MA261 **Calculus I: Notes for September 18**
Fall 2007 Kallfelz

- ***LOCATING ABSOLUTE/LOCAL EXTREMA*** (§ 4.1, 4.3)

Given a function $f(x)$ defined on a closed interval $[a, b]$,an absolute extremum (maximum or minimum) is a point $c \in [a, b]$ whose y-value happens to tower over (or under) all the other y-values for f in the interval $[a, b]$. Stated more formally:

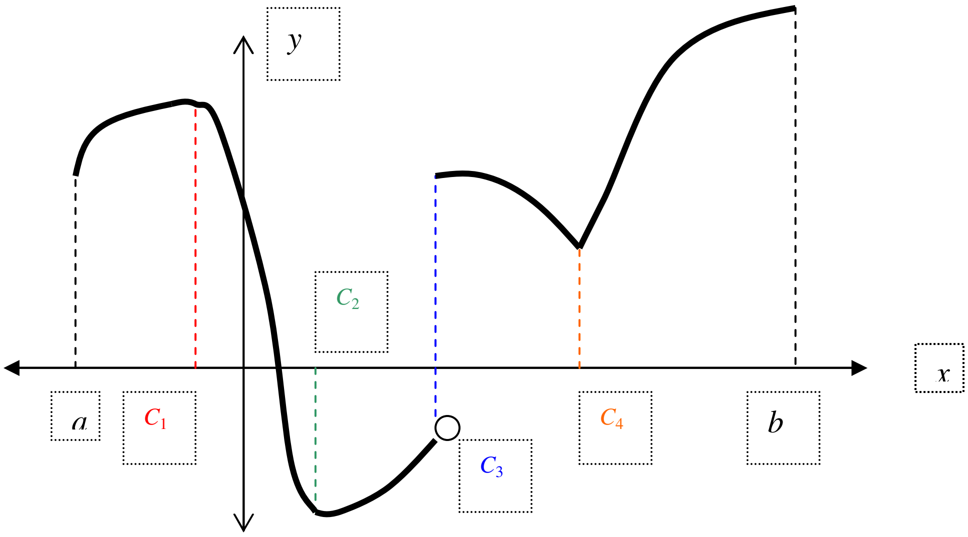
$c \in [a, b]$ is an absolute maximum if $f(c)$ if $f(c) \geq f(x)$ for all $x \in [a, b]$

$c \in [a, b]$ is an absolute mininum if $f(c)$ if $f(c) \leq f(x)$ for all $x \in [a, b]$

Obviously c doesn't need to be unique (there can be more than one absolute extremum). Section 4.1 details a relatively straightforward procedure for determining the absolute maxima of functions defined in finite, closed intervals.

To do this, necessarily involves the concept of derivative, however.

To see this, let's say you've got some complicated function $f(x)$ defined on a closed interval $[a, b]$, and you're interested in locating its absolute extrema. Below is a graph of such a case



We can immediately see from the graph what is the significance of notions like differentiability and continuity we learned in chapters 2 & 3. By inspection we observe the following:

- f is discontinuous at C_3 so f **cannot** be differentiable at C_3 ![1]
- f is not differentiable at C_4 .[2]
- f has zero-derivatives (its instantaneous rate of change is zero, i.e. has a flat tangent) at points C_1 and C_2

Basically, all those points C_1, C_2, C_3, C_4 are classified as critical points. That is to say, a **critical point c is a point where either $f'(c) = 0$ or $f'(c)$ fails to exist.**

Question: Why are such points classified as **critical**, and how would they be important in ascertaining what the absolute extrema are?

Answer: Because (as you can see the from the graph above), **critical points** are obviously places where the graph of a function *can* suddenly **reverse its trend…**that is to say, **suddenly *decrease* after the critical point** or vice versa. The word 'can' here is important: it doesn't mean that the graph of the function *must* alter its trend at a critical point (for reasons we'll see shortly), merely, that it *can*.

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[1] Recall the **necessary condition** for differentiability: **If** f is differentiable at c, **then** f has well-defined left handed and right handed derivatives that agree at c. That is to say, **if:**

$$f'(c) = \lim_{\Delta x \to 0} \frac{f(c+\Delta x) - f(c)}{\Delta x} = \lim_{x \to c} \frac{f(x) - f(x)}{x - c} \qquad \textbf{exists,}$$

Then: $$f'(c)^- = \lim_{\Delta x \to 0^-} \frac{f(c-\Delta x) - f(c)}{\Delta x} = \lim_{x \to c^-} \frac{f(x) - f(x)}{x - c}$$

$$f'(c)^+ = \lim_{\Delta x \to 0^+} \frac{f(c+\Delta x) - f(c)}{\Delta x} = \lim_{x \to c^+} \frac{f(x) - f(x)}{x - c}$$

…Must agree $f'(c)^- = f'(c)^+$

But as discussed in **September 6 Notes,** the converse is false! (In other words, **if** $f'(c)^- = f'(c)^+$**, it does not follow that** $f'(c) = \lim_{\Delta x \to 0} \frac{f(c+\Delta x) - f(c)}{\Delta x} = \lim_{x \to c} \frac{f(x) - f(x)}{x - c}$ **exists!** In other words, agreement of left and right handed derivatives at c is **not a sufficiency condition** for differentiability (only a necessary condition). To ensure differentiability at c we must (in addition to the condition $f'(c)^- = f'(c)^+$) ensure that f is **continuous** at c also! (Recall discussion on pages 1-4 of **Sept 6 Notes).**

[2] Since its left-hand and right-hand derivatives don't agree there (the graph has a 'kink'):

$$f'(c_4)^- \neq f'(c_4)^+$$

To state the matter more precisely note the definitions below:

- **Defn 1:** ***f*** is monotone increasing on an interval [*a*, *b*] whenever:

  $x_1 \le x_2 \Rightarrow f(x_1) \le f(x_2)$ (note: "⇒" means "implies")

  …For any $x_1, x_2 \in$ [*a*, *b*]

  In addition, we say *f* is strictly monotone increasing whenever:

  $x_1 \le x_2 \Rightarrow f(x_1) < f(x_2)$

- **Defn 2:** ***f*** is monotone decreasing on an interval [*a*, *b*] whenever:

  $x_1 \le x_2 \Rightarrow f(x_1) \ge f(x_2)$

  …For any $x_1, x_2 \in$ [*a*, *b*]. In addition, we say *f* is strictly monotone decreasing whenever: $x_1 \le x_2 \Rightarrow f(x_1) > f(x_2)$

So, based on **Defn1, Defn2** we can say that **critical points *can* disrupt the monotonicity** of a graph! This is of utmost importance, since once we know where the segments of the graph of a function are where the graph just climbs (i.e. is strictly monotone increasing) or just falls (i.e. is strictly monotone decreasing) we easily pick out the **extrema** (maxima or minima) of a graph, without needing to know what *all* its y-values are (i.e. without needed to graph the function).

