MA261 **Calculus I: Notes for October 9**
Fall 2007 Kallfelz

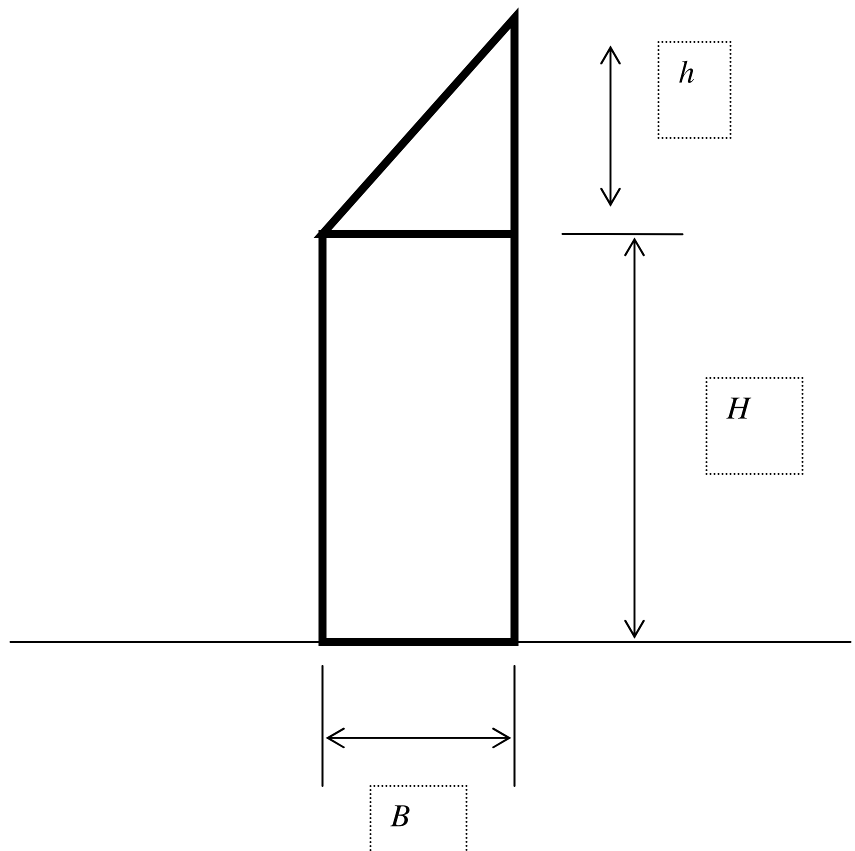
NUMERICAL INTEGRATION

- ***The Trapezoidal Rule***

Recall the Fundamental Theorem of Calculus, which derived and proved a result relating the definite integral in terms of a partition (whether regular or irregular) consisting of rectangles.

To approximate the value of a definite integral, there are various and sundry techniques more accurate than forming partitions consisting of rectangular objects. One can also reduce the error by squeezing in trapezoids (The Trapezoidal Rule) or rectangular objects with parabolic caps (Simpson's Rule). You'll be responsible for the former only (Trapezoidal Rule.)

Consider the following trapezoid:



By inspection it consists of a retangle + a right triangle. Its overall area then is the sum of the areas of these two objects:

$$Area = BH + \tfrac{1}{2}Bh = B(H + \tfrac{1}{2}h) = \tfrac{1}{2}B(2H + h) = \tfrac{1}{2}B[H + (H + h)]$$

Now, let's imagine attaching this figure to the *x*-axis. Let x_{k-1} be the coordinate of the point in which the left hand bottom corner of the trapezoid is anchored, and x_k be the coordinate of the point in which the right hand bottom corner of the trapezoid is anchored. Then certainly:

$$x_K - x_{k-1} = B \equiv \Delta x_k$$
$$y_{k-1} = H$$
$$y_K = H + h$$

So we can re-write: $Area_k = \tfrac{1}{2}\Delta x_h (y_{k-1} + y_k)$

Now if it's also the case that the *upper* corners of the trapezoid touch the graph of some function $y = f(x)$ at the tips, so that the 'roof' of the trapezoid is a *secant* line drawn between the points $P(x_{k-1}, f(x_{k-1})), Q(x_k, f(x_k))$ the we can re-write:

$$Area_k = \tfrac{1}{2}\Delta x_h (f(x_{k-1}) + f(x_k))$$

If the partition is regular, i.e. consisting of *n* equally spaced bases, where:

$$a = x_0 < x_1 < x_2 < ... < x_{k-1} < x_k < ... < x_{n-1} < x_n = b$$

Then: $\Delta x_k = \left(\dfrac{x_n - x_0}{n}\right) = \dfrac{b-a}{n}$ for all *k*: $1 \le k \le n$

