MATH 261   Notes for September 11 & 13   Kullfelz, Wm

I) Advanced Differentiation Rules

A) Product Rule $\frac{d}{dx}(f(x)g(x)) = f'(x)g(x) + f(x)g'(x)$

Proof: $\frac{d}{dx}(f(x)g(x)) = \lim_{\Delta x \to 0} \frac{1}{\Delta x}\left[f(x+\Delta x)g(x+\Delta x) - f(x)g(x)\right]$

$$= \lim_{\Delta x \to 0} \frac{1}{\Delta x}\Big[f(x+\Delta x)g(x+\Delta x) + \underbrace{f(x+\Delta x)g(x) - f(x+\Delta x)g(x)}_{\text{TRICK OF "0"}} - f(x)g(x)\Big]$$

$$= \lim_{\Delta x \to 0} \frac{1}{\Delta x}\left[f(x+\Delta x)g(x+\Delta x) - f(x+\Delta x)g(x) + f(x+\Delta x)g(x) - f(x)g(x)\right]$$

$$= \lim_{\Delta x \to 0} \frac{1}{\Delta x}\left\{f(x+\Delta x)\left[g(x+\Delta x) - g(x)\right]\right\} + \lim_{\Delta x \to 0} \frac{1}{\Delta x}\left\{\left[f(x+\Delta x) - f(x)\right]g(x)\right\}$$

$$= \lim_{\Delta x \to 0} f(x+\Delta x) \lim_{\Delta x \to 0}\left[\frac{g(x+\Delta x) - g(x)}{\Delta x}\right] + \lim_{\Delta x \to 0}\left[\frac{f(x+\Delta x) - f(x)}{\Delta x}\right] g(x)$$

$$= f(x)g'(x) + f'(x)g(x)$$

B) Quotient Rule: $\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}$

Proof: $\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \lim_{\Delta x \to 0} \frac{1}{\Delta x}\left[\frac{f(x+\Delta x)}{g(x+\Delta x)} - \frac{f(x)}{g(x)}\right]$

$$= \lim_{\Delta x \to 0} \frac{1}{\Delta x}\left[\frac{f(x+\Delta x)g(x) - g(x+\Delta x)f(x)}{g(x+\Delta x)g(x)}\right]$$

$$= \lim_{\Delta x \to 0} \frac{1}{\Delta x}\left[\frac{f(x+\Delta x)g(x) - g(x+\Delta x)f(x) + \overbrace{f(x)g(x) - f(x)g(x)}^{\text{"Trick of 0"}}}{g(x+\Delta x)g(x)}\right]$$

$$= \lim_{\Delta x \to 0} \frac{1}{\Delta x}\left[\frac{f(x+\Delta x)g(x) - f(x)g(x) - g(x+\Delta x)f(x) + f(x)g(x)}{g(x+\Delta x)g(x)}\right]$$

$$= \lim_{\Delta x \to 0} \frac{1}{g(x+\Delta x)g(x)}\left\{\lim_{\Delta x \to 0}\left[\frac{f(x+\Delta x) - f(x)}{\Delta x}\right]g(x) - \lim_{\Delta x \to 0}\left[\frac{g(x+\Delta x) - g(x)}{\Delta x}\right]f(x)\right\}$$

$$= \frac{1}{[g(x)]^2}\left\{ f'(x)g(x) - g'(x)f(x) \right\} = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}.$$

c) $\boxed{\text{CHAIN RULE}: \quad \frac{d}{dx}\left(f(g(x))\right) = f'(g(x))g'(x) = \frac{df}{dg}\frac{dg}{dx}}$

PROOF: $\frac{d}{dx}\left(f(g(x))\right) = \lim_{\Delta x \to 0}\left[\frac{f(g(x+\Delta x)) - f(g(x))}{\Delta x}\right]$

$$= \lim_{\Delta x \to 0}\left[\frac{f(g(x+\Delta x)) - f(g(x))}{\Delta x} \cdot \underbrace{\frac{g(x+\Delta x) - g(x)}{g(x+\Delta x) - g(x)}}_{\text{"TRICK OF 1"}}\right]$$

$$= \lim_{\Delta x \to 0}\left[\frac{f(g(x+\Delta x)) - f(g(x))}{g(x+\Delta x) - g(x)} \cdot \frac{g(x+\Delta x) - g(x)}{\Delta x}\right]$$

