MATH 262 **Name:** KEY
EXAMII
March 27, 2008

Directions: For maximum credit, please show all work in most reasonable detail. An answer, *even if correct*, without any accompanying problem-solving steps, utilizing in a clear and logical manner calculus principles and formalae, as well as algebra, will receive minimal credit (5 pts out of 20 at most). I am grading you primarily for your clearly demonstrated problem solving strategies and tactics If you run out of room, please write on the back page of exam sheet and clearly indicate with a note. No books or notes. Formula sheet provided. Calculator permitted, with programming/memory mode shut off. (However, you may use the graphing utility). Please choose FOUR from the following FIVE (worth 25 pts each.) If you do more, I will grade the best four. Bonus problem included. Good luck!

I. Determine the convergence/divergence of:

a.)(10) $\sum_{k=1}^{\infty} \frac{1}{\sqrt{n^2 - n}}$

BY SCT: $n^2 - n < n^2$ for all $n \geq 1$

$\therefore \frac{1}{\sqrt{n^2-n}} > \frac{1}{\sqrt{n^2}} = \frac{1}{n}$

But $\sum_{n=1}^{\infty} \frac{1}{n}$ is a diverging p-series $(p=1)$

$\therefore \sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2-n}} > \sum_{n=1}^{\infty} \frac{1}{n}$

$\boxed{\text{diverges}}$ diverges

b.) (15) $\sum_{k=1}^{\infty} (-1)^k \frac{1}{\sqrt{k-1}}$

$\sum_{k=1}^{\infty} |a_k| = \sum_{k=1}^{\infty} \frac{1}{\sqrt{k-1}}$

$\therefore$ Diverges absolutely, apply AST to check for conditional convergence:

ii.) $\cancel{\frac{\sqrt{k}}{\sqrt{k-1}}} = \frac{a_{k+1}}{a_k} = \cancel{\sqrt{\frac{k}{k-1}}} = \sqrt{\frac{k-1}{k}}$

$= \sqrt{1 - 1/k} < 1 \quad \therefore a_{k+1} < a_k$

iii.) $\lim_{k\to\infty} a_k = \lim_{k\to\infty} \frac{1}{\sqrt{k-1}} = 0$

$\therefore$ conditionally converges

Method 1: By SCT $k-1 < k$ for all $k \geq 1$

$\therefore \sum_{k=1}^{\infty} \frac{1}{\sqrt{k-1}} > \sum_{k=1}^{\infty} \frac{1}{k^{1/2}}$

Diverging; diverging p-series $(p = 1/2)$

Method 2: $\int_1^{\infty} \frac{dx}{\sqrt{x-1}} = \lim_{b\to\infty} \int_0^b \frac{du}{u^{1/2}} = \lim_{b\to\infty} 2u^{1/2}\Big|_1 = \infty$

$\therefore \sum |a_k|$ diverges by I.T.

II.) Given $\sum_{k=1}^{\infty} \frac{k}{1-k^2}$

a.) ($\cancel{10}$ 7) Use the integral test to examine convergence/divergence:

$$\int_1^{\infty} \frac{x\,dx}{1-x^2} = \lim_{b\to\infty} \int_1^b \frac{x\,dx}{1-x^2} \qquad u = 1-x^2 \quad du = -2x\,dx \to x\,dx = -\frac{1}{2}du$$

$$= \lim_{b\to\infty} \int_{0=u(1)}^{1-b^2} \frac{du}{u} = \lim_{b\to\infty} \ln|u|\Big|_0^{1-b^2} = \lim_{b\to\infty} \ln|1-b^2| - \ln 0 \quad \text{diverges}$$

b.) (10)Use the **SCT** or **LCT** to confirm your answer

BY SCT $\quad 1-k < k$ for all $k \geq 1$

$$\therefore \frac{1-k}{1-k^2} = \frac{\cancel{1-}k}{(1\cancel{-k})(1+k)} \Rightarrow \sum_{k=1}^{\infty} \frac{1}{k+1} < \sum_{k=1}^{\infty} \frac{k}{1-k^2}$$

$$\sum_{j=2}^{\infty} \frac{1}{j} < \sum \frac{k}{1-k^2} \to \infty \quad \therefore \text{diverges}$$

diverging ($p=1$) series

BY LCT $\quad$ when $k >> 1 \therefore \quad a_k = \frac{k}{-k^2+1} \approx -\frac{1}{k} = b_k$

$$\lim_{k\to\infty} \frac{b_k}{a_k} = \lim_{k\to\infty} \frac{1-k^2}{-k^2} = = \lim_{k\to\infty} \frac{k^2-1}{k^2} = \lim_{k\to\infty} (1 - 1/k^2)$$

$$= 1 \Rightarrow \quad \therefore 0 < \lim_{k\to\infty} \frac{a_k}{b_k} < \infty$$

c.) (8) Use partial fractions to express the series in a simpler manner.