As discussed in § 4.1, 4.3, the kinds of critical points that change the strict monotonicity of a graph of a function are **local (or relative) extrema.** For example, in the graph above:

- C_1 and C_3 are **local *maxima.***
- C_2 and C_4 are **local *minima.***

…since *f* is strictly monotone increasing to the left of C_1 & C_3 *and* strictly monotone decreasing to the right of C_1 & C_3. Conversely, *f* is strictly monotone decreasing to the left of C_2 & C_4 *and* strictly monotone increasing to the right of C_2 & C_4

These considerations form the basis of locating **local** (or **relative**) extrema (as discussed in detail in § 4.3,) known as the **first derivative test (FDT).** The underlying reasoning of the FDT is stated in **Thm 4.5** (p168, text). If *f* is differentiable on (*a* , *b*) then:

a) *If* $f^{/} > 0$ on (a, b)[3], *then* is **strictly monotone increasing** on (a, b).

b) *If* $f^{/} < 0$ on (a, b), *then* is **strictly monotone decreasing** on (a, b).

c) *If* $f^{/} = 0$ on (a, b), *then* is **constant** on (a, b).

The proof is supplied in the text, and is obvious enough. Adopting the **FDT** amounts to, therefore, **sketching a sign chart for the *derivative* of *f*.**

Recall the remark in page 2 of these notes. Critical points *can* disrupt the monotonicity of a graph. But they don't *have* to. For example, if a graph is strictly monotone increasing on either side of a zero-valued derivative critical point (stated more succinctly according to the FDT: if $f^{/} > 0$ on either side of c where $f^{/}(c) = 0$) then such a point is a **saddle-point** (neither maximum not minimum)

We can *combine* both approaches in sections 4.1 and 4.3 to adopt a full-blown procedure for locating both absolute and relative extrema, which is stated in the table below:

Step	***Procedure***
1	**Calculate** $f(a)$ and f(b) (i.e. determine the y values at the endpoints)
2	**Find all** critical points c (all points c where $f^{/}(c) = 0$ or $f^{/}(c)$ DNE)
3	**Sketch a sign chart for $f^{/}$** to determine which (if any) of the critical points obtained in 2. above are *local* (relative) *maxima or minima.*
4a)	Find the *absolute maxima* **by maximizing over the set of the y-values consisting of *f* 's endpoints as well as *f* 's local maxima**. That is to say: find: max{$f(a)$, f(b) , $f(c)$}, where c is a *local* maximum.
4b)	Find the ***absolute minima* by minimizing over the set of the y-values consisting of *f* 's endpoints as well as *f* 's local minima.** That is to say: find: min{$f(a)$, f(b) , $f(c)$}, where c is a *local* minimum.

The methods of section 4.1, 4.3 are combined below in the examples that follow, adopting the procedure as summarized in the table above

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- Example (#13, Sect. 4.1)

$$f(x) = 3x^{\frac{2}{3}} - 2x \quad \text{on } [-1, 1]$$

Step 1.) $f(-1) = 3(-1)^{\frac{2}{3}} - 2(-1) = 5$

$f(-1) = 3(1)^{\frac{2}{3}} - 2(1) = 1$

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[3] "on (a , b) is shorthand for "all points in (a , b)."

Step 2.)

2.a) Find the derivative

$$f(x) = 3x^{\frac{2}{3}} - 2x \Rightarrow f'(x) = \frac{d}{dx}\left(3x^{\frac{2}{3}} - 2x\right)$$

$$= 3\frac{d}{dx}x^{\frac{2}{3}} - 2\frac{d}{dx}x = 3 \cdot \tfrac{2}{3}x^{-\frac{1}{3}} - 2 = 2x^{-\frac{1}{3}} - 2$$

$$= 2\left(\tfrac{1}{\sqrt[3]{x}} - 1\right)$$

2.b) Find critical points of Type 2 (where f' DNE)

On [-1, 1], at $x_1 = 0$, $f'(0) = 2\left(\tfrac{1}{\sqrt[3]{0}} - 1\right) = \infty$, a vertical tangent

2.c) Find critical points of Type 1 (where $f' = 0$)

$f'(x) = 2\left(\tfrac{1}{\sqrt[3]{x}} - 1\right) = 0 \Rightarrow x_2 = 1$ (at one of *f*'s endpoints)

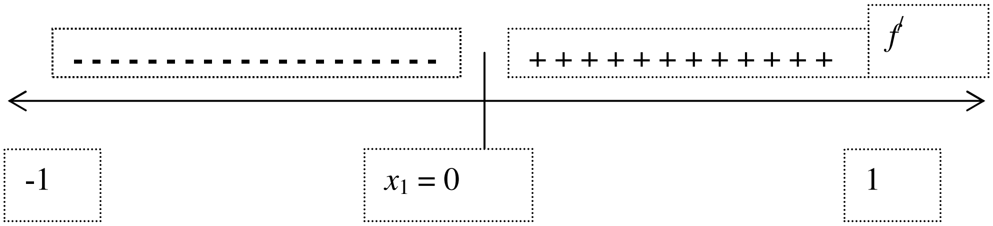
3.) Sketch a sign chart for f'

f' is partitioned by $x_1 = 0$ and $x_2 = 1$, so pick an easy number in the two regions $(-1,0)$ and $(0,1)$ to evaluate the sign of f'. For example, $\pm {}^1/_2$:

$$f'\left(-\tfrac{1}{2}\right) = 2\left(\frac{1}{\sqrt[3]{-\frac{1}{2}}} - 1\right) = 2\left(-\sqrt[3]{2} - 1\right) < 0$$

$$f'\left(\tfrac{1}{2}\right) = 2\left(\frac{1}{\sqrt[3]{\frac{1}{2}}} - 1\right) = 2\left(\sqrt[3]{2} - 1\right) > 0$$

So drawing the chart:



Hence *f* is **strictly monotone decreasing** on [-1, 0) and **strictly monotone increasing** on (0, 1]. So $x_1 = 0$ is a **local minimum.**

4.a) Find the absolute maximum:

$\max\{f(-1), f(1)\} = \max\{5,1\} = 5$ …which occurs at the endpoint a = -1.