Note 1: As $\Delta x \to 0$, $g(x+\Delta x) \to g(x)$, so $g(x+\Delta x) - g(x) \to 0$

$$= \lim_{g(x+\Delta x) \to g(x)}\left[\frac{f(g(x+\Delta x)) - f(g(x))}{g(x+\Delta x) - g(x)}\right] \cdot \lim_{\Delta x \to 0}\frac{g(x+\Delta x) - g(x)}{\Delta x}$$

$$= f'(g) \cdot g'(x)$$

Note how $\lim_{g(x+\Delta x) \to g(x)} \frac{f(g(x+\Delta x)) - f(g(x))}{g(x+\Delta x) - g(x)}$ is identical in form to the second definition of derivative: $f'(c) = \lim_{x \to c}\frac{f(x) - f(c)}{x - c}$

(To see this, simply label "x" $\mapsto g(x+\Delta x)$
"c" $\mapsto g(x)$)

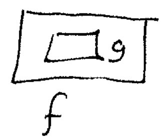
$\boxed{\text{D COROLLARY}}$ $\frac{d}{dx}(u(x))^n = n[u(x)]^{n-1}u'(x)$ for any differentiable function $u(x)$. PROOF: $f(x) = x^n \therefore f(u(x)) = [u(x)]^n$

$\therefore \quad \frac{d}{dx} f(u(x)) = f'(u)\,u'(x)$ according to the Chain Rule

$$f'(u) = \frac{d}{du}[u]^n = n[u(x)]^{n-1}$$

<u>NOTE 2</u>: The meaning of $f'(g) = \frac{df}{dy}$ is identical to the meaning of $f'(x) = \frac{df}{dx}$, <u>provided you replace the independent variable</u> x <u>with the "variable"</u> g, <u>and suppress the explicit x-dependency</u>.

- For example: If $f(x) = 10x^{2/3} - 5x^2$
  and $g(x) = x^4 + x^{-2}$



<u>then</u>: $f(g(x)) = 10(g(x))^{2/3} - 5(g(x))^2$

<u>So</u>: $$f'(g) = \frac{d}{dg}\left[10g^{2/3} - 5g^2\right] = \underbrace{\frac{20}{3}g^{-1/3} - 10g^1}_{\text{"opening large package first"}}$$

$\therefore$ According To Chain Rule:

$$\frac{d}{dx}(f(g(x))) = \frac{df}{dg}\frac{dg}{dx} = \left[\frac{20}{3}g^{-1/3} - 10g\right] \cdot \frac{d}{dx}\left[x^4 + x^{-2}\right]$$

$$= \left[\frac{20}{3}g^{-1/3} - 10g\right]\left[4x^3 - 2x^{-3}\right]$$

... or explicitly in terms of x:

$$\frac{d}{dx}(f(g(x))) = \underbrace{\left[\frac{20}{3}(x^4 + x^{-2})^{-1/3} - 10(x^4 + x^{-2})\right]}_{\text{Recall remark (Tuesday's lecture, 9/11/07)}}\left[4x^3 - 2x^{-3}\right]$$

Examples :

- P127, #7 $\qquad f(x) = (x^5 - 3x)(1/x^2)$

This can be reduced to : $\frac{x^5}{x^2} - \frac{3x}{x^2} = x^{-3} - 3x^{-1}$ and treated according to the more elementary rules :

$$\frac{d}{dx} f(x) = \frac{d}{dx}\left(x^{+3} - 3x^{-1}\right) = +3x^{+2} + 3x^{-2} = \cancel{3x^{-2}(x^{-2}+1)} = 3(x^2 + x^{-2})$$

Moral : Whenever possible, try to reduce to simpler form, to save on work!