$$\sum_{k=1}^{\infty} \frac{k}{1-k^2} \qquad \frac{k}{1-k^2} = \frac{k}{(1-k)(1+k)} = \frac{A_1}{(1-k)} + \frac{A_2}{(1+k)}$$

$$k = A_1(1+k) + A_2(1-k)$$

$$k=1 \Rightarrow 1 = 2A_1 \Rightarrow A_1 = 1/2$$

$$k=-1 \Rightarrow -1 = 2A_2 \Rightarrow A_2 = -1/2$$

$$\therefore \sum_{k=1}^{\infty} \frac{k}{1-k^2} = \sum_{k=1}^{\infty} \left[\frac{1/2}{(k+1)} + \frac{-1/2}{(1-k)} \right]$$

$$= \frac{1}{2} \sum_{k=1}^{\infty} \left[\frac{1}{(k+1)} + \frac{1}{(k-1)} \right]$$

III.) Determine the interval of convergence (which includes checking the convergence/divergence at the endpoints!) for:

a.) (15) $\sum_{k=0}^{\infty} (-1)^k \frac{(x-2)^k}{(k+1)^2}$

Method 1 Using RaT: $\rho(x) = \lim_{k\to\infty} \frac{|a_{k+1}(x)|}{|a_k(x)|} = \lim_{k\to\infty} \frac{|x-2|^{k+1}}{(k+2)^2} \cdot \frac{(k+1)^2}{|x-2|^k}$

$$= \lim_{k\to\infty} |x-2| \frac{(k+1)^2}{(k+2)^2} = |x-2| \lim_{k\to\infty} \left[\frac{k+1}{k+2}\right]^2 = |x-2| \lim_{k\to\infty} \left[\frac{1+1/k}{1+2/k}\right]^2$$

$$= |x-2| \cdot 1 < 1 \Rightarrow |x-2| < 1 \Rightarrow -1 < (x-2) < 1$$

$$\Rightarrow 1 < x < 3 \Rightarrow (1,3) \text{ int. of abs. convergence}$$

method 2 : Using RoT : $r(x) = \lim_{k\to\infty} |a_k|^{1/k} = \lim_{k\to\infty} \sqrt[k]{\frac{|x-2|^k}{(k+1)^2}} = |x-2| \lim_{k\to\infty} (k+1)^{-2/k}$

$$\ln\left[\lim_{k\to\infty} (k+1)^{-2/k}\right] = \lim_{k\to\infty} \ln (k+1)^{-2/k} = \lim_{k\to\infty} \frac{-2\ln(k+1)}{k} \xrightarrow{LHR} \lim_{k\to\infty} \frac{-2/k+1}{1} = 0$$

$$\therefore \lim_{k\to\infty} (k+1)^{-2/k} = e^0 = 1 \Rightarrow r(x) = |x-2| < 1 \Rightarrow (1,3) \text{ abs. conv.}$$

Endpts $x=1$ $\sum_{k=0}^{\infty} \frac{(-1)^k(-1)^k}{(k+1)^2} = \sum_{k=0}^{\infty} \frac{1}{(k+1)^2} = \sum_{j=1}^{\infty} \frac{1}{j^2}$ converging p-series $(p=2)$

$x=3$ $\sum_{k=0}^{\infty} (-1)^k \left[\frac{1}{(k+1)^2}\right] \Rightarrow \sum_{k=0}^{\infty} |a_k| = \sum_{j=0}^{\infty} \frac{1}{j^2}$ converging p-series $\therefore$ absolutely converge.

$\therefore$ Interval of convergence $= (1,3) \cup \{1,3\} = [1,3]$

b.) (10) $\sum_{k=1}^{\infty}(-1)^k \frac{(x-2)^k}{2^k}$

$$\sum_{k=1}^{\infty}|a_k| = \sum_{k=1}^{\infty} \frac{(x-2)^k}{2^k} = \sum_{k=1} \left[\frac{x-2}{2}\right]^k$$

Method 1 RoT: $r(x) = \lim_{k\to\infty} |a_k|^{1/k} = \lim_{k\to\infty} \left(\left|\frac{x-2}{2}\right|^k\right)^{1/k} = |x-2| \lim_{k\to\infty} \frac{1}{2}$

$$= \tfrac{1}{2}|x-2| < 1 = -2 < x-2 < 2 \Rightarrow 0 < x < 4 \Rightarrow (0,4)$$

Method 2 RaT: $\rho(x) = \lim_{k\to\infty} \frac{|a_{k+1}(x)|}{|a_k(x)|} = \frac{|x-2|^{k+1}}{|x-2|^k} \lim_{k\to\infty} \frac{2^k}{2^{k+1}} = |x-2| \lim_{k\to\infty} \frac{1}{2} = \frac{1}{2}|x-2|$

Endpts: $x=0 \quad \sum_{k=1}^{\infty}(-1)^k \frac{(-2)^k}{2^k} = \sum_{k=1}^{\infty} \frac{1}{2^k} \longleftrightarrow \int_1^{\infty} 2^{-x}dx = \lim_{b\to\infty} \frac{1}{\ln 2} -2^{-x}\Big|_1^b =$

$$= \frac{1}{\ln 2}\left[-\lim_{b\to\infty} 2^{-b} + 2^{-1}\right] = \frac{1}{2\ln 2} < \infty \therefore \text{converges}$$

$x = 4 \quad \sum_{k=1}^{\infty} (-1)^k \left[\frac{4-2}{2}\right]^k$

$" \sum_{k=1}^{\infty} (-1)^k \cdot 1^k$

$" \sum_{k=1}^{\infty} (-1)^k$ diverges

$$\boxed{\therefore IC = [0,4)}$$

IV.) Find a power series representation for:

a.) (13) $f(x) = \int_0^x \frac{\cos t}{t^2}\,dt$

$$f(x) = \int_0^x \frac{\cos t}{t^2} dt$$

$$= \int_0^x \frac{1}{t^2} \sum_{k=0}^{\infty} \frac{t^{2k}}{(2k)!} dt$$

$$= \sum_{k=0}^{\infty} \frac{1}{(2k)!} \int_0^x t^{2(k-1)} dt = \sum_{k=0}^{\infty} \frac{1}{(2k)!} \left[\frac{t^{2k-1}}{2k-1}\right]\Big|_0^x$$

$$= \sum_{k=0}^{\infty} \frac{x^{2k-1}}{(2k)!(2k-1)}$$

b.) (12) $g(x)=\dfrac{x}{2+x^2}$

$$= \frac{x}{2\left[1+x^2/2\right]} = \frac{x}{2}\cdot\frac{1}{1+\left(x/\sqrt{2}\right)^2}$$

$$= \frac{x}{2}\cdot\sum_{k=0}^{\infty}\left(-\frac{x^2}{\sqrt{2}}\right)^k$$

$$= \frac{x}{2}\sum_{k=0}^{\infty}(-1)^k\frac{x^{2k}}{2^{k/2}} = \sum_{k=0}^{\infty}(-1)^k\frac{x^{2k+1}}{2^{k/2+1}}$$

V.) Given $f(x)=e^{(3x-2)}=e^{-2}e^{3x}$

a.) (10) Find its McClaurin Series, given that $e^x=\sum_{k=0}^{\infty}\dfrac{x^k}{k!}$

$$f(x)= e^{-2}e^{3x} = e^{-2}\sum_{k=0}^{\infty}\frac{(3x)^k}{k!} = \sum_{k=0}^{\infty}\frac{3^k x^k}{k!\,e^2}$$

b.) (15) Find its Taylor Series, at any point c and show you answer agreeing with a.) in the $c \to 0$ limit.

$$e^{(3x-2)} = e^{-2} e^{3(x+c-c)} = e^{-2} e^{3(x-c)} e^{3c} = e^{3c-2} e^{3(x-c)}$$

$$= e^{3c-2} \sum_{k=0}^{\infty} \frac{[3(x-c)]^k}{k!} = e^{3c-2} \sum_{k=0}^{\infty} \frac{3^k (x-c)^k}{k!}$$

$$\lim_{c \to 0} e^{3c-2} \sum_{k=0}^{\infty} \frac{3^k (x-c)^k}{k!} = e^{-2} \sum_{k=0}^{\infty} \frac{3^k x^k}{k!} = \sum_{k=0}^{\infty} \frac{3^k x^k}{k! e^2}$$

Bonus) Represent $f(x)=\int_0^x (a+bt^2)^{3/2}\,dt$ as a power-series(where a, b are non-zero constants)

$$= a^{3/2}\int_0^x \left(1+\frac{b}{a}t^2\right)^{3/2} dt$$

$$(1+u)^\alpha = \sum_{k=0}^{\infty}\binom{\alpha}{k}u^k$$

$$\therefore \left(1+\frac{b}{a}t^2\right)^{3/2} = \sum_{k=0}^{\infty}\binom{3/2}{k}\left(\frac{bt^2}{a}\right)^k = \sum_{k=0}^{\infty}\binom{3/2}{k}\frac{b^k}{a^k}t^{2k}$$

$$\therefore f(x) = a^{3/2}\int_0^x \sum_{k=0}^{\infty}\binom{3/2}{k}\frac{b^k}{a^k}t^{2k}dt = \sum_{k=0}^{\infty}\frac{b^k}{a^{k-3/2}}\binom{3/2}{k}\int_0^x t^{2k}dt$$

$$= \sum_{k=0}^{\infty}\frac{b^k}{a^{k-3/2}}\binom{3/2}{k}\frac{x^{2k+1}}{2k+1} = \sum_{k=0}^{\infty}\frac{b^k}{a^{k-3/2}}\cdot\frac{\binom{3/2}{k}}{(2k+1)}x^{2k+1}$$