MA261 **Calculus I: Notes for October 30 (Part 1)**
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ELEMENTARY GROWTH/DECAY PHENOMENA

Now that you acquired thee techniques for integrating and differentiating exponential and logarithmic functions, you can apply them to an interesting class of problems dealing with ***compounding growth and decay***. Basically such phenomena are described by the observation that:

$^{d}/_{\mathrm{dt}}\, A(t) \propto A(t)$ (or some elementary combination of sums/ products of $^{d}/_{\mathrm{dt}}\, A(t)$)

where: $A(t)$: is the **amount** of whatever quantity is of interest at time t
$\propto$: means 'is directly proportional to'[1]

We'll apply the tools of calculus to look at three different classes of growth phenomena: **a.) exponential growth/decay, b.) limited growth, c.) logistical growth.**

- **Exponential Growth/Decay**

The above proportionality relation is simply translated as: $^{d}/_{\mathrm{dt}}\, A(t) = rA(t)$

where: r is the **rate constant.** If $r > 0$, we have **exponential growth.** If $r = 0$, we have **steady-state** (no growth or decay). If $r < 0$ we have **exponential decay.**

Example: Population growth in demographics: r is usually expressed as some complicated ratio involving average # of children per household (N_c). So if $N_c > 2$, then $r > 0$, since having more than two children per household, all things being equal, translates to a positive growth rate since more than two offspring means the next generation will experience a net gain. For example, having three children means a net gain of one extra person in the next generation, as the other two children replace the eventual death of the two parents. By the same token, $N_c = 2$, then $r = 0$, and if $N_c < 2$, then $r < 0$.

Using the techniques in Chapter 7, one solve the above differential equation[2] as follows:

$$\frac{dA}{dt} = rA \Rightarrow \frac{dA}{A} = rdt \Rightarrow \int \frac{dA}{A} = \int kdt \Rightarrow \ln|A| = kt + C$$

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[1] Recall from algebra: $x \propto y$ whenever there exists a constant k such that: x = ky.

[2] I.e. an equation relating some quantity (described as a function) with its own rates of change and higher-order rates of change (i.e., orders of its derivatives).

So the calculus was relatively straightforward: bring all the "A" terms to one side[3], then integrate both sides of the equation. So technically, we have our answer. However, it's not expressed in a very convenient or meaningful form. To make it more useful, it's necessary to isolate A **in terms of a formula:**

$$\ln|A| = rt + C \Rightarrow e^{\ln|A|} = |A| = e^{rt+C} = e^{C}e^{rt}$$

So now we're told that $|A|$ is basically some constant term (let's define it as $K = e^{C}$) times a simple exponential function e^{rt}. However, recall that the exponential function is always positive, i.e. $e^{x} > 0$, for all x. Hence, with no loss of generality, we can get rid of the absolute value bars around A: $\boldsymbol{A(t) = K\, e^{rt}}$.

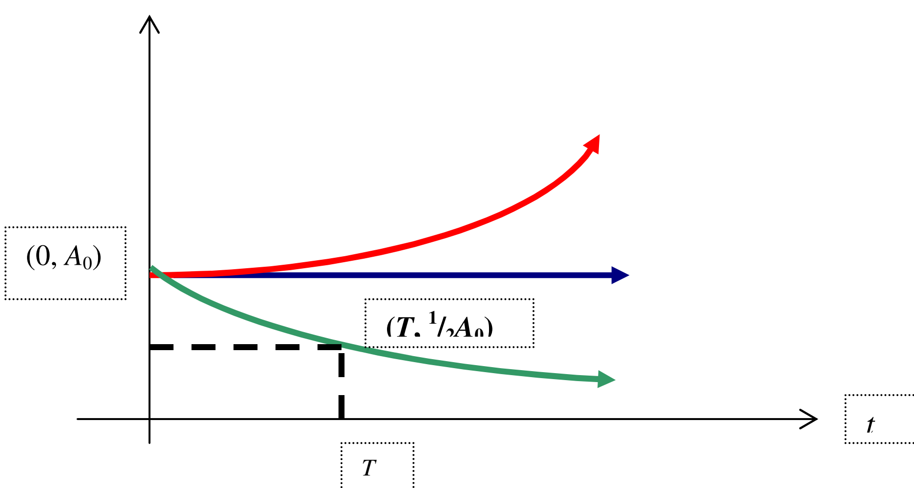
The unknown constant can be related to a physical initial conditions in the following way:

$\boldsymbol{A(0) = K\, e^{0}} = K,$ so define: $K = A(0) = A_0$, i.e. the **initial amount.**

So now we have completely solved the differential equation describing exponential growth, and cast in a useful, physically meaningful, form: $\boldsymbol{A(t) = A_0 e^{rt}}$

$$\boxed{\boldsymbol{A(t) = A_0 e^{rt}}}$$

Before focusing on a particular exercise, it's useful to graph this function:



The red curve is exponential growth ($r > 0$). The blue curve is steady-state ($r = 0$). The green curve is exponential decay ($r < 0$).

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[3] This is **necessary**, for if you tried to integrate the original form of the equation: $\int \frac{d}{dt}A(t)dt = r\int A(t)dt$, then you're basically stuck: The right hand side expresses an integral with respect to A, **but you don't know what A is (that's the quantity you're trying to find).**

Notice that all curves obviously start out at $t = 0$ at the *initial amount* A_0, and exist in the first quadrant (since not only are exponential functions always positive, but elapsed time t is interpreted as positive as well.)

Example: Determining half-life in radioactive decay

Note that the green curve has the vertical line drawn at a position T at precisely **half** the original amount $^1/_2 A_0$. This value for time is known as the **half-life *T*** in the case of exponential decay: **the elapsed time in which half the initial quantity disappeared, or decayed.**

T can be easily solved using algebra:

$$\boldsymbol{A(T)} = {}^1/_2 A_0 = A_0 \boldsymbol{e^{rT}} \Rightarrow {}^1/_2 = \boldsymbol{e^{rT}} \Rightarrow \ln({}^1/_2) = -\ln 2 = rT \Rightarrow T = -\frac{\ln 2}{r}$$

Note that the negative sign does *not* mean that T is negative, since we're dealing with exponential decay so $r < 0$ as well.

Consider an initial amount of 10mg of Rn (Radon) with a decay constant of $r = -1.2 \ \text{ms}^{-1} = -1.2 \times 10^{-3} \ \text{s}^{-1}$. (a) Determine the half-life of the sample. (b) How much Rn is left after 15 minutes?

(a) is easy to answer, just insert into the above formula: $T = -\frac{\ln 2}{r} = -\frac{\ln 2}{-.0012\,\text{sec}^{-1}} = 577\,\text{sec} = \mathbf{9.62min}$

(b) Just use the original equation (make sure you convert your units properly)

$A(15\text{min}) = A(900\text{sec}) = A_0\boldsymbol{e^{rt}} = (10\text{mg})\boldsymbol{e^{-(0.0012)(900)}} = \mathbf{3.4mg}$ (or 34 % left)

- **Limited Growth**

Limited growth involves a kind of exponential growth which however is bound by certain constraints: phenomena include saturation, bacterial growth in a petri dish, etc. The phenomenon is described by the following differential equation:

$$^d/_{dt}\, A(t) = r[M -- A(t)]$$

Where M is the ceiling or maximum amount , and r is the rate constant.