To illustrate the above, suppose I tried instead:

Method 2 : (Quotient Rule)

$$\frac{d}{dx}\left(\frac{x^5 - 3x}{x^2}\right) = \frac{x^2(5x^4 - 3) - (x^5 - 3x)2x}{x^4}$$

$$= \frac{5x^6 - 3x^2 - 2x^6 + 6x^2}{x^4} = \frac{3x^6 + 3x^2}{x^4} = 3x^2 + 3x^{-2} = 3(x^2 + x^{-2})$$

(More cumbersome!)

Method 3 (Product Rule + ~~Chain Rule~~)

$$\frac{d}{dx}\left[(x^5 - 3x)(1/x^2)\right] = \left[\frac{d}{dx}(x^5 - 3x)\right]\frac{1}{x^2} + (x^5 - 3x)\frac{d}{dx}x^{-2}$$

$$= (5x^4 - 3)\left(\frac{1}{x^2}\right) + (x^5 - 3x)(-2x^{-3})$$

$$= \frac{5x^4 - 3}{x^2} - \frac{2(x^5 - 3x)}{x^3} = 5x^2 - 3x^{-2} - 2x^2 + 6x^{-2}$$

$$= 3x^2 + 3x^{-2} = 3(x^2 + x^{-2})$$

(More cumbersome still!)

- P127, #10

$$f(x) = \frac{x^3 + 3x - 2}{x^2 - 1}$$

HINT : Always Reduce whenever degree (numerator) > degree (denominator)

THIS WILL SAVE YOU PLENTY OF WORK!

$$\begin{array}{r} x \\ (x^2-1)\,\overline{)\,x^3+3x-2} \\ -(x^3-x) \\ \hline 4x-2 \end{array} \quad = \quad x + \frac{4x-2}{x^2-1} \quad = \quad x + \frac{2(2x-1)}{(x^2-1)}$$

$$f'(x) = \frac{d}{dx}\left\{x + \frac{2(2x-1)}{(x^2-1)}\right\} = 1 + 2\frac{d}{dx}\left\{\frac{2x-1}{x^2-1}\right\} = 1 + 2\left\{\frac{(x^2-1)2-(2x-1)(2x)}{(x^2-1)^2}\right\}$$

$$= 1 + 2\left\{\frac{2x^2-2-4x^2+2x}{(x^2-1)^2}\right\} = 1+2\left\{\frac{-2x^2+2x-2}{(x^2-1)^2}\right\} = 1+4\left\{\frac{-x^2+x-1}{(x^2-1)^2}\right\}$$

This answer can be recombined into one fraction:

$$\frac{(x^2-1)^2}{(x^2-1)^2} + \frac{-4x^2+4x-4}{(x^2-1)^2} = \frac{x^4-2x^2+1-4x^2+4x-4}{(x^2-1)^2}$$

$$= \frac{-3x^4-6x^2+4x-3}{(x^2-1)^2}$$

Suppose, on the other hand, you didn't reduce the fraction first:

$$\frac{d}{dx}\left\{\frac{x^3+3x-2}{x^2-1}\right\} = \frac{(x^2-1)(3x^2+3)-(x^3+3x-2)(2x)}{(x^2-1)^2}$$

$$= \frac{3x^4-3x^2+3x^2-3-6x^4-6x^2+4x}{(x^2-1)^2} = \frac{-3x^4-6x^2+4x-3}{(x^2-1)^2}$$

- Exercise 15, P127

$$h(t) = \frac{t+1}{t^2+2t+2}$$

Use quotient rule only (degree (numerator) < degree (denom.))

$$h'(t) = \frac{d}{dt}\left\{\frac{t+1}{t^2+2t+2}\right\} = \frac{(t^2+2t+2)-(t+1)(2t+2)}{(t^2+2t+2)^2}$$

$$= \frac{t^2+2t+2-2t^2-4t-2}{(t^2+2t+2)^2} = \frac{-t^2-2t}{(t^2+2t+2)^2} = \frac{-t(t+2)}{(t^2+2t+2)^2}$$

