MA261 PRACTICE EXERCISES-EXAMII
Fall 2007 Kallfelz

Like last exam, you will choose 5 out of 6 problems . Some of the problems will have several parts, which depend on what answer you got in the previous part. If you do all 6 I'll grade the best 5. Bonus problem will be included. Calculators permitted, for graphing and arithmetic functions only.

I. (20 pts) Evaluate the following:

a.) $\int \frac{x}{(x^2+1)^6}dx$

$$u(x) = x^2 + 1 \Rightarrow \frac{du}{dx} = 2x \Rightarrow du = 2xdx \Rightarrow xdx = \tfrac{1}{2}du$$

$$\therefore \int \frac{xdx}{(x^2+1)^6} = \int u^{-6}\tfrac{1}{2}du = \tfrac{1}{2}\int u^{-6}du = \tfrac{1}{2}\cdot\left(-\tfrac{1}{5}\right)u^{-5} + C = -\tfrac{1}{10}u^{-5} + C$$

$$= -\tfrac{1}{10}(x^2+1)^{-5} + C$$

Check: $\frac{d}{dx}\left(-\tfrac{1}{10}(x^2+1)^{-5} + C\right) = \tfrac{5}{10}(x^2+1)^{-6}(2x) = \frac{2x}{2(x^2+1)^6} = \frac{x}{(x^2+1)^6}$

b.) $\int_0^1 (y-1)\sqrt{y+1}dy$

$$u(y) = y + 1 \Rightarrow \frac{du}{dy} = 1 \Rightarrow du = dy$$

$$u(y) = y + 1 \Rightarrow y = u - 1$$

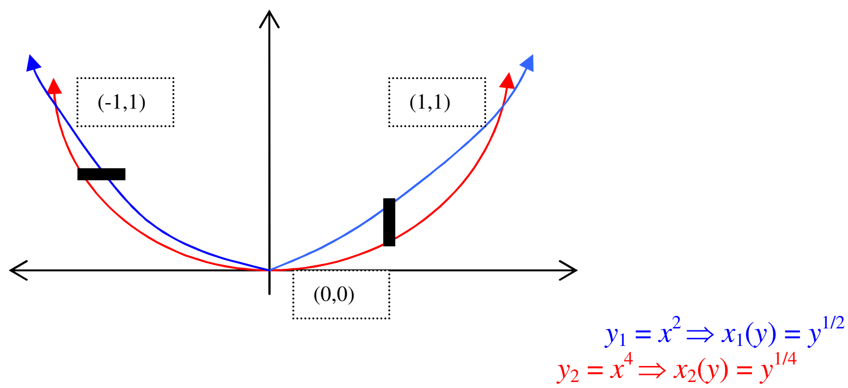
$$\therefore \int_0^1 (y-1)\sqrt{y+1}dy = \int_{u(0)}^{u(1)} (u-2)u^{1/2}du = \int_1^2 \left(u^{3/2} - 2u^{1/2}\right)du$$

$$= \left(\tfrac{2}{5}u^{5/2} - 2\cdot\tfrac{2}{3}u^{3/2}\right)\Big|_1^2 = 2u^{3/2}\left(\tfrac{1}{5}u - \tfrac{2}{3}\right)\Big|_1^2 = 2\cdot 2^{3/2}\left(\tfrac{2}{5} - \tfrac{2}{3}\right) - 2\left(\tfrac{1}{5} - \tfrac{2}{3}\right)$$

$$= 8\sqrt{2}\left(\tfrac{1}{5} - \tfrac{1}{3}\right) - 2\left(\tfrac{1}{5} - \tfrac{1}{3}\right) = \left(-\tfrac{2}{15}\right)\left(8\sqrt{2} - 2\right) = \tfrac{4\left(1-4\sqrt{2}\right)}{15}$$

