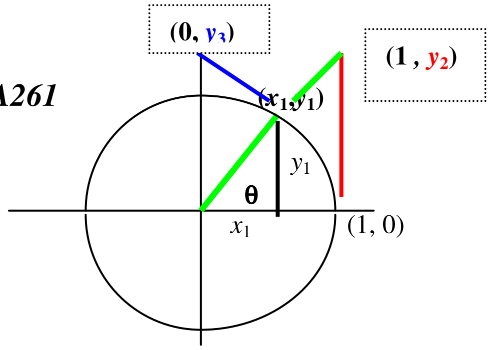
Trig. Formulae/Review: MA261

Nov 6-Kallfelz



$x_1 = \cos\theta$
$y_1 = \sin\theta$
$r = 1$

Equation of unit circle: $x^2 + y^2 = 1$ therefore $\cos^2\theta + \sin^2\theta = 1$ (*since* $x_1 = \cos\theta$, $y_1 = \sin\theta$)

Special Angles:

θ	$\cos\theta$	$\sin\theta$	θ	$\cos\theta$	$\sin\theta$
30° ($^{\pi}/_{6}$ rad)	$^{\sqrt{3}}/_{2}$	$^{1}/_{2}$	*210°* ($^{7\pi}/_{6}$ rad)	- $-^{\sqrt{3}}/_{2}$	$-^{1}/_{2}$
45° ($^{\pi}/_{4}$ rad)	$^{\sqrt{2}}/_{2}$	$^{\sqrt{2}}/_{2}$	*225°* ($^{5\pi}/_{4}$ rad)	$-^{\sqrt{1}}/_{2}$	$-^{1}/_{2}$
60° ($^{\pi}/_{3}$ rad)	$^{1}/_{2}$	$^{\sqrt{3}}/_{2}$	*240°* ($^{4\pi}/_{3}$ rad)	$-^{1}/_{2}$	$-^{\sqrt{3}}/_{2}$
90° ($^{\pi}/_{2}$ rad)	0	1	*270°* ($^{3\pi}/_{2}$ rad)	0	-1
120° ($^{2\pi}/_{3}$ rad)	$-^{1}/_{2}$	$^{\sqrt{3}}/_{2}$	*300°* ($^{5\pi}/_{3}$ rad)	$^{1}/_{2}$	$^{\sqrt{3}}/_{2}$
135° ($^{3\pi}/_{4}$ rad)	$-^{\sqrt{2}}/_{2}$	$^{\sqrt{2}}/_{2}$	*315°* ($^{7\pi}/_{4}$ rad)	$^{\sqrt{2}}/_{2}$	$-^{\sqrt{2}}/_{2}$
150° ($^{5\pi}/_{6}$ rad)	$-^{\sqrt{3}}/_{2}$	$^{1}/_{2}$	*270°* ($^{11\pi}/_{6}$ rad)	$^{\sqrt{3}}/_{2}$	$-^{1}/_{2}$
180° (π rad)	-1	0	*360°* (2πrad)	*1*	*0*

***Non-Uniqueness of measure of angle**: any angle* θ *with* $0° \leq \theta < 360°$ *can be written:* $\theta + n(360°)$
or in radians $(0 \leq \theta < 2\pi)$ *can be written:* $\theta + 2n\pi$

Periodicity and relation between sine/cosine: $\sin(\theta + {}^{\pi}/_{2}) = \cos\theta$ $\sin(\theta) = \sin(\pi - \theta)$

$\cos(\theta) = -\cos(\pi - \theta)$ $\cos(\theta) = -\cos(\pi + \theta)$ $\cos(\theta) = \cos(-\theta)$ $\sin(\theta) = -\sin(-\theta)$

$\sin(\theta + 2n\pi) = \sin\theta$ $\cos(\theta + 2n\pi) = \cos\theta$

Associated Trig Functions:

$\tan\theta = \dfrac{\sin\theta}{\cos\theta}$ $\cot\theta = \dfrac{\cos\theta}{\sin\theta}$ $\sec\theta = 1/\cos\theta$ $\csc\theta = 1/\sin\theta$

- **Note1: From the above illustration:** $y_2 = \tan\theta$ $y_3 = \csc\theta$

Pythagorean Identities for associated trig fnctions:

$\sec^2\theta = 1 + \tan^2\theta$ $\csc^2\theta = 1 + \cot^2\theta$

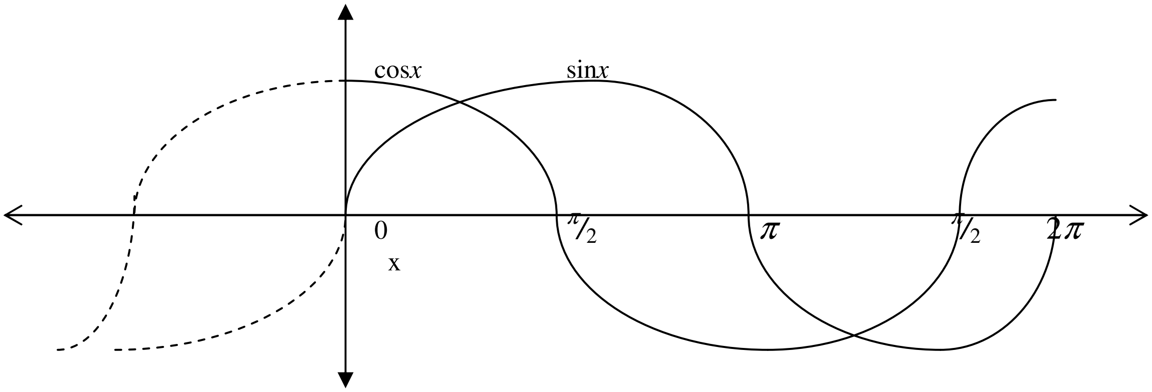
Since:

$$\sec^2\theta = \frac{1}{\cos^2\theta} = \frac{\sin^2\theta + \cos^2\theta}{\cos^2\theta} = \tan^2\theta + 1$$

$$\csc^2\theta = \frac{1}{\sin^2\theta} = \frac{\sin^2\theta + \cos^2\theta}{\sin^2\theta} = 1 + \cot^2\theta$$

- **Note2:** From the above illustration, the length of the green hypotenuse of the above triangle = $\sec^2\theta$ as is readily seen by computing the Pythagorean Theorem (opposite side = $\tan\theta$, and adjacent side = 1.)

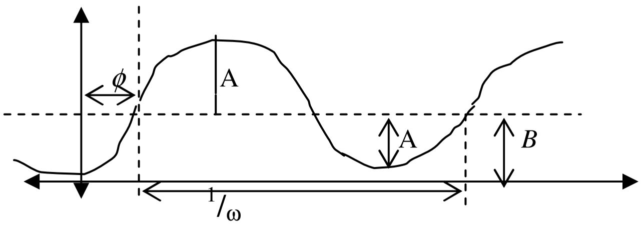
Simple Plot of* sin*x*, cos*x



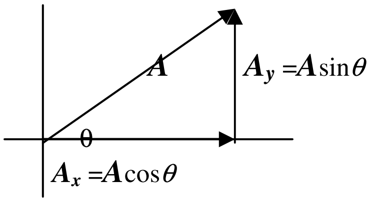
Graph of sin*x*, cos*x* (for all real x) not to scale

A more General Plot of sinx,

Suppose $y = A\sin(\omega x+\phi) + B$ *A is the **amplitude** (height of wave)*
*ω is the **angular frequency,** indicating number of angular cycles per unit time. Furthermore, $\omega = 2\pi f$, where f is the **frequency** (number of cycles per unit time) and f = 1/T is the **period,** duration of cycle)*
*B is the up/down **shift,** relative to x axis.* *ϕ is the **phase**, measuring shift away from y axis*



Components



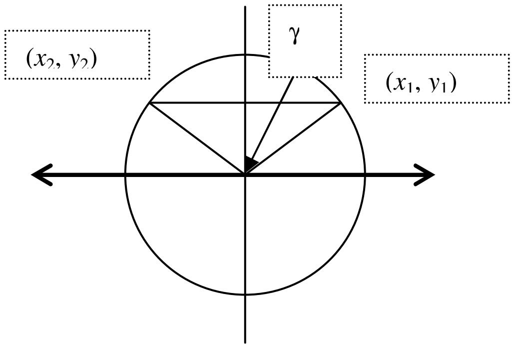
Pythagorean Law: $A^2 = A_x^{\,2} + A_y^{\,2}$
therefore: $A = \sqrt{A_x^{\,2} + A_y^{\,2}}$
$\tan\theta = A_y/A_x$ $\theta = \tan^{-1}(A_y/A_x)$

Angle Addition Formulae:

$\sin(\theta + \varphi) = \sin\theta\cos\varphi + \cos\theta\sin\varphi$ $\cos(\theta + \varphi) = \cos\theta\cos\varphi - \sin\theta\sin\varphi$
$\sin(\theta - \varphi) = \sin\theta\cos\varphi - \cos\theta\sin\varphi$ $\cos(\theta - \varphi) = \cos\theta\cos\varphi + \sin\theta\sin\varphi$
$\mathbf{\sin(2\theta)} = \sin(\theta + \theta) = \sin\theta\cos\theta + \cos\theta\sin\theta = \mathbf{2\sin\theta\cos\theta}$ $\cos(\theta + \theta) = \cos\theta\cos\theta - \sin\theta\sin\theta$
or: $\mathbf{\cos(2\theta) = \cos^2\theta - \sin^2\theta}$

(What's indicated in boldface are the special cases of addition as the double angle formulae)

- **A geometric derivation of the cosine addition rule:**



From the above illustration, $x_1 = \cos\theta$, $y_1 = \sin\theta$, $x_2 = \cos\varphi$, $y_2 = \sin\varphi$, $\gamma = (\varphi - \theta)$

To compute the length of the base of the inscribed isosoles triangle, on the left hand side use the Law of Cosines and on the right hand side use the distance formula:

$$1^2 + 1^2 - 2\cos\gamma = |x_2 - x_1|^2 + |y_2 - y_1|^2 \Rightarrow 2 - 2\cos\gamma = (\cos\varphi - \cos\theta)^2 + (\sin\varphi - \sin\theta)^2$$
$$\Rightarrow 2 - 2\cos\gamma = \cos^2\varphi - 2\cos\varphi\cos\theta + \cos^2\theta + \sin^2\varphi - 2\sin\varphi\sin\theta + \sin^2\theta$$
$$\Rightarrow 2 - 2\cos\gamma = \cos^2\varphi + \sin^2\varphi + \sin^2\theta + \cos^2\theta - 2(\cos\varphi\cos\theta + \sin\varphi\sin\theta)$$
$$\Rightarrow \cos\gamma = \cos(\varphi - \theta) = \cos\varphi\cos\theta - \sin\varphi\sin\theta$$

Half Angle Laws

From the double angle formula, $\cos(2\theta) = \cos^2\theta - \sin^2\theta = \cos^2\theta - (1 - \cos^2\theta) = 1 + 2\cos^2\theta$

↑**(since $\cos^2\theta + \sin^2\theta = 1$)**

Therefore: $\cos\theta = \pm {}^1/_2\sqrt{1 + \cos 2\theta}$ (isolating $\cos\theta$ from above expression) Renaming: $\phi = 2\theta$:

Therefore: $\boldsymbol{\cos(\phi/2) = \pm {}^1/_2\sqrt{1 + \cos\phi}}$

$\boldsymbol{\sin(\phi/2) = \pm {}^1/_2\sqrt{1 - \cos\phi}}$ (Performing the same procedure above for $\sin\theta$_

<u>Exercise:</u> *Use the above half angle laws to derive the sine, cosine for 15°,22.5°*

Product Sum Formulae: *Taking Appropriate sums and differences of the angle addition laws we can relate products of angles with sums:*

$\sin\theta\cos\varphi = {}^1/_2\,[\sin(\theta + \varphi) + \sin(\theta - \varphi)]$ $\qquad$ $\sin\theta\sin\varphi = {}^1/_2\,[\cos(\theta - \varphi) - \cos(\theta + \varphi)]$

$\cos\theta\cos\varphi = {}^1/_2\,[\cos(\theta + \varphi) + \cos(\theta - \varphi)]$

Sum Product Formulae: *Finally, renaming:* $\alpha = (\theta + \varphi)$ $\beta = (\theta - \varphi)$, *using the above formuale we can relate sums/differences of sine, cosine with products:*

$\sin\alpha + \sin\beta = 2\sin[(\alpha + \beta)/2]\cos[(\alpha - \beta)/2]$ $\qquad$ $\cos\beta + \cos\alpha = 2\sin[(\alpha + \beta)/2]\sin[(\alpha - \beta)/2]$

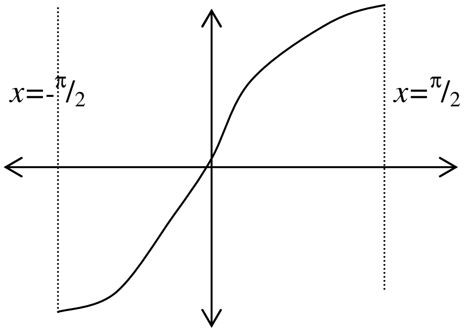
$\cos\alpha + \cos\beta = 2\cos[(\alpha + \beta)/2]\cos[(\alpha - \beta)/2]$

Inverse Trigonometric Functions

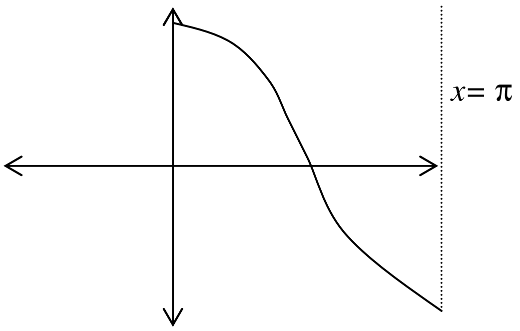
Recall the definition of an **inverse function:** For any function $f: X \rightarrow Y$ (where: X = Domain$f$ and Y = Range$f$), we can define: $f^{-1}: Y \rightarrow X$ where: $f^{-1}(f(x)) = x$ (for all x in X) and $f^{-1}(f(y)) = y$ (for all y in Y) **if f is a 1-1 correspondence.** (A 1-1 correspondence means: $f(x_1) = f(x_2)$ implies $x_1 = x_2$. For example, $f : \mathrm{R} \rightarrow \mathrm{R}$ [where: R are the real numbers] defined by rule: $f(x) = x^2$ is **not** a 1-1 correspondence. Consider $x_1 = 2$ and $x_2 = -2$. Obviously $f(x_1) = f(x_2)$, but this does not imply $x_1 = x_2$!)

- **Inverse sine, Inverse cosine**

  Looking at the graph of sine and cosine, obviously both of these functions aren't 1-1 correspondences as long as we let the domain=all the real numbers. (You probably remember the rule: 'Does the graph pass the horizontal line test?' [meaning: 'Can we cut the graph with a horizontal line at only one place?']) We'll have to shrink the domain in a way that doesn't affect the range! Note the illustrations below:



Shrinking domain of sine to interval: $[-\pi/2, \pi/2]$ makes it 1-1 (passes horiz. line test)



Shrinking domain of cosine to interval: $[0, \pi]$ makes it 1-1.

So now that we've turned sine and cosine into 1-1 correspondences by defining their domains and ranges accordingly as:

$$\sin : [-\pi/2, \pi/2] \rightarrow [-1,1] \qquad \cos : [0, \pi] \rightarrow [-1,1]$$

we define their inverse functions (sometimes denoted: 'arcsin,' 'arccos') respectively as:

$$\sin^{-1} : [-1,1] \rightarrow [-\pi/2, \pi/2] \qquad \cos^{-1} : [-1,1] \rightarrow [0, \pi]$$

which in a practical sense means when you evaluate inverse sine on your calculator, the answer is always bounded in the interval $[-\pi/2, \pi/2]$ (or $[-90^\circ, 90^\circ]$) and similarly $[0, \pi]$ (or $[0^\circ, 180^\circ]$) for inverse cosine. Care must be taken to produce final right answer, *which depends on the geometry of the problem!* For example, if you **know** (from your picture) that the unknown angle in a triangle is obtuse (greater than 90) and you produce its value by evaluating its inverse sine, ***you must subtract this answer from 180^o to produce your final answer! (Why?)***

Similarly the 1-1 correspondence versions of the associated trig functions are:

$\tan : (-\pi/2, \pi/2) \rightarrow (-\infty,\infty)$ $\quad cot : (0, \pi) \rightarrow (-\infty,\infty)$ $\quad \sec : [0, \pi/2) \cup (-\pi/2, 1] \rightarrow (-\infty,-1] \cup [1,\infty)$

$\csc : [-\pi/2, 0) \cup (0, \pi/2] \rightarrow (-\infty,-1] \cup [1,\infty)$

So their inverse functions are, respectively:

$$\tan^{-1} : (-\infty,\infty) \rightarrow (-\pi/2, \pi/2) \qquad \cot^{-1} : (0, \pi) \rightarrow (-\pi/2, \pi/2)$$

$$\sec^{-1} : (-\infty,-1] \cup [1,\infty) \rightarrow [0, \pi/2) \cup (-\pi/2, 1] \qquad \csc^{-1} : [-\pi/2, 0) \cup (0, \pi/2] \rightarrow [0, \pi/2) \cup (-\pi/2, 1]$$

LIMITS OF TRIGONOMETRIC FUNCTIONS

- **Exercise: sinx, cosx are continuous**

Show: for all c real: $\lim_{x\to c} \cos x = \cos c$

We use the result from text, that: $\lim_{x\to 0}\cos x = \cos 0 = 1$ (that cosx is continuous at origin.)
and: $\lim_{x\to 0}\sin x = \sin 0 = 0$ (that sinx is continuous at origin.)

To show continuity everywhere for cos$x$, use the addition formula for cosine:

cos(A+B) = cosAcosB - sinAsinB

To do this, let: $x = c + h$. Then, as $x\to c$, obviously $h\to 0$. Hence:

$$\lim_{x\to c} \cos x = \lim_{h\to 0} \cos(c+h) = \lim_{h\to 0} \cos(c)\cos(h) - \lim_{h\to 0} \sin(c)\sin(h)$$

$$=\cos(c) \lim_{h\to 0}\cos(h) - \sin(c)\lim_{h\to 0} \sin(h) \qquad \text{(Using limit theorems)}$$

$$\cos(c)*1 - \sin(c)*0 = \cos(c)$$

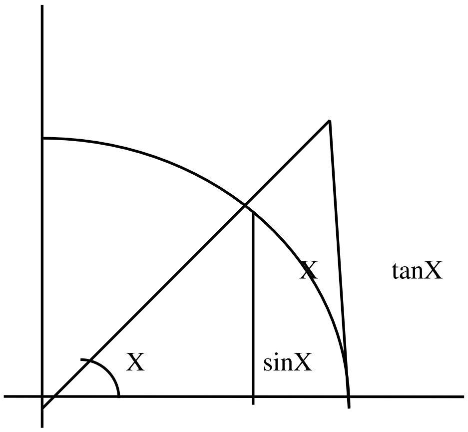
(or we've shown cosine is continuous everywhere)

- **Exercise: Show (as $x\to 0$) $\sin x/x \to 1$ and $(1 - \cos x)/x \to 0$**

These are well-known and interesting and useful limits, albeit counterintuitive. One way to understand them is to recall that as x <<1 (as x gets very small), sinx ≅ x, so sinx and x change at about the same rate as x gets very small. A similar argument can be made for cosx.

One can use an **area** argument to show that . $\lim_{x\to 0} \frac{\sin x}{x} = 1$ (as covered in class)

A simpler argument however can be made, which we do here (based on arclength):



- **From above illustration, certainly the length of the arc lies between sinX and tanX. The arclength, of course, is = RX (where X is measured in radians). However this is a unit circle, so R=1. Therefore the arclength = X, hence::**

$$\sin X \leq X \leq \tan X$$

...dividing this inequality by sinX:

$$1 \leq \frac{X}{\sin X} \leq \frac{1}{\cos X}$$

Using Sandwich (Pinching) Theorem:

$$1 \leq \lim_{x\to 0} \frac{x}{\sin x} \leq \lim_{x\to 0} \frac{1}{\cos x} = 1$$

Hence: $\lim_{x\to 0} \frac{x}{\sin x} = 1$ **so by limit theorems it's also true for the reciprocal.**[1]

To show the second result: $\lim_{x\to 0} \frac{(1 - \cos x)}{x} = 0$ **we use the aboce result and the "trick of 1":**

$$\frac{(1 - \cos x)}{x}\underset{\substack{\uparrow \\ \text{"Trick of 1"}}}{\frac{(1 + \cos x)}{(1 + \cos x)}} = \frac{(1 - \cos^2 x)}{x(1 + \cos x)} = \frac{\sin^2 x}{x(1 + \cos x)} = \underset{\substack{\uparrow \\ \mathbf{A}}}{\frac{\sin x}{x}}\ \underset{\substack{\uparrow \\ \mathbf{B}}}{\frac{\sin x}{(1 + \cos x)}}$$

Now, as x→0, we've seen already that Term A→1, while Term B→0 (continuity of sinx,cosx)

Exercise $\lim_{x\to 0} [(x^2 - 2x)/\sin 3x]$

- **In these excercises we can of course borrow from our previous results, but it's important that consistency is maintained regarding the arguments of sine and cosine and their divisors. To emphasize this, we can re-write the above limits in slightly more general form:**

$$\boxed{\lim_{u\to 0} \frac{(1 - \cos(u(x)))}{u(x)} = 0 \qquad \lim_{u\to 0} \frac{\sin(u(x))}{u(x)} = 1}$$

$$\lim_{x\to 0} \frac{(x^2 - 2x)}{\sin 3x} = \lim_{x\to 0} \frac{x}{\sin 3x} \lim_{x\to 0}(x - 2) = \underset{\substack{\uparrow \\ \textbf{We must express with proper consistency}}}{\lim_{x\to 0} \frac{3x}{\sin 3x}} \lim_{x\to 0} \frac{(x - 2)}{3}$$

$$= (1)*(-{}^{2}/_{3}) = (-{}^{2}/_{3}).$$

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[1] Precisely, because recall if limg=L≠0 then lim(1/g) = (1/L)

Exercise $\lim_{x\to 0}$ {$\sin^2 x/x[1 - \cos x]$}

- **We're tempted to split up $\sin^2 x = \sin x \sin x$ and use above results, the problem with this approach is we'll end up with an indeterminate expression of the form: $^0/_0$ which we have seen since limits were introduced means we have more work to do before we can come up with a definite answer! (IMPORTANT: Bear in mind that an answer of the form: $^a/_0$ (where: $0 < |a| < \infty$) <u>is</u> "determinate" (and equal to $\pm\infty$, depending on the sign of a) while an answer of the form $^0/_0$ is <u>not determinate.</u> In fact, as you've seen, the distinction between $^a/_0$ and $^0/_0$ is precisely the distinction between an <u>essential</u> and a <u>removable</u> singularity. Perhaps a greater insight into "removable" is possible for you to see now, for "removable" implies more work needs to be done to decide what's going on at the point.**

Thus in light of the above remark, it's better to rewrite expression in terms of the Pythagorean Identity:

$$\frac{\sin^2 x\ .}{x[1 - \cos x]\}} = \frac{(1 - \cos^2 x)}{x[1 - \cos x]} = \frac{(1 - \cos x)\,(1 + \cos x)}{x[1 - \cos x]\}} = \frac{(1 + \cos x)}{x}$$

- **As $x\to 0$, the expression $\to 2/0 \to\infty$**

Method2: L'Hopital's Rule:

$$\lim_{x\to 0}\frac{\sin^2 x}{x(1-\cos x)} = \lim_{x\to 0}\frac{\frac{d}{dx}\sin^2 x}{\frac{d}{dx}x(1-\cos x)} = \lim_{x\to 0}\frac{2\sin x\cos x}{(1-\cos x)+x\sin x} = \lim_{x\to 0}\frac{\sin 2x}{x\sin x-\cos x+1}$$

$$= \lim_{x\to 0}\frac{\frac{d}{dx}\sin 2x}{\frac{d}{dx}(x\sin x-\cos x+1)} = \lim_{x\to 0}\frac{2\cos 2x}{(\sin x+x\cos x-\sin x)} = \lim_{x\to 0}\frac{2\cos 2x}{x\cos x} = \tfrac{2}{0} = \infty$$

Exercise $\lim_{x\to\pi}$ {$\sin x/(x - \pi)$}

Remeniscient of the proof of the continuity of sinx, cosx, let $h = (x-\pi)$. Then: $x=h+\pi$:

$$\frac{\sin(h+\pi)}{h} = \frac{\sin h\cos\pi + \cos h\sin\pi}{h} = \frac{\sin h\cos\pi}{h} + \frac{\cos h\sin\pi}{h} = \frac{-\sin h}{h} + 0 \text{ ...which}\to -1$$

Exercise Prove that if $|f(x)/x| \leq B$ for all $x\neq 0$, then $\lim_{x\to 0} f(x)=0$

- <u>**Answer:**</u> $|f(x)/x| = |f(x)|/|x| = \leq B$ hence $|f(x)| = \leq B|x|$ which implies:

  $|f(x) - 0| \leq B|x-0| < B\delta = \varepsilon$