- Exercise 18, p127

$$f(x) = \left(\frac{x^2 - x - 3}{x^2 + 1}\right)(x^2 + x + 1)$$

Use product rule & quotient rule :

$$f'(x) = \left[\frac{d}{dx}\left(\frac{x^2 - x - 3}{x^2 + 1}\right)\right](x^2 + x + 1) + \left(\frac{x^2 - x - 3}{x^2 + 1}\right)\frac{d}{dx}\left[x^2 + x + 1\right]$$

$$= \left[\frac{(x^2+1)(2x-1) - (x^2 - x - 3)(2x)}{(x^2+1)^2}\right](x^2 + x + 1) + \left(\frac{x^2 - x - 3}{x^2 + 1}\right)(2x + 1)$$

$$= \left[\frac{(2x^3 - x^2 + 2x - 1) - (2x^3 - 2x^2 - 6x)}{(x^2+1)^2}\right](x^2 + x + 1) + \left(\frac{(x^2 - x - 3)(2x + 1)}{(x^2+1)}\right)$$

$$= \left[\frac{x^2 + 4x - 1}{(x^2+1)^2}\right](x^2 + x + 1) + \left(\frac{2x^3 - 2x^2 - 6x + x^2 - x - 3}{(x^2 + 1)}\right)$$

$$= \frac{(x^2 + 4x - 1)(x^2 + x + 1)}{(x^2+1)^2} + \frac{(2x^3 - x^2 - 7x - 3)}{(x^2 + 1)}$$

$$= \frac{x^4 + 4x^3 - x^2 + x^3 + 4x^2 - x + x^2 + 4x - 1}{(x^2+1)^2} + \frac{(2x^3 - x^2 - 7x - 3)}{(x^2 + 1)}$$

$$= \frac{x^4 + 5x^3 + 4x^2 + 4x - 1}{(x^2+1)^2} + \frac{(2x^3 - x^2 - 7x - 3)}{(x^2 + 1)}$$

$$= \frac{x^4 + 5x^3 + 4x^2 + 4x - 1 + (2x^3 - x^2 - 7x - 3)(x^2 + 1)}{(x^2+1)^2}$$

$$= \frac{\cancel{x^4} + \cancel{5x^3} + 4x^2 + 4x - 1 + 2x^5 - \cancel{x^4} - \cancel{7x^3} - 3x^2 + \cancel{2x^3} - x^2 - 7x - 3}{(x^2+1)^2}$$

$$= \frac{2x^5 + 3x^2 - 3x - 4}{(x^2+1)^2}$$

Note, however, you could have reduced the first term:

$$\begin{array}{r|l} & \;1 \\ x^2+1 & \overline{x^2 - x - 3} \\ & \underline{x^2 \quad\;\; + 1} \\ & \quad -x - 4 \end{array} \implies \frac{x^2 - x - 3}{x^2+1} = 1 - \frac{(4+x)}{x^2+1}$$

- *Exercise 33, p 135*

$$y = \frac{x}{\sqrt{x^2+1}}$$

Method 1 (Quotient & Chain Rule)

$$\frac{d}{dx}\left\{\frac{x}{\sqrt{x^2+1}}\right\} = \frac{\sqrt{x^2+1} - x\,\frac{1}{2}(x^2+1)^{-1/2}(2x)}{x^2+1}$$

$$= \frac{(x^2+1)^{1/2} - x^2(x^2+1)^{-1/2}}{(x^2+1)} = \frac{(x^2+1)^{-1/2}\left[(x^2+1) - x^2\right]}{(x^2+1)}$$

$$= \frac{1}{(x^2+1)^{3/2}} = (x^2+1)^{-3/2}$$

Method 2: Product & Chain Rule:

$$y = \frac{x}{\sqrt{x^2+1}} = x(x^2+1)^{1/2}$$

$$y' = \frac{d}{dx}\left[x(x^2+1)^{-1/2}\right] = \left(\frac{d}{dx}x\right)(x^2+1)^{-1/2} + x\frac{d}{dx}(x^2+1)^{1/2}$$

$$= (x^2+1)^{-1/2} + x\cdot\frac{1}{2}(x^2+1)^{-3/2}(2x) = (x^2+1)^{-3/2}\left[(x^2+1) - x^2\right]$$

$$= (x^2+1)^{-3/2}(1) = (x^2+1)^{-3/2}$$

II) Implicit Differentiation

How to obtain $\frac{dy}{dx}$ in some equation $a_n y^n + \dots + a_0 + b_m x^m + \dots + b_0 + c_p x^{k_p} y^{l_p} + \dots = 0$

where y may not even in principle be isolated as a formula for x.