Recall, however, from step 2.c that b =1 is a **critical point (Type 1)** Is this point a **local maximum**? To find out, we can adopt the FDT and determine the sign of f' at a point just to the right of b =1. For instance:

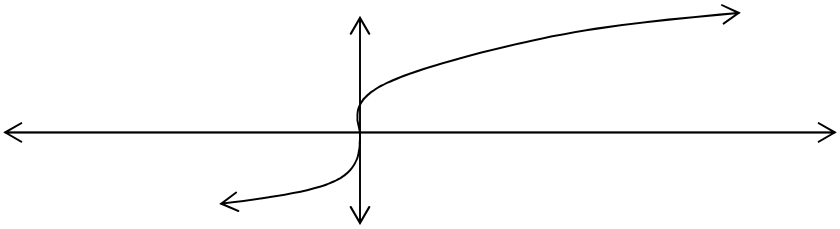
$$f'\left(\tfrac{3}{2}\right) = 2\left(\frac{1}{\sqrt[3]{\frac{3}{2}}} - 1\right) = 2\left(\sqrt[3]{\tfrac{2}{3}} - 1\right) < 0$$

So, yes, it's also a local maximum. However, keep in mind that since $^3/_2$ is not part of the original domain of this function (the function is defined on [-1,1]) the question is academic. However, if the domain were extended beyond x=1 on the right, then this additional information is useful.

4.a) Find the absolute minimum:

$\min\{f(-1), f(0), f(1)\} = \min\{5,0,1\} = -1$ …which occurs at the Type-2 critical point $x_1 = 0$ (which is a **local minimum** also)

Note: For being such a simple-looking function (algebraically speaking) its graph nevertheless has some pretty interesting geometric features. (Note the 'kink' at the origin). Kinks like these occur because one is taking an odd-root (i.e. the n-th root of a number, where n is odd, which is defined everywhere.[4] For example, the graph of $x^{1/3}$ looks like:



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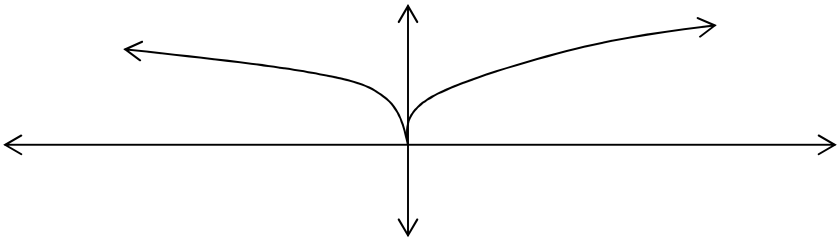
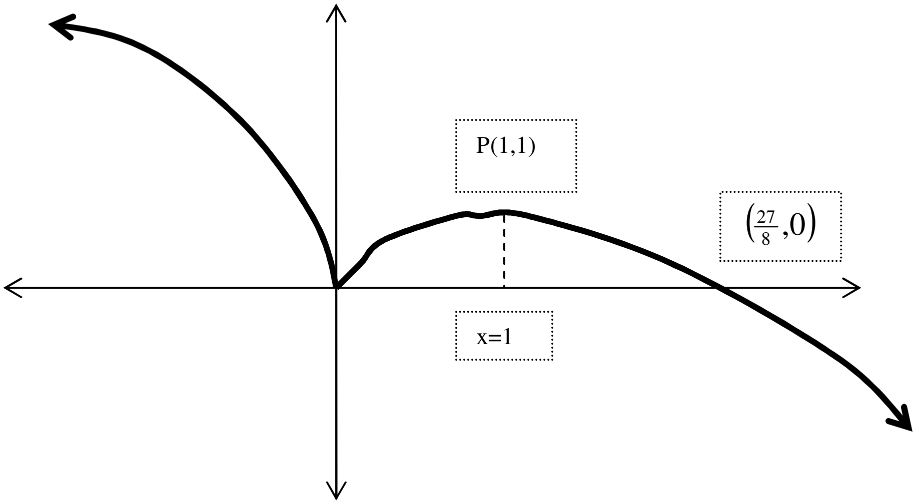
[4] In other words, taking an odd-valued n-th root of a negative number is well-defined and negative. However, of course, this is not true when n is even. Even n-th *real* roots are only well-defined for non-negative real numbers

..So note the ∞ derivative at the origin (a vertical tangent line) Squaring this resul, i.e. producing the graph of $x^{2/3}$:



…which of course yields a 'kink' at the origin. In our case, the graph of the function has the shape (if extended to all real numbers)



…of course, the whole point of these calculus-based methods is *not* to have to draw the graph of a function (and still obtain important information like local and absolute maxima and local minima). Nevertheless, the sketch of the graph is showed above just to illustrate its interesting features. (When extended the domain of the function to include all the real numbers, note that it no longer has an absolute maximum or minimum, but still has local minima and maxima.)

- Example (#16, Sect. 4.1)

$$f(x) = \frac{t^2}{t^2+3} \quad \text{on } [-1, 1]$$

Note: Since this function is *even*, i.e. $f(-t) = f(t)$. it's symmetric about the y-axis. So without further ado, we can cut this domain in half (since the domain [-1, 1] is symmetric about the y – axis) and examine the behavior of the function on [0 , 1] only

Step 1.)

$$f(1) = \frac{1^2}{1^2+3} = \tfrac{1}{4} = f(-1)$$

Step 2.)

Step 2.a) Find the derivative:

$$f(t) = \frac{t^2}{t^2+3} \Rightarrow f'(t) = \frac{d}{dt}\left(\frac{t^2}{t^2+3}\right) = \frac{(t^2+3)2t - t^2(2t)}{(t^2+3)^2}$$

$$= \frac{2t^3+6t-2t^3}{(t^2+3)^2} = 6\frac{t}{(t^2+3)^2}$$

Step 2.b) The above expression is well-defined for all t (no zero denominator expressions nor $^0/_0$ expressions) So there are no Type-2 critical points.

Step 2.c)

$$f'(t) = 6\frac{t}{(t^2+3)^2} = 0 \Rightarrow t_1 = 0$$

Critical point at the origin (Type 1)

Step 3)

Test the sign of f' on the interval (0, 1]:

By inspection, the formula $f'(t) = 6\frac{t}{(t^2+3)^2}$ tells us that *no matter what* point we pick to the **righ**t of 0, $f' > 0$. (The numerator and denominator terms are strictly positive). By the same token, *no matter what* point we pick to the **left** of 0, $f' < 0$. (The numerator term is strictly negative and the denominator term is strictly positive). But based on the remarks concerning the symmetry of *f*, we knew this already! So $t_1 = 0$ is a **local minimum.**

Step 4a)

The absolute maxima *have* to occur at the endpoints $a = -1$, $b = 1$ based on the argument above. The value of the absolute maximum is $^1/_4$

Step 4b)

$\min\{f(0), f(1)\} = \min\{0, \frac{1}{4}\} = 0$, which occurs at the **local minimum** $t_1 = 0$.