Hence we can write:

$$Area_k = \tfrac{1}{2}\Delta x_h (f(x_{k-1}) + f(x_k)) = \left(\frac{b-a}{2n}\right)(f(x_{k-1}) + f(x_k))$$

Summing over the *n* trapezoids in the partition:

$$a = x_0 < x_1 < x_2 < ... < x_{k-1} < x_k < ... < x_{n-1} < x_n = b$$

One arrives at the following expression:

$$A_{Total} = \sum_{k=1}^{n} Area_k = \sum_{k=1}^{n}\left(\frac{b-a}{2n}\right)\left(f(x_{k-1}) + f(x_k)\right) = \left(\frac{b-a}{2n}\right)\sum_{k=1}^{n}\left(f(x_{k-1}) + f(x_k)\right)$$

$$= \left(\frac{b-a}{2n}\right)\{(f(x_0) + f(x_1)) + (f(x_1) + f(x_2)) + (f(x_2) + f(x_3)) + \ldots + (f(x_{n-2}) + f(x_{n-1})) + (f(x_{n-1}) + f(x_n))\}$$

$$= \left(\frac{b-a}{2n}\right)\{f(a) + 2[f(x_1) + f(x_2) + f(x_3) + \ldots + f(x_{n-1})] + f(b)\}$$

$$= \left(\frac{b-a}{2n}\right)\{f(a) + 2[f(x_1) + f(x_2) + f(x_3) + \ldots + f(x_{n-1}) + f(b)] - f(b)\}$$

$$= \left(\frac{b-a}{2n}\right)\left\{f(a) + 2\sum_{k=1}^{n} f(x_k) - f(b)\right\}$$

$$= \left(\frac{b-a}{n}\right)\left\{\left[\frac{f(a) - f(b)}{2}\right] + \sum_{k=1}^{n} f(x_k)\right\}$$

The book gives all sorts of convergence proofs for the above result.

Its maximum error formula is denoted by:

$$E \le \frac{(b-a)^3}{12n^3}\max_{a \le x \le b}\left|f''(x)\right|$$

Consider the example:

$$f(x) = \sqrt{x+1}$$

Suppose on [0 , 1] we adopt a Trapezoidal Approximation with a desired accuracy of 10^{-4}

As discussed in class, $f(x) = \sqrt{x+1}$ is monotone decreasing. Its derivatives are:

$$f'(x) = \tfrac{1}{2}(x+1)^{-\frac{1}{2}}$$
$$f''(x) = -\tfrac{1}{4}(x+1)^{-\frac{3}{2}}$$

Which are also monotone on that interval. Hence:

$$\max_{0 \le x \le 1}\left|f''(x)\right| = \max_{0 \le x \le 1}\left(\tfrac{1}{4}(x+1)^{-\frac{3}{2}}\right) = \frac{1}{4}$$

So: $E \le \frac{(b-a)^3}{12n^3} \max_{a \le x \le b} |f''(x)| \Rightarrow 10^{-4} \le \frac{1}{12n^3} \cdot \frac{1}{4} \Rightarrow 48n^3 \ge 10^4 \Rightarrow n \ge \lceil 5.9218 \rceil = 6$

Running the results through on Excel:

n		f(x)	(f(a) - f(b))/2n
	1	1.080123	-0.034517797
	2	1.154701	
	3	1.224745	
	4	1.290994	
	5	1.354006	
	6	1.414214	
Sum		**7.518783**	
(Sum)(b-a)/n		1.253131	
RESULT		**1.218613**	

To check its accuracy, note the exact answer:

$$\int_0^1 (x+1)^{\frac{1}{2}} dx = \int_{u(0)}^{u(1)} u^{\frac{1}{2}} du = \tfrac{2}{3} u^{\frac{3}{2}} \Big|_1^2 = \tfrac{2}{3}\left(2\sqrt{2}-1\right) = 1.218591$$

…which is accurate to 10^(-4)