MA261 **Calculus I: Notes for October 30 (Part 2)**
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INDETERMINATE FORMS AND L'HOPITAL'S RULE

- Recall when you first learned about evaluating limits, that expressions like $\frac{0}{0}, \infty - \infty$ were **indeterminate**. That is to say, depending on how fast the numerator approaches zero versus that of the denominator term determines the answer, so there's an ambiguity. Same is true for the second case: depending on how fast the left hand term approaches infinite versus the right term influences the answer, so there's no determinate result in general.[1]

  As the footnote (n1) below indicates, to figure what the actual answer (which of course can't be rendered general) for these cases, one often compares growth rates of derivatives with respect to such terms. This procedure is known as **L'Hopital's Rule** and is stated as follows:

$$\lim_{x\to c}\frac{p(x)}{q(x)} = \frac{0}{0} \Rightarrow \lim_{x\to c}\frac{p'(x)}{q'(x)}$$

  Basically the procedure can be repeated until the ambiguity is eventually resolved.[2]

  The proof of L'Hopital's rule is quite straightforward:

$$\lim_{x\to c}\frac{p(x)}{q(x)} = \lim_{x\to c}\frac{p(x)-p(c)}{q(x)-q(c)} \qquad \text{(Since } p(c) = q(c) = 0.\text{)}$$

  Using the trick of 1 and the limit theorem for quotients as well as the definition of

derivative: $: \lim_{x\to c}\frac{p(x)-p(c)}{q(x)-q(c)} = \dfrac{\lim_{x\to c}\dfrac{p(x)-p(c)}{x-c}}{\lim_{x\to c}\dfrac{q(x)-q(c)}{x-c}} = \lim_{x\to c}\frac{p'(x)}{q'(x)}$

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[1] For example: $\lim_{x\to 0}\frac{x}{x^2}, \lim_{x\to 0}\frac{3x}{x}, \lim_{x\to 0}\frac{10x^2+5x^3}{x}$ are all simple indeterminate forms. Of course by simple canceling one discovers that the respective limits are: ∞, 3, 0. By the same token, $\lim_{x\to\infty}(3x^2 - x), \lim_{x\to\infty}(x^2 - x^2), \lim_{x\to\to\infty}(x - x^{10})$ are all indeterminate and converge to ∞, 0, -∞

[2] In which, for example, the answer might be finite, or +∞ or -∞.

- Example (#9, §7.8)

$$\lim_{x\to 0}\frac{e^x-(1-x)}{x}=\lim_{x\to 0}\frac{\frac{d}{dx}\left(e^x-(1-x)\right)}{\frac{d}{dx}x}=\lim_{x\to 0}\frac{e^x+1}{1}=2$$

- Example (#11, §7.8)

$$\lim_{x\to 0}\frac{e^x-(1+x)}{x^n}=\lim_{x\to 0}\frac{\frac{d}{dx}\left(e^x-(1+x)\right)}{\frac{d}{dx}x^n}=\lim_{x\to 0}\frac{e^x-1}{nx^{n-1}}$$

$$=\lim_{x\to 0}\frac{\frac{d}{dx}\left(e^x-1\right)}{\frac{d}{dx}\left(nx^{n-1}\right)}=\lim_{x\to 0}\frac{e^x}{n(n-1)x^{n-2}}=\frac{1}{0}=\infty$$

(Notice that in this example, L'Hopital's Rule had to be applied twice to get rid of the indeterminacy.)

OTHER INDETERMINACIES

Note that $\frac{1}{0}"="\infty$ (where the quotation marks indicate that this answer is an infinity limit.[3] This automatically entails that limits producing $\infty\cdot 0, \frac{\infty}{\infty}$ are of the same class as the $\frac{0}{0}$ indeterminacy, and hence can be resolved as well by L'Hopital's Rule.

