MATH 262
EXAMI
February 19, 2008

Name: *KEY*

Directions: For maximum credit, please show all work in most reasonable detail. <u>An answer, *even if correct*, without any accompanying problem-solving steps, utilizing in a clear and logical manner calculus principles and formalae, as well as algebra, will receive minimal credit (5 pts out of 20 at most). I am grading you primarily for your clearly demonstrated problem solving strategies and tactics</u> If you run out of room, please write on the back page of exam sheet and clearly indicate with a note. No books or notes. Formula sheet provided. Calculator permitted, <u>with programming/memory mode shut off.</u> (However, you may use the graphing utility). Please choose FIVE from the following SIX (worth 20 pts each.) If you do more, I will grade the best <u>five</u>. Bonus problem included. Good luck!

Choose4 out of 5 problems (worth 25 pts each) + 1 bonus problem

I, a.) (15) Use integration by parts to evaluate:

$$\int e^{-st}t^3dt \qquad \text{(where: } s \text{ is a constant)}$$

u		dv
t^3	$+$	e^{-st}
$3t^2$	$-$	$-\frac{1}{s}e^{-st}$
$6t$	$+$	$\frac{1}{s^2}e^{-st}$
6	$-$	$-\frac{1}{s^3}e^{-st}$
0		$\frac{1}{s^4}e^{-sr}$

$$\int e^{-st}t^3dt = -\frac{1}{s}e^{-st}t^3 - \frac{3t^2}{s^2}e^{-st} - \frac{6t}{s^3}e^{-st} - \frac{6}{s^4}e^{-st} + C$$

$$= -\frac{1}{s}e^{-st}\left\{t^3 + \frac{3t^2}{s} + \frac{6t}{s^2} + \frac{6}{s^3}\right\} + C$$

b.)(10) Use your answer in a.) to find:

$$L\{t^3\} = \int_0^\infty e^{-st} t^3\,dt = \lim_{b\to\infty} \int_0^b e^{-st} t^3\,dt$$

$$= \lim_{b\to\infty} -\frac{1}{s} e^{-st} \left\{ t^3 + \frac{3t^2}{s} + \frac{6t}{s^2} + \frac{6}{s^3} \right\} \Big|_0^b$$

$$= \lim_{b\to\infty} -\frac{1}{s} e^{-sb} \left\{ b^3 + \frac{3b^2}{s} + \frac{6b}{s^2} + \frac{6}{s^3} \right\} + \frac{1}{s}\left\{\frac{6}{s^3}\right\}$$

$$= \lim_{b\to\infty} \frac{\left[b^3 + 3b^2/s + 6b/s^2 + 6/s^3 \right]}{-e^{sb}\cdot s} + \frac{6}{s^4}$$

L.HR*

$$= \lim_{b\to\infty} \frac{\left[3b^2 + 6b/s + 6/s^2 \right]}{-s^2 e^{sb}} + \frac{6}{s^4}$$

LHR

$$= \lim_{b\to\infty} \frac{6b + 6/s}{-s^3 e^{sb}} + \frac{6}{s^4}$$

LHR

$$= \lim_{b\to\infty} \frac{6}{-s^4 e^{sb}} + \frac{6}{s^4}$$

$$= 0 + \frac{6}{s^4} = 6s^{-4}$$ (Recall Feb 14 Notes $\mathcal{L}\{t^n\} = \frac{2n}{s^{n+1}}$

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* LHR: L'Hopital's Rule

II.) (25)

a.) (15) Use a rationalizing substitution to solve: $\int_1^2 \frac{dx}{\sqrt{x}-\sqrt[3]{x}}$

$u = x^{1/6}$

$\therefore \ x = u^6$

$dx = 6u^5 du$

$$\int_1^2 \frac{dx}{\sqrt{x}-\sqrt[3]{x}} = \int_{u(1)}^{u(2)} \frac{6u^5\,du}{u^3-u^2} = \int_1^{64} \frac{6\cancel{u^2}\cdot u^3}{\cancel{u^2}(u-1)}du$$

$$6\int_1^{64} \frac{u^3}{u-1}\,du \qquad w = u-1 \quad dw = du$$
$$u = w+1$$

$$6\int_0^{63} \frac{(w+1)^3}{w}du = 6\int_0^{63}\left[\frac{w^3+3w^2+3w+1}{w}\right]dw$$
$$= 6\int_0^{63}\left[w^2+3w+3+\frac{1}{w}\right]dw$$
$$= 6\left\{\frac{1}{3}w^3+\frac{3}{2}w^2+3w+\ln|w|\right\}\Big|_0^{63} \text{ diverges}$$

b.) (10) $\int \frac{\cos^2 \sqrt{x}}{\sqrt{x}}\, dx$

$u = x^{1/2}$

$du = \frac{1}{2} x^{-1/2} dx \rightarrow x^{-1/2} dx = 2du$

$\int \frac{\cos^2 x^{1/2}}{x^{1/2}} dx = 2\int \cos^2 u\, du = \int (1 + \cos 2u)\, du$

$= u + \frac{1}{2}\sin 2u + C$

$= \sqrt{x} + \frac{1}{2}\sin 2\sqrt{x} + C$

$= \sqrt{x} + \sin\sqrt{x}\cos\sqrt{x} + C$

III.a)(15) Evaluate: $\int_{\pi/4}^{\pi/2} \sec^4 x \tan x dx$

$u = \sec x$

$du = \sec x \tan x \, dx$

$$\int_{\frac{\pi}{4}}^{\pi/2} \sec^3 x \cdot \sec x \tan x \, dx = \int_{u(\pi/4)}^{u(\pi/2)} u^3 \, du = \int_{\sqrt{2}}^{\infty} u^3 du$$

$$= \lim_{b\to\infty} \int_{\sqrt{2}}^{b} u^3 du = \lim_{b\to\infty} \frac{1}{4} u^4 \Big|_{\sqrt{2}}^{b} = \frac{1}{4} \lim_{b\to\infty} u^4 - \frac{\sqrt{2}}{4}$$

diverges.

b.)(10) $\int \cos^5 x dx$

$$\int \cos^4 x \cos x \, dx = \int (1-\sin^2 x)^2 \cos x \, dx$$

$$= \int (1-u^2)^2 du = \int (1 - 2u^2 + u^4) \, du$$

$$= u - \frac{2}{3} u^3 + \frac{1}{5} u^5 + C$$

$$= \sin \cancel{x} - \frac{2}{3} \sin^3 x + \frac{1}{5} \sin^5 x + C$$

$$= \sin x \left\{ 1 - \frac{2}{3} \sin^2 x + \frac{1}{5} \sin^4 x \right\} + C$$

IV.) a.) Evaluate using a trigonometric substitution

a.)(13) $\int_0^4 x\sqrt{16-x^2}\,dx$

$x = 4\sin\theta \rightarrow \theta = \arcsin(x/4)$

$dx = 4\cos\theta\, d\theta$

$$\int_{\arcsin(0)}^{\arcsin(1)} 4\sin\theta\sqrt{16-16\sin^2\theta}\,4\cos\theta\, d\theta$$

$$= 4^3\int_0^{\pi/2}\sin\theta\cdot\cos^2\theta\, d\theta$$

$$= 64\int_0^{\pi/2}\cos^2\theta\,\sin\theta\, d\theta$$

$v = \cos\theta \quad dv = -\sin\theta\, d\theta$

$$= -64\int_{\cos(0)}^{\cos(\pi/2)} v^2\,dv = -64\int_1^0 v^2\,dv = 64\int_0^1 v^2\,dv$$

$$= \frac{64}{3}v^3\Big|_0^1 = \frac{64}{3}$$

b.) (12) $\int \frac{dx}{\sqrt{x^2-2x+2}}$

$$\int \frac{dx}{\sqrt{x^2-2x+1+1}} = \int \frac{dx}{\sqrt{(x-1)^2+1^2}}$$

$$= \int \frac{du}{\sqrt{u^2+1^2}}$$

$u = (x-1) = \tan\theta \Rightarrow \tan\theta = \frac{x-1}{1} = \frac{OPP}{ADJ}$

$du = \sec^2\theta \, d\theta$ $\quad\quad \sec\theta = \frac{HYP}{ADJ} = \frac{\sqrt{(x-1)^2+1}}{1}$

$$= \int \frac{\sec^2\theta \, d\theta}{\sqrt{\tan^2\theta+1}} = \int \frac{\sec^2\theta \, d\theta}{\sec\theta} = \int \sec\theta \, d\theta = \ln|\sec\theta + \tan\theta| + C$$

$$= \ln\left|\sqrt{x^2-2x+2} + (x-1)\right| + C$$

V.) (25) Evaluate Using partial fractions (Do a $u = \sin\theta$ substitution first)

$$\int_0^{\pi/2} \frac{\sin\theta\cos\theta}{\sin^2\theta - 1} d\theta$$

$u = \sin\theta$

$$\int_{u(0)}^{u(\pi/2)} \frac{u\,du}{u^2-1}$$

$$\frac{u}{u^2-1} = \frac{u}{(u-1)(u+1)} = \frac{A_1}{(u-1)} + \frac{A_2}{(u+1)}$$

$$\Rightarrow u = A_1(u+1) + A_2(u-1)$$

$$u=1 \Rightarrow 1 = 2A_1 \Rightarrow A_1 = 1/2$$

$$u=-1 \Rightarrow -1 = -2A_2 \Rightarrow A_2 = 1/2$$

$$\therefore \frac{u}{u^2-1} = \frac{1}{2}\left\{\frac{1}{u-1} + \frac{1}{u+1}\right\} = \underbrace{\frac{1}{2}\left\{\frac{2u}{(u-1)(u+1)}\right\}}_{\text{Check } \checkmark}$$

$$\therefore \int_{u(0)}^{u(\pi/2)} \frac{u\,du}{u^2-1} = \int_0^1 \frac{1}{2}\left\{\frac{1}{u-1} + \frac{1}{u+1}\right\} du = \frac{1}{2}\left\{\ln|u-1| + \ln|u+1|\right\}\Big|_0^1$$

$$= \ln\sqrt{|u^2-1|}\,\Big|_0^1 \quad \underline{\text{div}}$$

BONUS

Given: $\int \frac{du}{u^2 - a^2}$

a.) (10) Use Trig substitution to solve

$$u = a \sec\theta$$

$$du = a \sec\theta \tan\theta \, d\theta$$

$$\int \frac{du}{u^2 - a^2} = \frac{1}{a} \int \frac{\cancel{a}\sec\theta \cancel{\tan\theta}\, d\theta}{\tan^2\theta}$$

$$= \frac{1}{a} \int \frac{\sec\theta}{\tan\theta} d\theta = \frac{1}{a} \int \frac{1}{\cancel{\cos\theta}} \frac{\cancel{\cos\theta}}{\sin\theta} d\theta$$

$$= \frac{1}{a} \int \csc\theta \, d\theta = \frac{1}{a} \ln|\csc\theta - \cot\theta| + C$$

$$\sec\theta = u/a = \frac{HYP}{ADJ} \therefore OPP = \sqrt{u^2 - a^2}$$

$$= \frac{1}{a} \ln\left| \frac{u}{\sqrt{u^2 - a^2}} - \frac{a}{\sqrt{u^2 - a^2}} \right| + C$$

$$= \frac{1}{a} \ln\left| \frac{u - a}{\sqrt{u^2 - a^2}} \right| + C$$

$$= \frac{1}{a} \ln\left| \frac{u - a}{\sqrt{(u-a)(u+a)}} \right| + C$$

$$= \frac{1}{a} \ln \sqrt{\frac{u-a}{u+a}} + C$$

b.) (10) Check your answer using partial fractions

$$\int \frac{du}{u^2-a^2} \qquad \frac{1}{u^2-a^2} = \frac{A_1}{u-a} + \frac{A_2}{u+a}$$

$$1 = A_1(u+a) + A_2(u-a)$$

$$u=a \Rightarrow 1 = 2aA_1 \Rightarrow A_1 = 1/2a$$

$$u=-a \Rightarrow 1 = -2aA_2 \Rightarrow A_2 = -1/2a$$

$$\int \frac{du}{u^2-a^2} = \frac{1}{2a}\left\{\int \frac{du}{u-a} - \int \frac{du}{u+a}\right\}$$

$$= \frac{1}{2a}\left\{\ln|u-a| - \ln|u+a|\right\} + C$$

$$= \frac{1}{2a}\ln\left|\frac{u-a}{u+a}\right| + C$$

$$= \frac{1}{a}\ln\sqrt{\frac{u-a}{u+a}} + C$$