Solving this above differential equation involves the same trick as in the previous case concerning exponential growth/decay:

$$\frac{dA}{dt} = r(M - A) \Rightarrow \frac{dA}{(M - A)} = rdt \Rightarrow \int \frac{dA}{(M - A)} = \int rdt \Rightarrow u(A) = M - A$$

$$\therefore du = -dA \Rightarrow dA = -du \Rightarrow -\int \frac{du}{u} = r\int dt \Rightarrow -\ln|u| = rt + C$$

$$= -\ln|M - A| = \ln\left(\frac{1}{|M - A|}\right) = rt + C$$

Here, based on calculus, the equation is formally solved. However, as in the previous case, we seek to cast it in a more useful form by isolating A as a formula:

$$\ln\left(\frac{1}{|M - A|}\right) = rt + C \Rightarrow |M - A| = e^{-C}e^{-rt} \Rightarrow M - A = Ke^{-rt} \Rightarrow A = M - Ke^{-rt}$$

where, similar to the previous case, we redefine $K = e^{-C}$ and drop the absolute values (with no loss of generality), since exponentials are always positive.

We can relate K in terms of the initial conditions by examining the $t = 0$ case:

$A(0) = A_0 = M - Ke^0 = M - K \Rightarrow K = M - A_0$. Hence:

$$A(t) = M - (M - A_0)\, e^{-rt} = M[\ 1 - (1 - {}^{A_0}\!/_{M})e^{-rt}\ ]$$

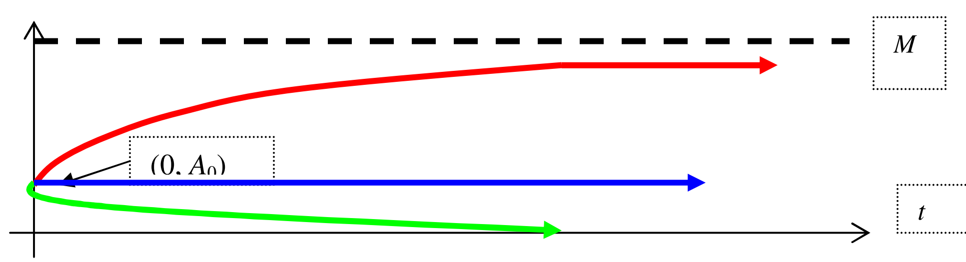
…which relates the formula in terms of physical quantities.

- **<u>Note1</u>**: $A(0) = M[\ 1 - (1 - {}^{A_0}\!/_{M})e^{0}\ ] = M[\ 1 - (1 - {}^{A_0}\!/_{M})\ ] = M[\ {}^{A_0}\!/_{M}\ ] = A_0$ (as expected)

- **<u>Note2</u>**: $\lim_{t\to\infty} A(t) = \lim_{t\to\infty} M[\ 1 - (1 - {}^{A_0}\!/_{M})e^{-rt}\ ]$

  $= M[\ 1 - (1 - {}^{A_0}\!/_{M}) \lim_{t\to\infty} e^{-rt}\ ] = M[1 - (1 - {}^{A_0}\!/_{M})\cdot 0] = M$ (as expected)

- Graphically, limited growth has the following behavior:



Which reveals the asymptotic nature of this form of growth, revealing that A cannot exceed this upper bound. Note that the red curve corresponds to the $r > 0$ case, the blue curve corresponds to the $r = 0$ case, and the green curve corresponds to the $r < 0$ case. Obviously, in the case of limited growth, only the $r>0$ case is interesting (the other two are trivial).

- *Example*: A sample of bacteria is cultured in a petri dish, and exhibits limited growth. a.) Given that the initial population of the sample was estimated = 1.2 million, and that the rate constant = 5.5 $\mu s^{-1} = 5.5 \times 10^{-6}\ s^{-1}$ and after 2days and 6 hours (54 hours ≅ 1.9 × 10 s) the estimated population grew to 3.0 million, what is the maximum amount M that the sample can reach? b) How much time would it take the sample to reach an estimated population of 90% M ?