- Example (#31, Sect. 4.1)

To find the maximum value of $|f^{(4)}(x)|$, one must first compute $f^{(4)}(x)$:

$$f(x) = 15x^4 - \left(\frac{2x-1}{2}\right)^6 = 15x^4 - \left(x - \tfrac{1}{2}\right)^6$$

$$\therefore\ f'(x) = \frac{d}{dx}\left(15x^4 - \left(x-\tfrac{1}{2}\right)^6\right) = 60x^3 - 6\left(x - \tfrac{1}{2}\right)^5 = 6\left[10x^3 - \left(x-\tfrac{1}{2}\right)^5\right]$$

$$f''(x) = \frac{d}{dx}6\left[10x^3 - \left(x-\tfrac{1}{2}\right)^5\right] = 6\left[30x^2 - 5\left(x-\tfrac{1}{2}\right)^4\right] = 30\left[6x^2 - \left(x-\tfrac{1}{2}\right)^4\right]$$

$$f^{(3)}(x) = \frac{d}{dx}30\left[6x^2 - \left(x-\tfrac{1}{2}\right)^4\right] = 30\left[12x - 4\left(x-\tfrac{1}{2}\right)^3\right] = 120\left[3x - \left(x-\tfrac{1}{2}\right)^3\right]$$

$$f^{(4)}(x) = \frac{d}{dx}120\left[3x - \left(x-\tfrac{1}{2}\right)^3\right] = 120\left[3 - 3\left(x-\tfrac{1}{2}\right)^2\right] = 360\left[1 - \left(x-\tfrac{1}{2}\right)^2\right]$$

Now define: $g(x) = \left|f^{(4)}(x)\right| = 360\left|1 - \left(x-\tfrac{1}{2}\right)^2\right|$

Important: Absolute-value functions can be tricky in calculus, since they introduce 'kinks' or non-differentiable points. What you must do to ensure that you don't accidentally overlook such points, from the use of the formalism alone (i.e. without graphing the function), is to transform all absolute-value expressions in terms of a piecewise continuous function defined as follows:

For any function $u(x)$:

$$|u(x)| = \begin{cases} u(x) & for\_all\_x\_where\_u(x) \ge 0 \\ -u(x) & for\_all\_x\_where\_u(x) < 0 \end{cases}$$

Hence, transforming:

$$g(x) = \left|f^{(4)}(x)\right| = 360\left|1-\left(x-\tfrac{1}{2}\right)^2\right|$$
$$= \begin{cases} 360\left[1-\left(x-\tfrac{1}{2}\right)^2\right] & for\_all\_x\_where:\_\left[1-\left(x-\tfrac{1}{2}\right)^2\right] \geq 0 \\ -360\left[1-\left(x-\tfrac{1}{2}\right)^2\right] & for\_all\_x\_where:\_\left[1-\left(x-\tfrac{1}{2}\right)^2\right] < 0 \end{cases}$$

Now the next step would be simplifying those inequalities, which amounts to solving[5]:

$$\left[1-\left(x-\tfrac{1}{2}\right)^2\right] \geq 0 \Rightarrow -\left(x-\tfrac{1}{2}\right)^2 \geq -1$$
$$\Rightarrow \left(x-\tfrac{1}{2}\right)^2 \leq 1 \Rightarrow -1 \leq \left(x-\tfrac{1}{2}\right) \leq 1 \Rightarrow -\tfrac{1}{2} \leq x \leq \tfrac{3}{2}$$

Hence:

$$g(x) = \begin{cases} 360\left[1-\left(x-\tfrac{1}{2}\right)^2\right] & -\tfrac{1}{2} \leq x \leq \tfrac{3}{2} \\ -360\left[1-\left(x-\tfrac{1}{2}\right)^2\right] & otherwise, i.e.: x < -\tfrac{1}{2}\_OR\_x > \tfrac{3}{2} \end{cases}$$

Reducing the function to the above piecewise-continuous form above is a crucial step! In this case, since the function is defined on the puny interval [0,1] we can immedidately simplify g(x) to:

$$g(x) = 360\left[1-\left(x-\tfrac{1}{2}\right)^2\right] \qquad \text{since obviously}^6\ [0,\ 1] \subseteq \left[-\tfrac{1}{2}, \tfrac{3}{2}\right]$$

However, had the function been defined on some *other* interval like: [-1, 1] we would have *no choice* but to represent $g(x)$ by:

$$g(x) = \begin{cases} 360\left[1-\left(x-\tfrac{1}{2}\right)^2\right] & -\tfrac{1}{2} \leq x \leq 1 \\ -360\left[1-\left(x-\tfrac{1}{2}\right)^2\right] & -1 \leq x < -\tfrac{1}{2} \end{cases}$$

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[5] Note the crucial step: $\left(x-\tfrac{1}{2}\right)^2 \leq 1 \Rightarrow -1 \leq \left(x-\tfrac{1}{2}\right) \leq 1$. This was obtained by the following case-reasoning: $\left(x-\tfrac{1}{2}\right)^2 \leq 1 \Rightarrow \left(x-\tfrac{1}{2}\right) \leq 1$ OR $\left(x-\tfrac{1}{2}\right) \geq -1$. Make sure you're clear on how to solve quadratic inequalities!

[6] Using the subset notation. I.e. the interval [0,1] is contained in the interval $\left[-\tfrac{1}{2}, \tfrac{3}{2}\right]$ where $\left[1-\left(x-\tfrac{1}{2}\right)^2\right] \geq 0$

…since part of g's domain in this case overlaps with the interval $\left(-\infty,-\frac{1}{2}\right)$, which pertains to the case: -360[1 – (x – $^1/_2)^2$] as shown above.

So now we adopt the procedures on $g(x)=360\left[1-\left(x-\frac{1}{2}\right)^2\right]$ as discussed:

Step 1:

$$g(0)=360\left[1-\left(0-\tfrac{1}{2}\right)^2\right]=270$$
$$g(1)=360\left[1-\left(1-\tfrac{1}{2}\right)^2\right]=270$$

Step 2a:

$$g'(x)=360\frac{d}{dx}\left[1-\left(x-\tfrac{1}{2}\right)^2\right]=360\left[-2\left(x-\tfrac{1}{2}\right)\right]=-720\left(x-\tfrac{1}{2}\right)$$

2b: Obviously a well-behaved linear function, so no Type-2 critical points.

2c: $g'(x)=-720\left(x-\frac{1}{2}\right)=0\Rightarrow x_1=\frac{1}{2}$

Step 3: Obviously $g'(x)=-720\left(x-\frac{1}{2}\right)>0$ & $-720\left(x-\frac{1}{2}\right)<0$ for **all** $x<{}^1/_2$ and for **all** $x>{}^1/_2$, respectively. So $x_1={}^1/_2$ is a **local *as well as a global* maximum.**

Step 4a: The **absolute maximum**, based on the remarks above, is $x_1={}^1/_2$, with value: $g(0)=360\left[1-\left(\frac{1}{2}-\frac{1}{2}\right)^2\right]=360$

Step 4b: $\min\{g(0),g(1)\}=270$**,** so the absolute minima occur at the endpoints 0, 1.

