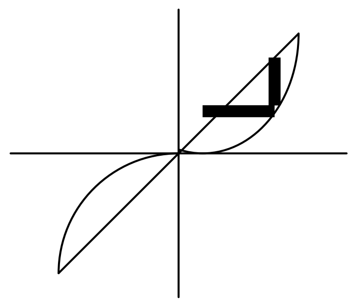
MA261 Final Exam Review Problems: Answers

Format: Choose 5 out of 6 problems (worth 20 pts each) + 1 bonus problem

I. Choose any problem from Exam 1 and rework it. (Do them all if necessary)

II. Given the functions $y_1 =$ ~~$x + 2$~~ $y_1 = x$ $y_2 = x^3$. Find the area of the region by evaluating both a *dx* and a *dy* integral

$$y_1 = y_2 \Rightarrow x = x^3 \Rightarrow x^3 - x = 0 \Rightarrow x(x^2 - 1) = 0 \Rightarrow x(x-1)(x+1) = 0$$
$$\Rightarrow x_1 = -1, x_2 = 0, x_3 = 1 \Rightarrow y_1 = -1, y_2 = 0, y_3 = 1$$



Based on the symmetry of the intersecting graphs,

$$A = 2\int_0^1 (x - x^3)dx = 2\left(\tfrac{1}{2}x^2 - \tfrac{1}{4}x^4\right)\Big|_0^1 = x^2\left(1 - \tfrac{1}{2}x^2\right)\Big|_0^1 = 1\left(1 - \tfrac{1}{2}\right) = \tfrac{1}{2}$$

$$A = 2\int_0^1 \left(y^{1/3} - y\right)dy = 2\left(\tfrac{3}{4}y^{4/3} - \tfrac{1}{2}y^2\right)\Big|_0^1 = \left(\tfrac{3}{2}y^{4/3} - y^2\right)\Big|_0^1 = \tfrac{3}{2} - 1 = \tfrac{1}{2}$$

III. Given the function $f(x) = \dfrac{\ln x}{\sqrt{x}}$ find the volume generated by rotating the graph around the *x*-axis from $x = 1$ to $x = 2$

Method of Washers:
$$V = \pi\int_1^2 \left(\frac{\ln x}{\sqrt{x}}\right)^2 dx = \pi\int_1^2 \ln^2 x\frac{dx}{x} \Rightarrow u = \ln x \therefore du = \frac{dx}{x}$$
$$\Rightarrow V = \pi\int_{u(1)}^{u(2)} u^2 du = \pi\int_0^{\ln 2} u^2 du = \tfrac{\pi}{3}u^3\Big|_0^{\ln 2} = \tfrac{\pi}{3}(\ln 2)^3$$

IV.a) Use L'Hopital's Rule to evaluate: $\lim_{x\to k} \dfrac{\sqrt[3]{x}-\sqrt[3]{k}}{x-k}$

$$\lim_{x\to k} \frac{\sqrt[3]{x}-\sqrt[3]{k}}{x-k} = \lim_{x\to k} \frac{\frac{d}{dx}\left(x^{1/3}-k^{1/3}\right)}{\frac{d}{dx}\left(x-k\right)} = \lim_{x\to k} \frac{\frac{1}{3}x^{-2/3}}{1} = \tfrac{1}{3}\lim_{x\to k} x^{-2/3} = \frac{1}{3\sqrt[3]{k^2}}$$

Check: Note that $x-k=\left(x^{1/3}-k^{1/3}\right)\left(x^{2/3}+k^{1/3}x^{1/3}+k^{2/3}\right)$

Hence:

$$\lim_{x\to k} \frac{\sqrt[3]{x}-\sqrt[3]{k}}{x-k} = \lim_{x\to k} \frac{x^{1/3}-k^{1/3}}{\left(x^{1/3}-k^{1/3}\right)\left(x^{2/3}+(kx)^{1/3}+k^{2/3}\right)} = \lim_{x\to k} \frac{1}{\left(x^{2/3}+(kx)^{1/3}+k^{1/3}\right)}$$

$$= \frac{1}{k^{2/3}+k^{2/3}+k^{2/3}} = \frac{1}{3k^{2/3}} = \frac{1}{3\sqrt[3]{k^2}}$$

b.) Use L'Hopital's Rule to evaluate: $\lim_{x\to\infty} \dfrac{e^{3x}}{x^3}$

$$\lim_{x\to\infty} \frac{e^{3x}}{x^3} = \lim_{x\to\infty} \frac{\frac{d}{dx}e^{3x}}{\frac{d}{dx}x^3} = \lim_{x\to\infty} \frac{3e^{3x}}{3x^2} = \lim_{x\to\infty} \frac{e^{3x}}{x^2} = \lim_{x\to\infty} \frac{\frac{d}{dx}e^{3x}}{\frac{d}{dx}x^2}$$

$$\lim_{x\to\infty} \frac{3e^{3x}}{2x} = \tfrac{3}{2}\lim_{x\to\infty} \frac{e^{3x}}{x} = \tfrac{3}{2}\lim_{x\to\infty} \frac{\frac{d}{dx}e^{3x}}{\frac{d}{dx}x} = \tfrac{3}{2}\lim_{x\to\infty} \frac{3e^{3x}}{1} = \tfrac{9}{2}\lim_{x\to\infty} e^{3x} = \infty$$

c.) Evaluate: $\displaystyle\int_a^b \frac{x+1}{3x^2+6x+3}dx$

$$u=3x^2+6x+3 \Rightarrow du=(6x+6)dx=6\left(x+1\right)dx \Rightarrow \left(x+1\right)dx=\tfrac{1}{6}du$$

$$\therefore \int_a^b \frac{x+1}{3x^2+6x+3}dx = \tfrac{1}{6}\int_{u(a)}^{u(b)} \frac{du}{u} = \tfrac{1}{6}\ln|u|\Big|_{3\left(a^2+2a+1\right)}^{3\left(b^2+2b+1\right)} = \tfrac{1}{6}\left(\ln\left|3(b+1)^2\right| - \ln\left|3(a+1)^2\right|\right)$$

$$= \tfrac{1}{6}\ln\left|\frac{b+1}{a+1}\right|^2 = \tfrac{1}{3}\ln\left|\frac{b+1}{a+1}\right| = \ln\left(\sqrt[3]{\frac{|b+1|}{|a+1|}}\right)$$

V.a) Show that: $\left[a+\ln|\cos x|\right]\cos x+(b+x)\sin x$ satisfies the differential equation: $y''+y=\sec x$

$$y=\left[a+\ln|\cos x|\right]\cos x+(b+x)\sin x \Rightarrow y'=\frac{d}{dx}\left(\left[a+\ln|\cos x|\right]\cos x+(b+x)\sin x\right)$$
$$=\left(\frac{-\sin x}{\cos x}\right)\cos x+\left(a+\ln|\cos x|\right)(-\sin x)+\sin x+(b+x)\cos x$$
$$=-\sin x-a\sin x-\ln|\cos x|\sin x+\sin x+b\cos x+x\cos x$$
$$=-\left(a+\ln|\cos x|\right)\sin x+(x+b)\cos x$$
$$y''=\frac{d}{dx}y'=\frac{d}{dx}\left(-\left(a+\ln|\cos x|\right)\sin x+(x+b)\cos x\right)=-\left(a+\ln|\cos x|\right)\cos x-\frac{-\sin x}{\cos x}\sin x$$
$$+\cos x-(x+b)\sin x=-a\cos x-\ln|\cos x|\cos x+\frac{\sin^2 x}{\cos x}+\cos x-(x+b)\sin x$$
$$=-\left(a+\ln|\cos x|\right)\cos x+\tan x\sin x+\cos x-(x+b)\sin x$$
$$\Rightarrow y''+y=\sin x\tan x+\cos x=\cos x\left(\tan^2 x+1\right)=\cos x\left(\sec^2 x\right)=\sec x$$

b.) Evaluate $\int\frac{1-x}{\sqrt{1-x^2}}dx$

$$\int\frac{1-x}{\sqrt{1-x^2}}dx=\int\frac{1}{\sqrt{1-x^2}}dx-\int\frac{x}{\sqrt{1-x^2}}dx=\arcsin x+\tfrac{1}{2}\int\frac{du}{u^{1/2}}$$
$$=\arcsin x+\tfrac{1}{2}\cdot 2u^{1/2}+C=\arcsin x+\sqrt{1-x^2}+C$$

c.) Evaluate: $\int\frac{\arcsin\left(\frac{x}{2}\right)}{\sqrt{4-x^2}}dx$

$$u=\arcsin\left(\frac{x}{2}\right)\Rightarrow du=\frac{dx}{\sqrt{2^2-x^2}}\Rightarrow\int\frac{\arcsin(x/2)}{\sqrt{4-x^2}}dx=\int u\,du$$
$$=\tfrac{1}{2}u^2+C=\tfrac{1}{2}\left(\arcsin(x/2)\right)^2+C$$

VI.a) Evaluate $\int x^2\cos 2x\,dx$ **by parts**

u	dv
x^2	cos2x
$2x$	(1/2)sin2x
2	-(1/4)cos2x
0	-(1/8)sin2x

$$\therefore \int x^2 \cos 2x dx = \tfrac{1}{2} x^2 \sin 2x + \tfrac{1}{2} x \cos 2x - \tfrac{1}{4} \sin 2x + C = \tfrac{1}{2}\left[\left(x^2 - 1\right)\sin 2x + x \cos 2x\right] + C$$

b.) Evaluate: $\int \frac{x}{x^2 + 2x + 5} dx$

$$\int \frac{x}{x^2 + 2x + 5} dx = \int \frac{x}{x^2 + 2x + 1 - 1 + 5} dx = \int \frac{x}{(x+1)^2 + 2^2} dx$$

$$= \int \frac{u-1}{u^2 + 2^2} du = \int \frac{u}{u^2 + 4} du - \int \frac{1}{u^2 + 2^2} du = \tfrac{1}{2} \int \frac{dw}{w} - \tfrac{1}{2} \arctan\left(\frac{u}{2}\right)$$

$$= \tfrac{1}{2} \ln w - \tfrac{1}{2} \arctan\left(\frac{x+1}{2}\right) + C = \tfrac{1}{2} \ln\left|u^2 + 4\right| - \tfrac{1}{2} \arctan\left(\tfrac{1}{2}(x+1)\right) + C$$

$$= \tfrac{1}{2}\left[\ln\left(x^2 + 2x + 5\right) - \arctan\left(\tfrac{1}{2}(x+1)\right)\right] + C$$