a.) $A(t) = M[\ 1 - (1 - {}^{A_0}/_{M})e^{-rt}\ ]$

$$\Rightarrow A(t) = M(1-e^{-rt}) + A_0 e^{-rt} \Rightarrow M = \frac{A(t) - A_0 e^{-rt}}{(1-e^{-rt})} = \frac{(3.0\times10^6) - (1.2\times10^6)e^{-(5.5\times10^{-6}\cdot1.9\times10^5)}}{1-e^{-(5.5\times10^{-6}\cdot1.9\times10^5)}}$$

$= 3.976 \times 10^6 \cong 4$ million.

b.)

$$A(t) = .9M = M\left[1-\left(1-\frac{A_0}{M}\right)e^{-rt}\right] \Rightarrow \left(1-\frac{A_0}{M}\right)e^{-rt} = 0.1$$

$$\Rightarrow e^{-rt} = \frac{0.1}{\left(1-\frac{A_0}{M}\right)} \Rightarrow t = -\tfrac{1}{r}\left(\ln 0.1 - \ln\left(1-\tfrac{1.2}{4}\right)\right)$$

= 353,802 sec ≅ 98.2hrs ≅ 4.1days (i.e. 4 days, 2 hrs, 24min)

- **Logistical Growth**

This last case is perhaps the most interesting, for it involves the *competition* of the tow aforementioned forms (exponential and limited). Its differential equation is described by:

$^{d}/_{dt}\, A(t) \ = rA(t)[M -- A(t)]$

where: $rA(t)$ is the competing exponential growth/decay term

[M -- $A(t)$] is the limited growth term.[4]

Solving this above differential equation involves an algebraic trick known as the **method of partial fractions**.[5] After bringing all the *A*-terms to one side, we have:

$$\frac{dA}{A(M-A)} = rdt$$

Trying to integrate the left hand side at this stage is an intractable process (we have, as of yet, no technique for integrating products of functions.[6] The left hand side can be broken down into two fractions via the following procedure:

Solve for constants α, β such that:

$$\frac{1}{A(M-A)} = \frac{\alpha}{A} + \frac{\beta}{(M-A)} \Rightarrow 1 = (M-A)\alpha + A\beta$$

The technique is relatively straightforward to solve for α and β:

-Set $A = M$: Then: $1 = M\beta \Rightarrow \beta = {}^{1}/_{M}$
-Set $A = 0$: Then $1 = M\alpha \Rightarrow \alpha = {}^{1}/_{M}$

Hence:

$$\int \frac{dA}{A(M-A)} = \tfrac{1}{M}\int\left(\frac{1}{A} + \frac{1}{(M-A)}\right)dA = \int rdt$$

$$\Rightarrow \tfrac{1}{M}\left(\int\frac{dA}{A} + \int\frac{dA}{(M-A)}\right) = rt + C$$

$$\Rightarrow \tfrac{1}{M}\left(\ln|A| - \ln|M-A|\right) = rt + C$$

$$\Rightarrow \ln\left|\left(\frac{A}{M-A}\right)^{1/M}\right| = rt + C$$

Again, after doing the calculus we have an expression which should be translated into a formula for $A(t)$:

[4] Consider the growth/decay term to be, for example, a population of predator and prey populations. Predators, for example, can freely exponentially reproduce whereas prey are bound by constraints of resources

[5] It's basically the inverse process of finding a common denominator. In **Calculus II**, you'll learn the method in greater detail (though you may have studied it already in pre-calculus).