- ***LOCATING LOCAL EXTREMA*** (§ 4.4) *VIA THE SECOND-DERIVATIVE TEST (SDT)*

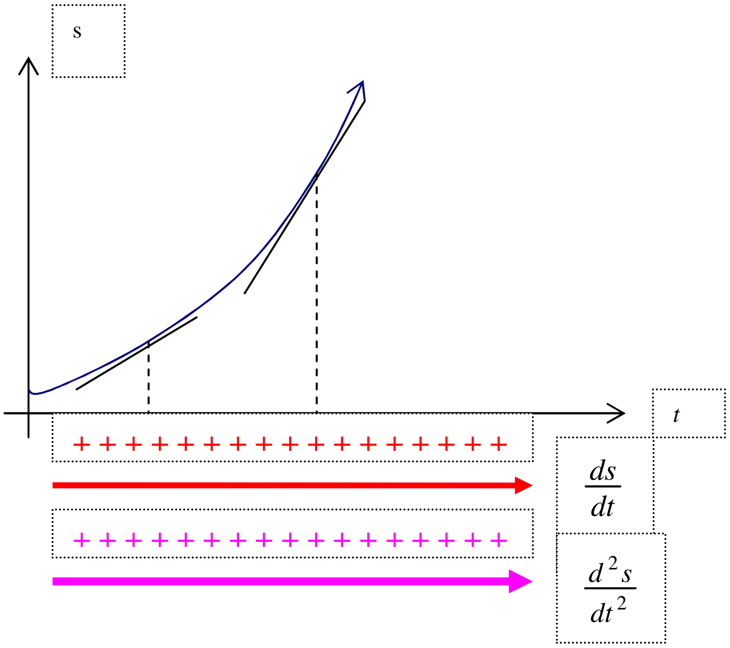
According to the FDT, we gain information concerning the *strict monotonicity* of f by examining the sign of f'. It turns out that f'' gives us more nuanced information as well. For starters, ask yourself: what does f'' actually *mean* ? It's describing the **rate of a rate of change.** Consider, for example, acceleration. Acceleration is the **second time derivative** of displacement. **Velocity** is the **first time derivative of displacement**. Velocity then tells us what the instantaneous rate of change of our *displacement* is, in time. Acceleration, on the other hand, tells what the instantaneous rate of change is of our *velocity*, in time. **Therefore,** our acceleration tells us ***what is the instantaneous rate of change of our instantaneous rate of change of our displacement, with respect to time.***

The velocity example is useful here. For you well know that there's no necessary connection between the *sign* , let alone the *value* of our acceleration and velocity. In this respect, they're independent. For example:

A) Both my acceleration and my velocity could be positive. This occurs when my car is moving *forward* and I'm stepping on the *gas.*
B) Both my acceleration and my velocity could be *negative*. This occurs when my car is moving in *reverse* and I'm hitting the *brakes*.
C) My acceleration could be positive and my velocity could be negative. This occurs when my car is moving in *reverse* and I'm stepping on the *gas*.
D) My acceleration could be negative and my velocity could be positive. This occurs when my car is moving *forward* and I'm hitting the *brakes*.

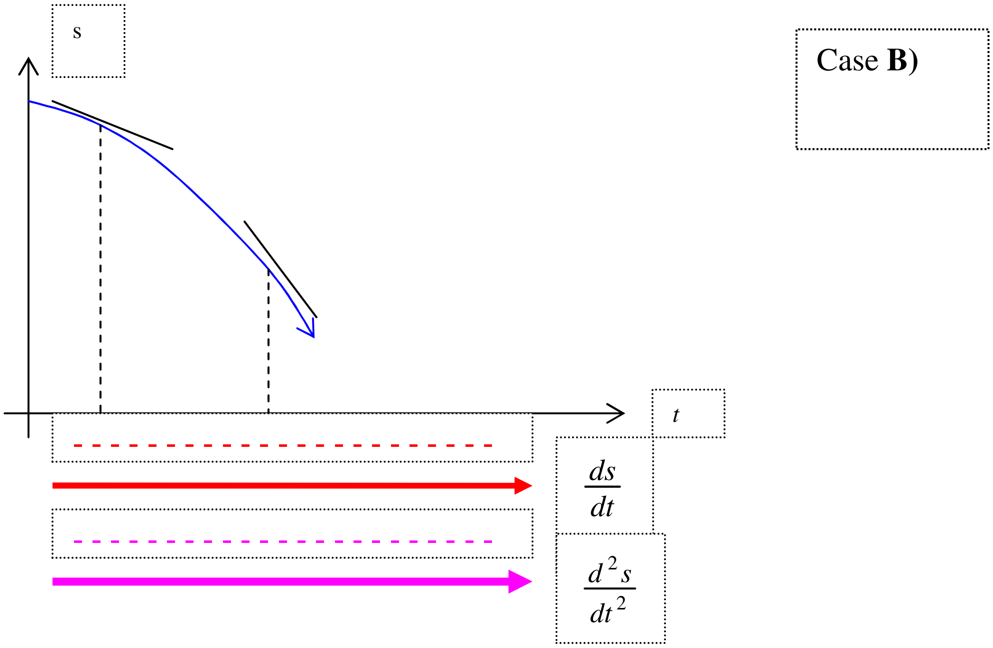
Now consider the above cases graphically. The graphs qualitatively represent *displacement* (s) versus time (t).

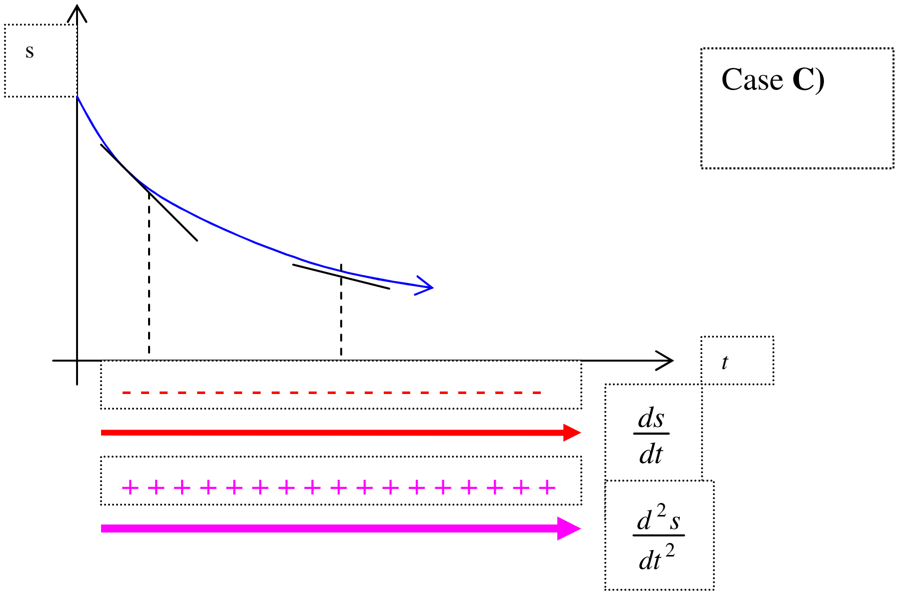
Case **A)**

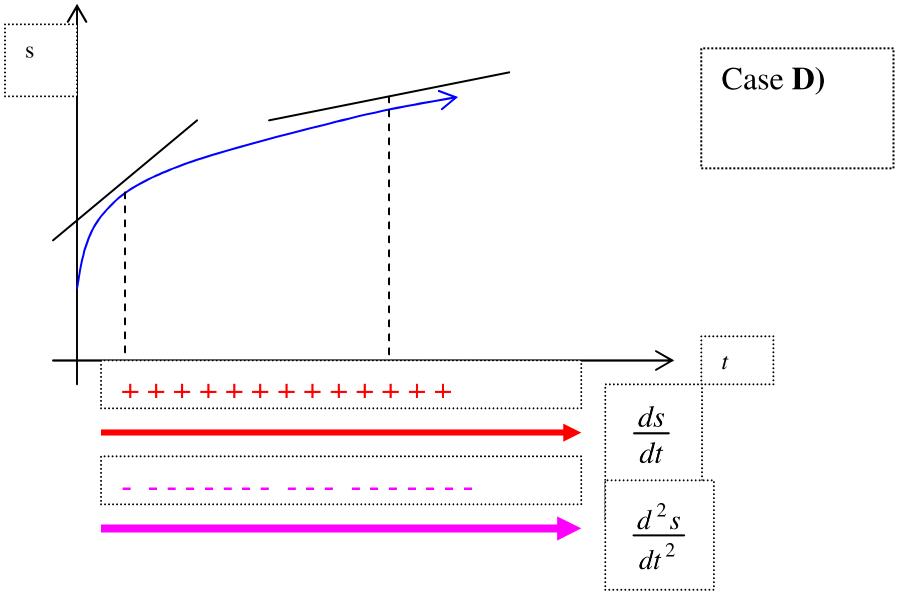


Sign charts are sketched below for the derivative of displacement (velocity) (sketched in red) as well as the second derivative (acceleration) (sketched in violet). Since the slope of the tangent lines of s's graph is always increasing, hence the sign of the second derivative is positive (my rate of increase of displacement is also increasing.) Now observe the shape of the graph here, as well. It seems to be opening with the concave side **up.** We say its **curvature is concave up.** In general, the second derivative gives us information concerning curvature. ***Positive* second derivative means concave *up* curvature. Negative second derivative means concave *down* curvature. *Zero* second derivative means *zero* curvature (neither concave up nor down)**

Consider now the graphs of the rest of the cases:







These illustrations form the intuitive basis of the *Second Derivative Test* (SDT) for testing the existence of local maxima and minima. Since we'll be discussing this in greater detail Sept 20, I'll be brief here. Informally the test states:

- Suppose c is a **Type-1** critical point. Then:
  a) c is a **local maximum** if $f''(c) < 0$ (curvature is concave down at c)
  b) c is a **local minimum** if $f''(c) > 0$ (curvature is concave up at c)
  c) **Test fails** if $f''(c) = 0$ (other tests like the FDT need to be invoked to determine if c is a saddle point (special case of an inflection point, a point where the curvature changes) or otherwise.

Warning: It's *very important* to understand the *logic* of the SDT! Especially in case c). Do not *conclude* you're dealing with a saddle-point just because $f''(c) = 0$ & $f'(c) = 0$. Literally, the test <u>fails</u> in this case (it's inconclusive). For example, consider the function: $f(x) = x^4$. Now, at the point $c = 0$:

$$f'(x) = 4x^3 \Rightarrow f'(0) = 0$$
$$f''(x) = 12x^2 \Rightarrow f''(0) = 0$$

But it's pretty obvious that $c = 0$ is *not* a saddle-point. In fact, $c = 0$ is an *absolute minimum*. You can determine that by using the FDT.

So here's how to think of the SDT, in reference to the FDT procedures discussed above: When you get an inconclusive case (like c)) then resort to the FDT. In short, though the SDT is more elegant (less labor-intensive like the FDT) it's also not as *general*. It only applies to Type-1 critical points, and *even then*, it can fail. On the other hand, the FDT is more laborious, but also more general. It can handle Type1 *and* Type2 critical points, and it's *never* inconclusive.

So, given the above mentioned remarks concerning the limitations of the SDT, how can we identify points of inflection? (Where the curvature changes).

Answer: It's not enough to just find all points c, such that $f^{\prime\prime}(c) = 0$, **if $f^{\prime}(c) = 0$ as well!** In that case, you must additionally test the *sign* of $f^{\prime\prime}(x)$ to the left and right of c.

- Example (#33)**:** Identify all extrema and points of inflection for the function:

$$f(x) = \frac{x-2}{x^2-4x+3}$$

Step 1: Calculate first and second derivatives:

$$f'(x) = \frac{d}{dx}\left(\frac{x-2}{x^2-4x+3}\right) = \frac{(x^2-4x+3)-(x-2)(2x-4)}{(x^2-4x+3)^2} = \frac{(x^2-4x+3)-(2x^2-8x+8)}{(x^2-4x+3)^2}$$
$$= \frac{-x^2+4x-5}{(x^2-4x+3)^2} = \frac{-(x^2-4x+5)}{(x^2-4x+3)^2}$$

Note: **An alternative to using the quotient rule compute the second derivative involves the following trick[7]:**

$$f'(x) = \frac{-(x^2-4x+5)}{(x^2-4x+3)^2} = \frac{-(x^2-4x+3+2)}{(x^2-4x+3)^2} = \frac{-(x^2-4x+3)}{(x^2-4x+3)^2} - \frac{2}{(x^2-4x+3)^2}$$
$$= -(x^2-4x+3)^{-1} - 2(x^2-4x+3)^{-2}$$

Then: $f''(x) = -\frac{d}{dx}\left[(x^2-4x+3)^{-1} + 2(x^2-4x+3)^{-2}\right]$

---

[7] Or in general for an expression like: $\dfrac{-(x^2+ax+b)}{(x^2+cx+d)^2}$, express a and b in terms of c and d, i.e. $a = -c + k$, $b = -d + m$, then: $\dfrac{-(x^2+ax+b)}{(x^2+cx+d)^2} = \dfrac{-(x^2+(-c+k)x+(-d+m))}{(x^2+cx+d)^2}$

$$= -\left[-\left(x^2-4x+3\right)^{-2}(2x-4) - 4\left(x^2-4x+3\right)^{-3}(2x-4)\right] = (2x-4)\left[\left(x^2-4x+3\right)^{-2} + 4\left(x^2-4x+3\right)^{-3}\right]$$

$$= 2(x-2)\left(x^2-4x+3\right)^{-2}\left[1+4\left(x^2-4x+3\right)^{-1}\right] = 2(x-2)\left(x^2-4x+3\right)^{-2}\left[\frac{x^2-4x+3}{x^2-4x+3}+\frac{4}{x^2-4x+3}\right]$$

$$= 2(x-2)\left(x^2-4x+3\right)^{-2}\left[\frac{x^2-4x+7}{\left(x^2-4x+3\right)}\right] = \frac{2(x-2)\left(x^2-4x+7\right)}{\left(x^2-4x+3\right)^3}$$