II. (20) Given the functions $y = x^2,\ y = x^4$

a.) Sketch the region these two graphs define



b.) Find the area of a.) by evaluating a *dx* integral

$A = \int_{-1}^{1} (x^2 - x^4)dx = 2\int_0^1 \left(x^2 - x^4\right)dx$ (since integrand *f*(*x*) is symmetric about *y*-axis, i.e.: $f(x) = x^2 - x^4 = f(-x)$)

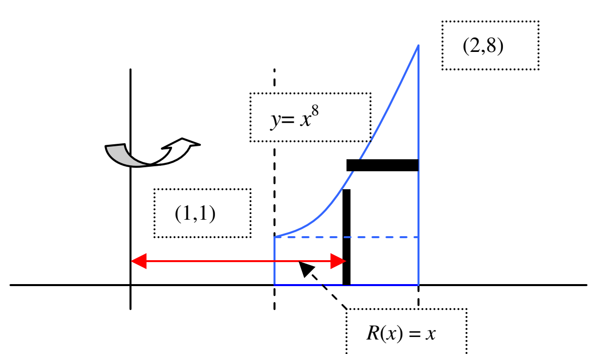
So: $A = 2\int_0^1 \left(x^2 - x^4\right)dx = 2\left(\tfrac{1}{3}x^3 - \tfrac{1}{5}x^5\right)\Big|_0^1 = 2\left(\tfrac{1}{3} - \tfrac{1}{5}\right) - 0 = 2\left(\tfrac{2}{15}\right) = \tfrac{4}{15}$

c.) Repeat b) by evaluating a *dy* integral

$$A = 2\int_0^1 \left(x_2(y) - x_1(y)\right)dy = 2\int_0^1 \left(y^{1/4} - y^{1/2}\right)dy = 2\left(\tfrac{4}{5}y^{5/4} - \tfrac{2}{3}y^{3/2}\right)\Big|_0^1$$

$$= 4y\left(\tfrac{2}{5}y^{1/4} - \tfrac{1}{3}y^{1/2}\right)\Big|_0^1 = 4\left(\tfrac{2}{5} - \tfrac{1}{3}\right) - 0 = 4 \cdot \tfrac{1}{15} = \tfrac{4}{15}$$

III. (20) Given the region $y = x^3,\ y = 0, x = 1, x = 2,$
Find the volume of the figure generated by revolving the region around the *y*-axis using the Method of Shells



$$V = 2\pi\int_1^2 r(x)\,y(x)dx = 2\pi\int_1^2 x\cdot x^3dx = 2\pi\int_1^2 x^4dx = 2\pi\cdot\tfrac{1}{5}x^5\Big|_1^2 = \tfrac{2\pi}{5}\left(2^5-1\right) = \tfrac{62}{5}\pi$$

IV. (25) Given the region $y = x^3$, $y = 0$, $x = 1$, $x = 2$,
(The same region as in III) above.) Find the volume using the method of washers

$$V = \pi\left\{\int_0^1\left(2^2-1^2\right)dy + \int_1^8\left(2^2-\left(y^{1/3}\right)^2\right)dy\right\} = \pi\left\{\int_0^1 3dy + \int_1^8\left(4-y^{2/3}\right)dy\right\}$$

$$= 3\pi\int_0^1 dy + 4\pi\int_1^8 dy - \pi\int_1^8 y^{2/3}dy = \pi\left\{3y\Big|_0^1 + 4y\Big|_1^8 - \tfrac{3}{5}y^{5/3}\Big|_1^8\right\}$$

$$= \pi\left\{3(1-0)+4(8-1)-\tfrac{3}{5}\left(8^{5/3}-1\right)\right\} = \pi\left\{3+28-\tfrac{3}{5}(8\cdot 4-1)\right\}$$

$$= \pi\left(31-\tfrac{3}{5}(31)\right) = 31\pi\cdot\left(1-\tfrac{3}{5}\right) = 31\pi\cdot\tfrac{2}{5} = \tfrac{62}{5}\pi$$

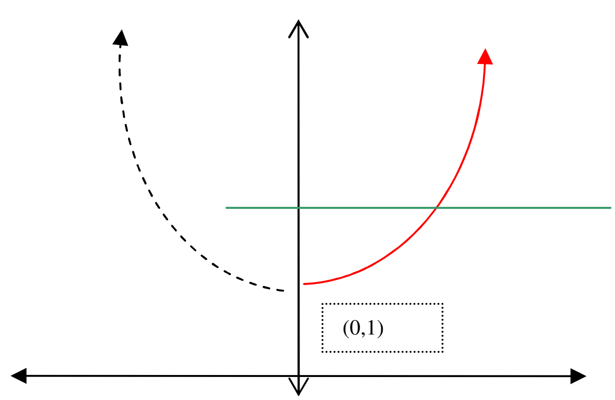
V. (15) Given the function: $f(x) = x^4 + 1$

a.) (3) Find the domain and range of the function

Domain$f(x)$ = $\{x \mid f(x)$ is well-defined$\}$ = $\boldsymbol{R}$ = $(-\infty,\infty)$
Range$f(x)$ = $\{y \mid y = f(x)\}$ = $\{y \mid y = x^4 + 1\}$ = $[1, \infty)$

b.) (2) Find the largest subdomain necessary to make the function 1-1

The largest subdomain would leave the range of the function unaffected, while at the same time enabling the graph of the function to be monotone or pass the horizontal line test. This is accomplished by cutting the domain in half: Domain$f(x)$ = $\{x \mid f(x)$ is well-defined and 1-1$\}$ = $[0, \infty)$ (see graph below)



c.) (5) Find the inverse function $f^{-1}(x)$

$$y = x^4 + 1 \Rightarrow x = (y-1)^{1/4}$$
$$\therefore f^{-1} : [1,\infty) \to [0,\infty)$$
$$f^{-1}(x) = (x-1)^{1/4}$$

d.) (5) Verify your answer in c.) by showing $f\left(f^{-1}(x)\right) = x = f^{-1}\left(f(x)\right)$ (both ways)

$$f\left(f^{-1}(x)\right) = \left[(x-1)^{1/4}\right]^4 + 1 = (x-1)+1 = x$$
$$f^{-1}\left(f(x)\right) = \left[(x^4+1)-1\right]^{1/4} = \left(x^4\right)^{1/4} = x$$

VI. (20) a.) (10) Given $f(x) = x^2 e^{x^2} + \ln \sqrt[5/2]{x^2-4}$

Find its derivative. Make sure you express your final answer in the simplest algebraic detail

$$f'(x) = \frac{d}{dx}\left(x^2 e^{x^2} + \ln\left(x^2-4\right)^{5/2}\right) = \frac{d}{dx}\left(x^2 e^{x^2}\right) + \frac{d}{dx}\tfrac{5}{2}\ln\left(x^2-4\right)$$
$$= \left(2xe^{x^2} + x^2 e^{x^2}\cdot 2x\right) + \tfrac{5}{2}\left(x^2-4\right)^{-1}\cdot 2x$$
$$= 2xe^{x^2}\left(1+x^2\right) + \frac{5x}{\left(x^2-4\right)}$$

b.) (10) Find: $\int_2^5 (\ln 2x)^3 \frac{dx}{x}$

$$u(x) = \ln 2x \Rightarrow \frac{du}{dx} = \frac{1}{2x}\cdot 2 = \frac{1}{x} \Rightarrow du = \frac{dx}{x}$$
$$\therefore \int_2^5 (\ln 2x)^3 \frac{dx}{x} = \int_{u(2)}^{u(5)} u^3 du = \int_{\ln 4}^{\ln 10} u^3 du = \tfrac{1}{4}u^4\Big|_{\ln 4}^{\ln 10} = \tfrac{1}{4}\left[(\ln 10)^4 - (\ln 4)^4\right]$$
$$\approx 6.104$$

BONUS: A Logartihmic differentiation problem (A surprise ☺)