[6] Though later, you'll learn the method of **integration by parts**, which can handle some of this situations

$$\left|\left(\frac{A}{M-A}\right)^{1/M}\right| = e^{rt+C} \Rightarrow \left(\frac{A}{M-A}\right)^{1/M} = e^{C}e^{rt}$$

$$\Rightarrow \left(\frac{A}{M-A}\right) = e^{MC}e^{Mrt} = Ke^{Mrt}$$

$$\Rightarrow A = (M-A)Ke^{Mrt} \Rightarrow A\left(1+Ke^{Mrt}\right) = MKe^{Mrt}$$

$$\Rightarrow A(t) = \frac{MKe^{Mrt}}{\left(1+Ke^{Mrt}\right)} = \frac{M}{\left(1+ke^{-Mrt}\right)}$$

(where $k = K^{-1} = e^{-MC}$)

Inserting the initial conditions $A(0) = {}^{M}/_{(1+k)} = A_0 \Rightarrow k = \frac{M}{A_0} - 1 = \frac{M-A_0}{A_0}$

Hence: $A(t) = \dfrac{M}{\left[1+\left(\dfrac{M-A_0}{A_0}\right)e^{-rMt}\right]} = \dfrac{MA_0}{\left[A_0+(M-A_0)e^{-rMt}\right]}$

- **Note 1:** $A(0) = \dfrac{MA_0}{\left[A_0+(M-A_0)\right]} = \dfrac{MA_0}{M} = A_0$ (as expected)
- **Note 2:** $\lim_{t\to\infty} A(t) = \lim_{t\to\infty} \dfrac{MA_0}{\left[A_0+(M-A_0)e^{-rMt}\right]} = \dfrac{MA_0}{\left[A_0+0\right]} = M$ (as expected)

So logistical growth exhibits in certain ways the same behavior as **limited growth** (at least with respect to the appropriate $t = 0$ and $t \to\infty$ extremes). But there's a crucial and subtle difference: **Note that logistical growth has an inflection point[7], while limited growth doesn't.**

To find the inflection point:

$$A'(t) = \frac{d}{dt}\frac{MA_0}{\left[A_0+(M-A_0)e^{-rMt}\right]} = -MA_0\left[A_0+(M-A_0)e^{-rMt}\right]^{-2}(M-A_0)e^{-rMt}(-rM)$$

$$= \frac{rM^2A_0\left(M-A_0\right)e^{-rMt}}{\left[A_0+(M-A_0)e^{-rMt}\right]^2}$$

[7] As you recall, this is a point (f defined) where the curvature = 0, or the second derivative = 0.

$$A''(t) = \frac{d}{dt}\frac{rM^2 A_0(M - A_0)e^{-rMt}}{\left[A_0 + (M - A_0)e^{-rMt}\right]^2}$$

$$= rM^2 A_0(M - A_0)\left\{\frac{-rMe^{-rMt}\left[A_0 + (M - A_0)e^{-rMt}\right]^2 - e^{-rMt}2\left[A_0 + (M - A_0)e^{-rMt}\right](M - A_0)e^{-rMt}(-rM)}{\left[A_0 + (M - A_0)e^{-rMt}\right]^4}\right\}$$

$$= rM^2 A_0(M - A_0)\left\{\frac{-rMe^{-rMt}\left[A_0 + (M - A_0)e^{-rMt}\right] + 2rM(M - A_0)e^{-2rMt}}{\left[A_0 + (M - A_0)e^{-rMt}\right]^3}\right\}$$

So: $A^{//}(t) = 0$ implies:

$$rM^2 A_0(M - A_0)\left\{\frac{-rMe^{-rMt}\left[A_0 + (M - A_0)e^{-rMt}\right] + 2rMe^{-2rMt}(M - A_0)}{\left[A_0 + (M - A_0)e^{-rMt}\right]^3}\right\} = 0$$

$$\Rightarrow -rMe^{-rMt}\left[A_0 + (M - A_0)e^{-rMt}\right] + 2rMe^{-2rMt}(M - A_0) = 0$$