Note that we arrive at the same answer when applying the Quotient Rule:

$$f''(x) = \frac{d}{dx}\frac{-\left(x^2-4x+5\right)}{\left(x^2-4x+3\right)^2} = -\left\{\frac{\left(x^2-4x+3\right)^2(2x-4) - \left(x^2-4x+5\right)2\left(x^2-4x+3\right)(2x-4)}{\left(x^2-4x+3\right)^4}\right\}$$

$$= -\left\{\frac{(2x-4)\left(x^2-4x+3\right)\left[\left(x^2-4x+3\right) - 2\left(x^2-4x+5\right)\right]}{\left(x^2-4x+3\right)^4}\right\} = -\frac{(2x-4)\left(-x^2+4x-7\right)}{\left(x^2-4x+3\right)^3}$$

$$= \frac{2(x-2)\left(x^2-4x+7\right)}{\left(x^2-4x+3\right)^3}$$

Step 2a: To locate extrema, find all Type-2 critical points:

$$f'(x) = \frac{-\left(x^2-4x+5\right)}{\left(x^2-4x+3\right)^2} = \frac{-x^2+4x-5}{[(x-3)(x-1)]}$$

When $x = 3$: $f'(3) = \frac{-3^2+12-5}{[0\cdot(3-1)]} = \frac{-2}{0} = \infty$

When $x = 1$: $f'(1) = \frac{-1^2+12-5}{[0]} = \frac{6}{0} = \infty$

So there are two Type-2 critical points: $x_1 = 3$, $x_1 = 1$. (Note that these occur at the vertical *asymptotes* of f, for obvious reasons)

Step 2b: To locate extrema, find all Type-1 critical points:

$$f'(x) = \frac{-\left(x^2-4x+5\right)}{\left(x^2-4x+3\right)^2} = \frac{-x^2+4x-5}{[(x-3)(x-1)]} = 0 \Rightarrow x^2-4x+5 = 0$$

There are none, since this is an irreducible quadratic. (Its discriminant: $b^2-4ac = 16-20 < 0$)

Step 3: To locate inflection points solve:

$$f''(x) = \frac{2(x-2)(x^2-4x+7)}{(x^2-4x+3)^3} = 0 \Rightarrow 2(x-2)(x^2-4x+7) = 0 \Rightarrow x_3 = 2$$

Note that $x^2 - 4x + 7$ is a quadratic irreducible: Its discriminant: $b^2 - 4ac = 16 - 28 < 0$

So the graph has two Type 2 critical points (located, not surprisingly, at its vertical asymptotes) as well on inflection point (*x* = 2), which is also where it crosses the *x*-axis.

If we wanted to sketch this graph, we have some valuable information now. Using techniques from chapters 2, and 3, recall:

- **Locating the horizontal asymptotes:**

  $$\lim_{x\to\pm\infty} f(x) = \lim_{x\to\pm\infty} \frac{x-2}{x^2-4x+3} = \lim_{x\to\pm\infty} \frac{\frac{1}{x} - \frac{2}{x^2}}{1 - \frac{4}{x} + \frac{3}{x^2}} = \frac{0}{1} = 0$$

  In other words, the *x*-axis (*y* = 0) is the horizontal asymptote.

- **To test the curvature, since there is only *one* inflection point (*x* = 2) pick some easy point like *x* =0 and x=3 (to the left and right of 2) to exmine the sign of *f*$^{//}$:**

$$f''(0) = \frac{2(0-2)(0^2-4\cdot 0+7)}{(0^2-4\cdot 0+3)^3} = \tfrac{-28}{27} < 0 \qquad \text{Concave down}$$

$$f''(4) = \frac{2(4-2)(4^2-4\cdot 4+7)}{(4^2-4\cdot 4+3)^3} > 0 \qquad \text{Concave up}$$

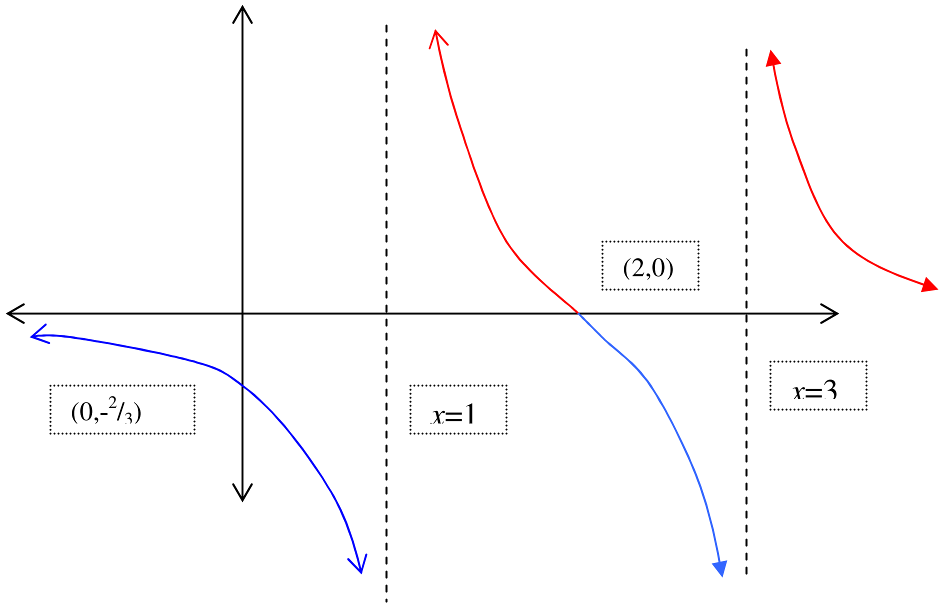
- **To test the behavior of *f* at its vertical asymptotes, we must evalue the left and right hand limits there:**

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} \frac{x-2}{x^2-4x+3} = \lim_{x\to 1^-} \frac{x-2}{(x-3)(x-1)} = -\infty$$

