)(cont.)

- From the classical solution of the Dirac Equation, Hestenes derives the result: $\dot{R} = \frac{d}{d\tau} R = \frac{e}{2m} FR$
or the (electron's proper time) rate of change of R (the spinor representation of the Lorentz transformation Λ in the solution of canonical form $\psi = (\rho e^{i\beta})^{1/2} R$) as proportional to the product of the electron's charge with R and the magnitude of the electromagnetic field F .

David Hestenes (1984-6)(cont.)

- Hestenes interprets as an expression of the *precession* of the comoving frame $e_\mu = e_\mu(x(\tau))$ with an additional rotation determined by a gauge factor. “It should be of genuine physical interest to identify and analyze any deviations from this classical rotation which QM might imply.” (333)
- Hestenes (1985, 13) suggests that such expressions of precession provide adequate models for the supposed *Zitterbewegung* mechanism of a free electron.

Elio Conte (1994-2000)

- Provides a toy model of wave-packet reduction for a spin- $1/2$ system using a Biquaternionic characterization of QM (i.e. a grade 3 Clifford algebra)
- Uses the above result to modify Arthur Fine's solution to the QM measurement problem.
- Generalizes Schroedinger's Eqn in a Biquaternionic formalism, which enables one to represent effective short-range potentials $V'(R)$ typically encountered in fusion phenomena.

Elio Conte (1994-2000) (cont.)

- Consider a spin- $1/2$ system S interacting with measurement apparatus A. Then the wavefunction describing S is given by:

$$\psi = c_+ \chi_+ + c_- \chi_- \equiv \begin{bmatrix} c_+ \\ c_- \end{bmatrix}$$

- The density matrix is given by:

$$\hat{\rho} = \psi \psi^\dagger \equiv \begin{bmatrix} c_+ c_+^* & c_+ c_-^* \\ c_+^* c_- & c_- c_-^* \end{bmatrix}$$

Elio Conte (1994-2000) (cont.)

- Consider the biquaternion:

$$\eta = a + be_1 + ce_2 + de_3 \equiv \begin{bmatrix} a + d & b - ic \\ b + ic & a - d \end{bmatrix}$$

- Where:

$$e_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, e_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, e_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Elio Conte (1994-2000) (cont.)

- Hence it follows:

$$a = \frac{|c_+|^2 + |c_-|^2}{2} = \frac{1}{2}$$

$$b = \frac{c_+ c_-^* + c_+^* c_-}{2}$$

$$c = \frac{i(c_+ c_-^* - c_+^* c_-)}{2}$$

$$d = \frac{|c_+|^2 - |c_-|^2}{2}$$

- (if we represent the density matrix ρ with biquaternion η)
- **Thm:** $\rho^2 = \rho$ iff $\|\eta\| = 0$ and $a = 1/2$

Elio Conte (1994-2000) (cont.)

- “During interaction of S with A, the coefficients c_{\pm} change from being complex numbers to biquaternions.” (1993, 534):

$$c_{+} = m_0 + m_1 e_1 + m_2 e_2 + m_3 e_3$$

$$c_{-} = p_0 + p_1 e_1 + p_2 e_2 + p_3 e_3$$

- With interaction density matrix:

$$\bar{\rho} \equiv \begin{bmatrix} -m_3^2 \pm m_3 p_1 & 0 \\ 0 & -p_1^2 \mp m_3 p_1 \end{bmatrix}$$

Elio Conte (1994-2000) (cont.)

- At the end of the measurement interaction:

$$\bar{\rho}_f \equiv \begin{bmatrix} 1/2 & 0 \\ 0 & 1/2_1 \end{bmatrix}$$

- Hence the passage: $\rho \rightarrow \bar{\rho} \rightarrow \bar{\rho}_f$ “is evidence of the passage from a density matrix of a pure state to a density matrix of a mixture...we have provided an example of wave packet reduction, using biquaternions.” (534)

Elio Conte (1994-2000) (cont.)

- **Response to Fine:**
- **Measurement interactions between A and S involve two kinds of information transfer:**
- **SIC (Selective information content) (governed by the Boolean logic of observer + A)**
- **DIC (Descriptive information content) (the actual information content of S , which need not be Boolean.**
- **“It should be clear that this transfer from DIC to SIC is not a trivial transformation. **Instead, a generalization of physics is required in order to actually describe such a transformation.**” (1196, 144)**

Finkelstein (2001)

- Clifford Algebraic Characterizations of QM can accommodate a quantum spacetime and field structure in a way that C^* algebraic methods cannot
- Clifford algebraic characterizations exhibit a robust regularization feature that other field theoretic formalisms do not.
- Clifford algebraic characterizations \rightarrow Normal relativity groups \rightarrow stable Lie algebra (generalization of the results of Wigner and Inonu)

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