Use chain rule (where y is the implicit function of x)

· Example (#9 §3.6, p.141)

$$y^2 = \frac{x^2-9}{x^2+9} \quad \text{at} \quad P(3,0)$$

Method 1: Implicit differentiation & quotient rule:

$$\frac{d}{dx} y^2 = 2y\,dy/dx = \frac{d}{dx}\left[\frac{x^2-9}{x^2+9}\right] = \frac{(x^2+9)(2x)-(2x)(x^2-9)}{(x^2+9)^2}$$

$$\therefore\ 2yy' = \frac{\cancel{2x^3}+18x-\cancel{2x^3}+18x}{(x^2+9)^2}$$

$$2yy' = \frac{36x}{(x^2+9)^2} \Rightarrow y' = \frac{18x}{y(x^2+9)^2}$$

$$\therefore \text{ At } P(3,0):\ y' = \frac{18(3)}{0(27)} = \infty \quad \text{undefined}$$

Method 2: Implicit differentiation & product rule:

$$(x^2+9)y^2 = x^2-9$$

$$\frac{d}{dx}\left[(x^2+9)y^2\right] = \frac{d}{dx}(x^2-9) \Rightarrow 2xy^2+2(x^2+9)yy' = 2x$$

$$xy^2 + (x^2+9)yy' = x$$

But $y^2 = \dfrac{x^2-9}{x^2+9}$ so: $\dfrac{x(x^2-9)}{(x^2+9)} + (x^2+9)yy' = x$

$$\therefore\ (x^2+9)yy' = x - \frac{x(x^2-9)}{x^2+9} = x\left\{1 - \frac{(x^2-9)}{x^2+9}\right\}$$

$$(x^2+9)yy' = x\left\{\frac{\cancel{x^2}+9-\cancel{x^2}+9}{x^2+9}\right\}$$

$$(x^2+9)yy' = \frac{18x}{(x^2+9)}$$

$$\boxed{y' = \frac{18x}{(x^2+9)^2 y}}$$

<u>#12, Section 3.6</u> $\sqrt{xy} = x - 2y$ at $P(4,1)$

$$\frac{d}{dx}(xy)^{1/2} = \frac{d}{dx}(x-2y)$$

$$\frac{1}{2}(xy)^{-1/2}[xy'+y] = 1-2y'$$

$$\frac{xy'+y}{2\sqrt{xy}} = 1-2y'$$

$$\frac{xy'}{2\sqrt{xy}} + 2y' = 1 - \frac{y}{2\sqrt{xy}}$$

$$y'\left[\frac{x}{2\sqrt{xy}} + 2\right] = 1 - \frac{y}{2\sqrt{xy}}$$

$$y'\left[\frac{1}{2}\sqrt{\frac{x}{y}} + 2\right] = 1 - \frac{1}{2}\sqrt{\frac{y}{x}}$$

$$y' = \frac{1 - \frac{1}{2}\sqrt{y/x}}{\frac{1}{2}\sqrt{\frac{x}{y}} + 2} = \frac{\left(\sqrt{x} - \frac{1}{2}\sqrt{y}\right)\sqrt{y}}{\sqrt{x}\left(\frac{1}{2}\sqrt{x} + 2\sqrt{y}\right)} = \frac{\sqrt{xy} - \frac{1}{2}y}{\frac{1}{2}x + 2\sqrt{xy}}$$

$$\therefore y'\Big|_{P(4,1)} = \frac{\sqrt{4}-\frac{1}{2}}{\frac{1}{2}\cdot 4+2\sqrt{4}} = \frac{2-\frac{1}{2}}{2+4} = \frac{3/2}{6} = \frac{3}{12} = \frac{1}{4}$$

• #15, §3.6 $\quad x^3 - 2x^2y + 3xy^2 = 38$ at $(2,3)$

$$3x^2 - 2(2xy + x^2y') + 3(y^2 + 2xyy') = 0$$

$$3x^2 - 4xy - 2x^2y' + 3y^2 + 6xyy' = 0$$

$$6xyy' - 2x^2y' = 4xy - 3x^2 - 3y^2$$

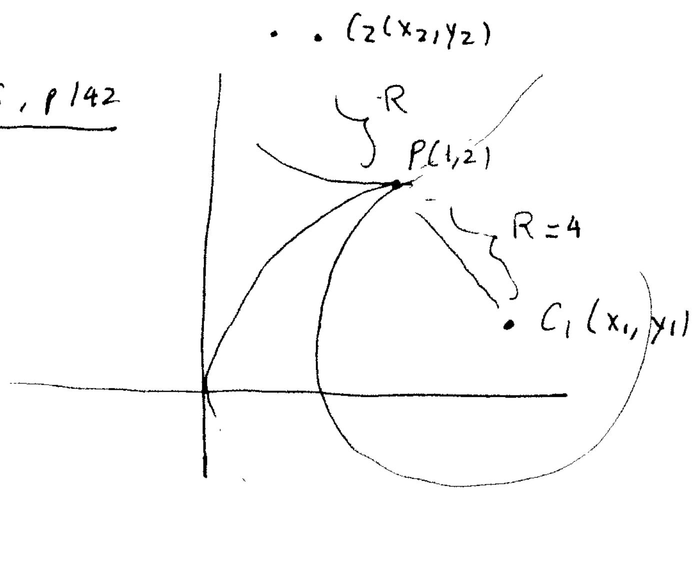
$$(6xy - 2x^2)y' = 4xy - 3x^2 - 3y^2$$

$$\boxed{y' = \frac{4xy - 3(x^2+y^2)}{2x(3y-x)}}$$

at $(2,3)$: $\quad y' = \dfrac{4\cdot 6 - 3(4+9)}{4(9-2)} = \dfrac{24-39}{28}$

$$\boxed{y' = \frac{-15}{28}}$$

• #39 §3.6, p 142



At P, the tangent lines must all be parallel

The equations for the circles are: $(x-x_1)^2+(y-y_1)^2=r^2=16$
$(x-x_2)^2+(y-y_2)^2=r^2=16$

- Implicitly differentiating them: $2(x-x_1)^2+2(y-y_1)^2y'=0$
$2(x-x_2)^2+2(y-y_2)^2y'=0$

- Implicitly differentiating the equation $y^2=4x$

$$\frac{d}{dx}y^2 = 2yy' = 4 \Rightarrow y' = \frac{2}{y}$$

At $P(1,2) \Rightarrow y' = \frac{2}{2} = 1 = m_t$

Now, $m_n = -\frac{1}{m_t} = -1$ (m_n is the slope of the line normal to the curve at $P(1,2)$)

This line must pass through the centers of the two circles!

- Using point-slope formula: $y-y_1=m_n(x-x_1)$

$$\Rightarrow \boxed{y-y_1=-(x-x_1)}$$

(where (x_1,y_1) is the center of circle 1

- According to the distance formula:

$$(1-x_1)^2+(2-y_1)^2=4^2=16$$

Since $y-y_1=-(x-x_1)$ for all (x,y) on line, since line passes through $P(1,2)$:

$$(2-y_1)=-(1-x_1)$$

Inserting into distance formula:

$$(2-y_1)^2 + (2+y_1)^2 = 2(2-y_1)^2 = 16 \Rightarrow (2-y_1)^2 = 8$$

$$\Updownarrow$$

$$(2-y_1) = -(1-x_1)^2$$

$$\therefore (2-y_1)^2 = (1-x_1)^2$$

$$2 - y_1 = \pm 2\sqrt{2}$$

$$y_1 = 2 \mp 2\sqrt{2}$$

$$= 2(1 \mp \sqrt{2})$$

and $2 - y = x_1 - 1$

$$\therefore x_{1,2} = 3 - y_{1,2}$$

$$x_1 = 3 - 2(1+\sqrt{2})$$

$$x_1 = 1 - 2\sqrt{2}$$

$$x_2 = 3 - 2(1-\sqrt{2})$$

$$= 1 + 2\sqrt{2}$$

$$\therefore C_1\ (1+2\sqrt{2},\ 2-\sqrt{2})$$

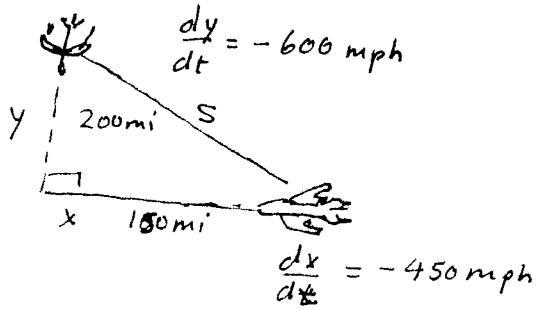
$$C_2\ (1-2\sqrt{2},\ 2+\sqrt{2})$$

Equations: $(x-1-2\sqrt{2})^2 + (y-2+\sqrt{2})^2 = 16$

$(x-1+2\sqrt{2})^2 + (y-2-\sqrt{2})^2 = 16$

Related Rates

- Example (#25, p150)



a) $s^2 = x^2 + y^2$

$$\frac{d}{dt} s^2 = \frac{d}{dt} x^2 + \frac{d}{dt} y^2$$

$$2s\dot{s} = 2x\dot{x} + y\dot{y}$$

(where $\dot{}$ is short-hand for "$\frac{d}{dt}$", time-derivative)

$$s\dot{s} = x\dot{x} + y\dot{y}$$

$$\dot{s} = \frac{x\dot{x} + y\dot{y}}{s} = \frac{x\dot{x} + y\dot{y}}{\sqrt{x^2+y^2}}$$

when $x = 150\text{mi}$, $y = 200$ mi $\Rightarrow$ $\dot{s} = \dfrac{150(-450) + 200(-600)}{\sqrt{150^2 + 200^2}}$ mph

$$= -750 \text{ mph}$$

(The minus sign indicates the planes are <u>approaching</u> one another)

b) The planes are $\Delta S = \sqrt{150^2 + 200^2} = 250$ miles apart. Assuming constant velocity: $\dot{s} = \dfrac{\Delta s}{\Delta t} = \lim\limits_{\Delta t \to 0} \dfrac{\Delta s}{\Delta t} = \dfrac{ds}{dt} = -750$

then $\Delta t = \dfrac{\Delta s}{|\dot{s}|} = \dfrac{250 \text{ mi}}{750 \text{ mph}} = 0.33$ hrs $= \cancel{16}\ 20$ minutes

(Not much time!)

- <u>Example #32, p151</u>

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} = \frac{R_1+R_2}{R_1R_2} \Rightarrow R = \frac{R_1R_2}{R_1+R_2}$$

$$\dot{R}_1 = 1\,k\Omega/\text{sec} \qquad \dot{R}_2 = 1.5\,k\Omega/\text{sec}$$

Find $\dot{R}$ when $R_1 = 50k\Omega$ and $R_2 = 75k\Omega$

$$\frac{d}{dt}R = \dot{R} = \frac{d}{dt}\left[\frac{R_1R_2}{R_1+R_2}\right] = \frac{(R_1+R_2)(R_1\dot{R}_2 + \dot{R}_1R_2) - R_1R_2(\dot{R}_1+\dot{R}_2)}{(R_1+R_2)^2}$$

$$= \frac{R_1^2\dot{R}_2 + R_1\cancel{R_2}\dot{R}_2 + R_1\cancel{R_2}\dot{R}_1 + \dot{R}_1R_2^2 - R_1\cancel{R_2}\dot{R}_1 - R_1\cancel{R_2}\dot{R}_2}{(R_1+R_2)^2}$$

$$\dot{R} = \frac{R_1^2\dot{R}_2 + \dot{R}_1R_2^2}{(R_1+R_2)^2} \Rightarrow \dot{R} = \frac{50^2(1.5) + 1\cdot(75)^2}{(125)^2} = 0.52\,k\Omega/\text{sec}$$