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} \frac{x-2}{x^2-4x+3} = \lim_{x\to 1^+} \frac{x-2}{(x-3)(x-1)} = \infty$$

$$\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} \frac{x-2}{x^2-4x+3} = \lim_{x\to 3^-} \frac{x-2}{(x-3)(x-1)} = -\infty$$
$$\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+} \frac{x-2}{x^2-4x+3} = \lim_{x\to 3^+} \frac{x-2}{(x-3)(x-1)} = \infty$$

Armed with this information:



(The blue segments indicate concave down curvature, the red segments indicate concave up curvature)

- ***ROLLE'S THEOREM AND THE MEAN-VALUE THEOREM***

As I mentioned Tuesday, (September 18), this section of the text deals with some important theory we'll visit later when doing integration. Here I state the two theorems without proving them (I proved them in class) and focus on a few applications:

- **Rolle's Theorem**: If f is differentiable on $[a\ ,b]$ and $f(a) = f(b)$ then there exists (at least one) point $c \in (a, b)$ such that: $f'(c) = 0$

Corollary (p163 text). The differentiability condition can be relaxed to continuity alone. Then Rolle's theorem reads:

- **Corollary Rolle's Theorem**: If f is continuous on $[a , b]$ and $f(a) = f(b)$ then there exists (at least one) point $c \in (a, b)$ such that c is a **critical point (either Type 2 or Type 1)**.

- Exercise 26 (pg 167) asks for a proof of the Corrolary. Here it is:

Proof (by contradiction) Suppose : If f is continuous on $[a , b]$ and $f(a) = f(b)$ and there exists **no** point $c \in (a, b)$ such that c is a **critical point (either Type 2 or Type 1)**. As discussed according to the FDT, if there are no critical points (of either type) on [a, b], then f is strictly monotone increasing or strictly monotone decreasing on $[a, b]$, But then $f(a) \neq f(b)$ unless f were discontinuous on $[a , b]$. But f is continuous on $[\boldsymbol{a}, b]$. **Contradiction**. Therefore, there must exist at least one critical point (of either type) on $[a , b]$

- **Mean Value Theorem (MVT)**

  If f is differentiable on $[a , b]$ and $f(a) = f(b)$ then there exists (at least one) point $c \in [a, b]$ such that: $f'(c) = \dfrac{f(b) - f(a)}{b - a}$

Note1: What the MVT is basically saying is that as long as f is differentiable (therefore continuous[8] there will be point(s) such that the instantaneous rate of change at those points will equal the average rate of change for the function at that interval. For example, suppose I drive to work. Suppose for the trip that my average overall speed was 28mph (lots of traffic!) The MVT states that there will be at least *one* instance when my speedometer will read that average value, i.e. there will be at least one instance when my momentary speed = 28 mph.

Note2: Like a lot of important and central theorems in math, the MVT is an *existence* theorem. These existence theorems are very important because they tell us, for example, that a solution is worth looking for (because at least one exists, so to speak) in a given problem-context. Moreover, note that on eh one hand the MVT states something which *seems* obvious, but on further reflection, turns out to be quite subtle. Many important theorems in math are like that!

- **Example (problem 11, section 4.2)**

Can Rolle's Theorem be applied to:

$f(x) = \dfrac{x^2 - 2x - 3}{x + 2}$ on [-1 , 3]

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[8] Recall from chapter 3

Answer: Yes! Note that a singularity occurs at x = -2, but that's outside the domain.

Moreover: $f(-1) = \dfrac{-1^2 - 2\cdot -1 - 3}{-1+2} = 0 = \dfrac{3^2 - 2\cdot 3 - 3}{3+2} = f(3)$

Examining the derivative[9]:

$$f(x) = \frac{x^2 - 2x - 3}{x+2} = x - \frac{4x+3}{x+2} \Rightarrow f'(x) = 1 - \frac{(x+2)4 - (4x+3)}{(x+2)^2} = 1 - \frac{5}{(x+2)^2}$$
$$= \frac{(x+2)^2 - 5}{(x+2)^2} = \frac{x^2 + 4x - 1}{(x+2)^2}$$

…we see it's everywhere well-behaved in the interval [-1, 3]

To show that such a point exists, (i.e. to verify Rolle's Theorem) set the derivative =0 and solve:

$$f'(x) = \frac{x^2 + 4x - 1}{(x+2)^2} = 0 \Rightarrow x^2 + 4x - 1 = 0$$
$$\Rightarrow x_{1,2} = \frac{-4 \pm \sqrt{16+4}}{2} = \frac{-4 \pm \sqrt{20}}{2} = \tfrac{1}{2}\left(-4 \pm 2\sqrt{5}\right) = -2 \pm \sqrt{5}$$

Certainly $x_1 = -2 + \sqrt{5}$ lies in the interval [-1, 3]

- **<u>Example (problem 20, section 4.2)</u>**

  Apply the MVT to $f(x) = x^3 - 2x$ on [0 , 2]

  Note that f is a simple polynomial, so it's everywhere differentiable.

  Calculating: $\dfrac{f(2) - f(0)}{2-0} = \dfrac{4-0}{2-0} = 2$

  We seek point(s) $c \in [0, 2]$ such that $f'(c) = \dfrac{f(2) - f(0)}{2-0} = 2$

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[9] Recall trick discussed last week: Use polynomial long division to simplify a rational expression whenever the degree of the numerator > degree of the denominator.

Calculating: $f'(x) = \dfrac{d}{dc}(x^3 - 2x) = 3x^2 - 2$

We therefore seek $c \in (0, 2)$ such that $f'(c) = 3c^2 - 2 = \dfrac{f(2) - f(0)}{2 - 0} = 2$

Solving: $3c^2 - 2 = 2 \Rightarrow c = 0$. (Which occurs at one of the endpoints)

- **Example (30, Section 4.2)**

Use the MVT to show that for any quadratic polynomial, $p(x) = Ax^2 + Bx + C$ defined on [a, b] the point c satisfying the condition of the MVT lies at the midpoint of the interval.

- The midpoint of [a, b] is: $c = \dfrac{a+b}{2} = \tfrac{1}{2}(a+b)$

- According to the MVT, there exits c such that:

$p'(c) = \dfrac{p(b) - p(a)}{b - a}$, which in this case means:

$$p'(c) = \frac{p(b) - p(a)}{b - a} \Rightarrow 2Ac + B = \frac{\left(Ab^2 + Bb + C\right) - \left(Aa^2 + Ba + C\right)}{b - a}$$

$$\Rightarrow 2Ac + B = \frac{A\left(b^2 - a^2\right) + B(b - a)}{b - a}$$

$$\Rightarrow 2Ac + B = \frac{A(b - a)(b + a)}{b - a} + \frac{B(b - a)}{b - a}$$

$$\Rightarrow 2Ac + B = A(b + a) + B$$

$$\Rightarrow 2Ac = A(b + a)$$

$$\Rightarrow c = \frac{b + a}{2} = \tfrac{1}{2}(a + b)$$

Hence we've proved the claim.