- Example (#37, §7.8)

$$\lim_{x\to\infty}\frac{(\ln x)^n}{x^m}\qquad\text{(a } \frac{\infty}{\infty} \text{ indeterminacy)}$$

[3] Stated formally: $\lim_{x\to 0^+}\frac{1}{x}=+\infty$

$$\lim_{x\to\infty}\frac{(\ln x)^n}{x^m}=\lim_{x\to\infty}\frac{\frac{d}{dx}(\ln x)^n}{\frac{d}{dx}x^m}=\lim_{x\to\infty}\frac{n(\ln x)^{n-1}\left(\frac{1}{x}\right)}{mx^{m-1}}=\tfrac{n}{m}\lim_{x\to\infty}\frac{(\ln x)^{n-1}}{x^m}$$

$$=\tfrac{n}{m}\lim_{x\to\infty}\frac{\frac{d}{dx}(\ln x)^{n-1}}{\frac{d}{dx}x^m}=\tfrac{n}{m}\lim_{x\to\infty}\frac{(n-1)(\ln x)^{(n-2)}\left(\frac{1}{x}\right)}{mx^{m-1}}=\tfrac{n(n-1)}{m^2}\lim_{x\to\infty}\frac{(\ln x)^{(n-2)}}{x^m}$$

$$=\ldots(n\_times)=\tfrac{n(n-1)(n-2)\ldots1}{m^n}\lim_{x\to\infty}\frac{(\ln x)^{(n-n)}}{x^m}=\tfrac{n!}{m^n}\lim_{x\to\infty}\frac{1}{x^m}=0$$

- Example (#38, §7.8)

$$\lim_{x\to\infty}\frac{x^m}{e^{nx}} \qquad \text{(a } \frac{\infty}{\infty} \text{ indeterminacy)}$$

$$\lim_{x\to\infty}\frac{x^m}{e^{nx}}=\lim_{x\to\infty}\frac{\frac{d}{dx}x^m}{\frac{d}{dx}e^{nx}}=\lim_{x\to\infty}\frac{mx^{m-1}}{ne^{nx}}=\tfrac{m}{n}\lim_{x\to\infty}\frac{x^{m-1}}{e^{nx}}$$

$$=\tfrac{m}{n}\lim_{x\to\infty}\frac{\frac{d}{dx}x^{m-1}}{\frac{d}{dx}e^{nx}}=\tfrac{m}{n}\lim_{x\to\infty}\frac{(m-1)x^{m-2}}{ne^{nx}}=\tfrac{m(m-1)}{n^2}\lim_{x\to\infty}\frac{x^{m-2}}{e^{nx}}=$$

$$\ldots(m\_times)=\tfrac{m!}{n^m}\lim_{x\to\infty}\frac{1}{e^{nx}}=0$$

Aside from the above indeterminacies $\frac{0}{0},\infty\cdot 0,\frac{\infty}{\infty}$, the role of logarithm and exponential functions reveal yet another kind: Consider: $\ln\left(\frac{\infty}{\infty}\right)=\infty-\infty$, as well as $e^{0\cdot\infty}=1^{\infty}$. Hence $\infty-\infty,1^{\infty}$ are indeterminacies as well

- Example (#24, §7.8)

$$\lim_{x\to1^+}\left(\frac{3}{\ln x}-\frac{2}{x-1}\right) \qquad \text{(a } \infty-\infty \text{ indeterminacy)}$$

$$\lim_{x\to1^+}\left(\frac{3}{\ln x}-\frac{2}{x-1}\right)=\lim_{x\to1^+}\left(\frac{3(x-1)-2\ln x}{\ln x(x-1)}\right)$$

$$=\lim_{x\to1^+}\left(\frac{\frac{d}{dx}(3(x-1)-2\ln x)}{\frac{d}{dx}\ln x(x-1)}\right)=\lim_{x\to1^+}\left(\frac{3-\frac{2}{x}}{\ln x+\frac{x-1}{x}}\right)$$

$$=\lim_{x\to1^+}\left(\frac{3x-2}{x\ln x+x-1}\right)=\frac{1}{0}=\infty$$

- Example (28)

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x \qquad \text{(a } 1^\infty \text{ indeterminacy)}$$

$$\ln\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x=\lim_{x\to\infty} x\ln\left(1+\tfrac{1}{x}\right)$$

$$=\lim_{x\to\infty}\frac{\ln\left(1+\frac{1}{x}\right)}{\frac{1}{x}}=\lim_{x\to\infty}\frac{\frac{d}{dx}\ln\left(1+\frac{1}{x}\right)}{\frac{d}{dx}\left(\frac{1}{x}\right)}=\lim_{x\to\infty}\frac{\left(1+\frac{1}{x}\right)^{-1}\left(-\frac{1}{x^2}\right)}{\left(-\frac{1}{x^2}\right)}$$

$$\lim_{x\to\infty}\frac{1}{\left(1+\frac{1}{x}\right)}=1\Rightarrow\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x=e^1=e$$