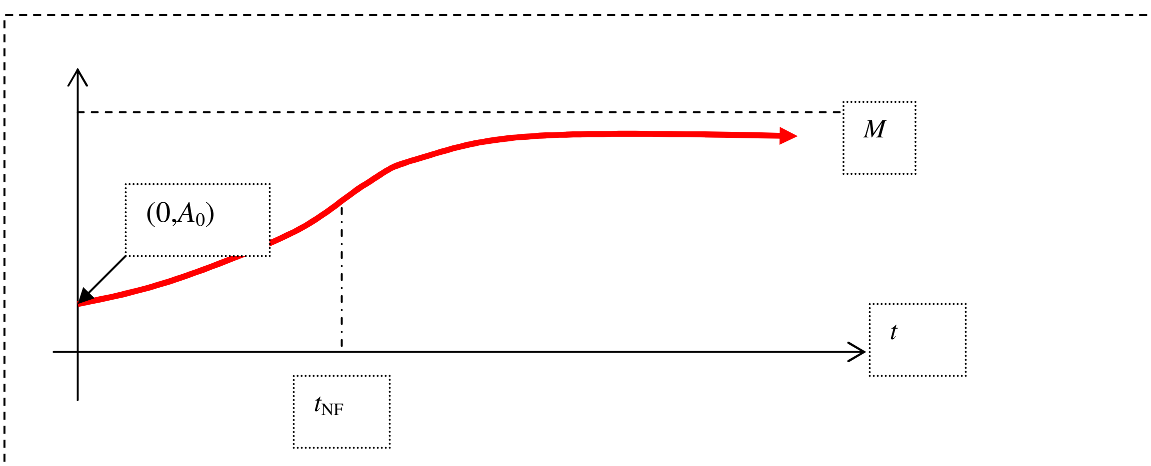
$$\Rightarrow \left[A_0 + (M - A_0)e^{-rMt}\right] - 2e^{-rMt}(M - A_0) = 0$$

$$\Rightarrow (M - A_0)\left(e^{-rMt} - 2e^{-rMt}\right) = -A_0$$

$$\Rightarrow e^{-rMt} = \frac{A_0}{(M - A_0)} \Rightarrow -rMt = \ln\left(\frac{A_0}{M - A_0}\right)$$

$$\Rightarrow t_{INF} = -\tfrac{1}{Mr}\ln\left(\frac{A_0}{M - A_0}\right) = \ln\left[\left(\frac{M - A_0}{A_0}\right)^{1/rM}\right] = \ln\left[\left(\frac{M}{A_0} - 1\right)^{1/rM}\right]$$

This inflection point is crucial to understanding the phenomenon of logistical growth: for it represents the **break-even point**, or point in which the two competing species are in *homeostatic equilibrium*. Graphically it is illustrated below:



Example: A population of wolves exhibits logistical growth with respect to a population of deer. Initially the wolves numbered 12,000. Their population growth was determined to have a rate constant $r = 1.0\times10^{-5}$ (wolves days) $^{-1}$, and a maximum population in the in the long term limit ($t \to \infty$) of 40,000. a) Find the population of wolves after 15 days.. b.) Determine the inflection point and the population of wolves at that point.

a.)

$$A(t) = \frac{MA_0}{\left[A_0 + (M - A_0)e^{-rMt}\right]}$$

$$A(15) = \frac{(40{,}000)(12{,}000)}{\left[12{,}000 + (40{,}000 - 12{,}000)e^{-(.00001\cdot 40{,}000\cdot 15)}\right]}$$

$$= 39{,}769$$

b.) $$t_{INFL} = \ln\left[\left(\frac{M}{A_0} - 1\right)^{1/rM}\right] = \ln\left[\left(\frac{40{,}000}{12{,}000} - 1\right)^{1/(.00001\cdot 40{,}000)}\right] = 2.12\,days$$

$$A(t_{INFL}) = \frac{MA_0}{\left[A_0 + (M - A_0)e^{-rMt_{INF}}\right]} = \frac{(40{,}000\cdot 12{,}000)}{\left[12{,}000 + 28{,}000e^{-(.00001\cdot 40{,}000\cdot 2.12)}\right]} = 19{,}967$$

So in this case, the graph of the wolf population looks like: