MA261 **Calculus I: Notes for October 25**
Fall 2007 Kallfelz

EXPONENTIAL AND LOGARITHMIC DIFFERENTIATION/INTEGRATION (cont.)

…More examples:

- Exercise 21, § 7.2

$$f(x) = \frac{2}{e^x + e^{-x}}$$
$$f'(x) = \frac{d}{dx}2\left(e^x + e^{-x}\right)^{-1} = 2\frac{d}{dx}\left(e^x + e^{-x}\right)^{-1} = -2\left(e^x + e^{-x}\right)^{-2}\left(e^x - e^{-x}\right)$$
$$= \frac{-2\left(e^x - e^{-x}\right)}{\left(e^x + e^{-x}\right)^2} = \frac{2\left(e^{-x} - e^x\right)}{\left(e^x + e^{-x}\right)^2}$$

- Exercise 24, § 7.2

$$y = x^2e^x - 2xe^x + 2e^x$$

Method 1 (easier!) Factor out e^{-x} and then use product rule

$$y = x^2e^x - 2xe^x + 2e^x = \left(x^2 - 2x + 2\right)e^x$$
$$\frac{dy}{dx} = \frac{d}{dx}\left[\left(x^2 - 2x + 2\right)e^x\right] = \left[\frac{d}{dx}\left(x^2 - 2x + 2\right)\right]e^x + \left(x^2 - 2x + 2\right)\left[\frac{d}{dx}e^x\right]$$
$$= (2x - 2)e^x + \left(x^2 - 2x + 2\right)e^x = \left[(2x - 2) + \left(x^2 - 2x + 2\right)\right]e^x = x^2e^x$$

Method 2 (more tedious!) Apply product rule separately on all three terms

$$y = x^2e^x - 2xe^x + 2e^x$$
$$\frac{dy}{dx} = \frac{d}{dx}\left(x^2e^x\right) + \frac{d}{dx}\left(-2xe^x\right) + \frac{d}{dx}\left(2e^x\right) = \frac{d}{dx}\left(x^2e^x\right) - 2\frac{d}{dx}\left(xe^x\right) + 2\frac{d}{dx}e^x$$
$$= \left(2xe^x + x^2e^x\right) - 2\left(e^x + xe^x\right) + 2e^x = x^2e^x$$

- Exercise 26, § 7.2

Use implicit differentiation to find the derivative:

$$e^{xy} + x^2 - 2y^2 = 10$$

$$\frac{d}{dx}(e^{xy} + x^2 - 2y^2) = \frac{d}{dx}10 \Rightarrow e^{xy}\frac{d}{dx}(xy) + 2x - 2y\frac{dy}{dx} = 0$$

$$e^{xy}(x\frac{dy}{dx} + y) + 2x - 2y\frac{dy}{dx} = 0 \Rightarrow (xe^{xy} - 2y)\frac{dy}{dx} + ye^{xy} + 2x = 0$$

$$(xe^{xy} - 2y)\frac{dy}{dx} = -ye^{xy} - 2x \Rightarrow \frac{dy}{dx} = \frac{-(ye^{xy} + 2x)}{(xe^{xy} - 2y)} = \frac{ye^{xy} + 2x}{2y - xe^{xy}}$$

- <u>Exercise 33, § 7.2</u>

Find extrema, pts. of inflection (if they exist) and sketch graph of function

$$f(x) = x^2 e^{-x}$$

Extrema:

$$f'(x) = \frac{d}{dx}\left(x^2 e^{-x}\right) = 2xe^{-x} + x^2\left(-e^{-x}\right) = 0$$

$$\Rightarrow \left(2x - x^2\right)e^{-x} = 0 \Rightarrow x(2-x) = 0 \Rightarrow x_1 = 2, x_2 = 0$$

(since $e^{-x} > 0$ for all x, one can factor it out of the above equation)

Using Second Derivative Test:

$$f''(x) = \frac{d}{dx}\left[\left(2x - x^2\right)e^{-x}\right] = (2-2x)e^{-x} + \left(2x - x^2\right)\left(-e^{-x}\right)$$

$$= \left[(2-2x) - (2x - x^2)\right]e^{-x} = \left(x^2 - 4x + 2\right)e^{-x}$$

$f''(x_1) = \left(x_1^2 - 4x_1 + 2\right)e^{-x_1} = (4 - 4\cdot 2 + 2)e^2 = -2e^2 < 0$ so $x_1 = 2$ is a maximum

$f''(x_2) = \left(x_2^2 - 4x_2 + 2\right)e^{-x_2} = (0 - 4\cdot 0 + 2)e^0 = 2 > 0$ so $x_2 = 0$ is a minimum

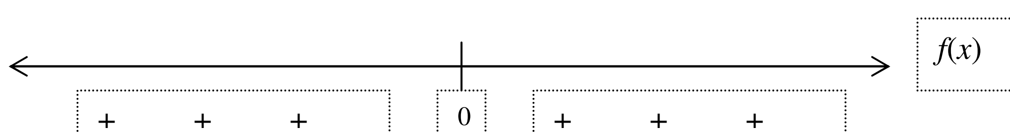
Inflection points:

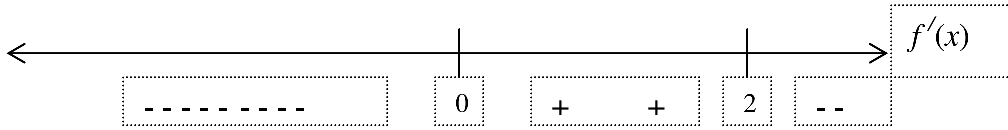
$$f''(x) = \left(x^2 - 4x + 2\right)e^{-x} = 0$$

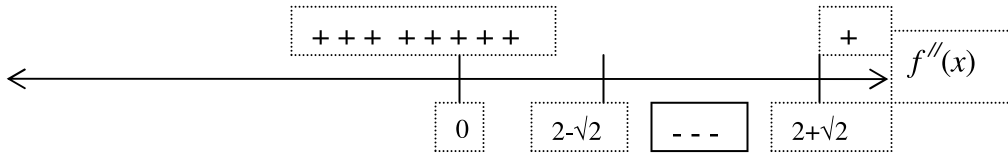
$$\Rightarrow x^2 - 4x + 2 = 0$$

$$\Rightarrow x_{3,4} = \frac{4 \pm \sqrt{16-8}}{2} = \frac{4 \pm 2\sqrt{2}}{2} = 2 \pm \sqrt{2}$$

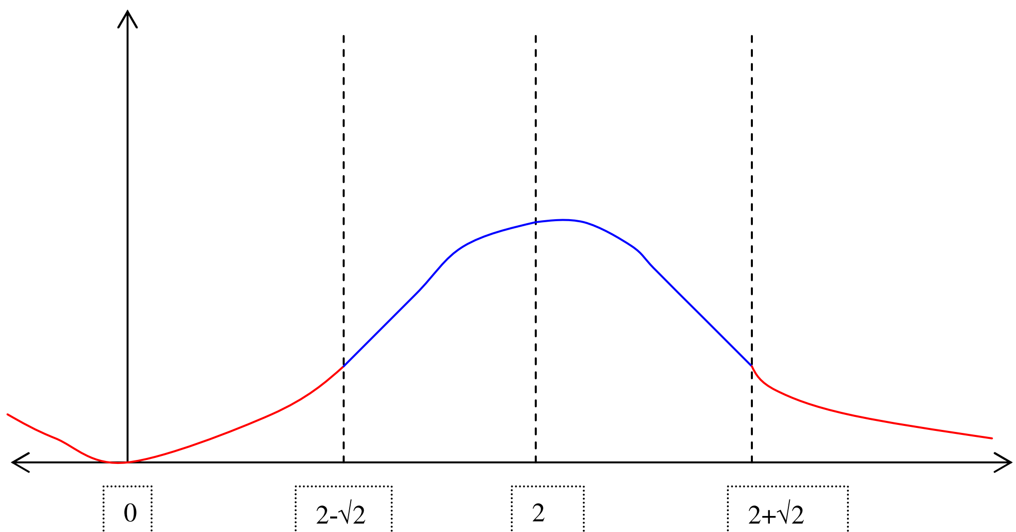
Sketching sign charts for:







Using the sign charts, the graph takes on the following shape:



The red segments indicate concave up curvature. The blue segments indicate concave down curvature.

- <u>Exercise 45, § 7.2</u>

$$\int_0^2 \left(x^2-1\right)e^{x^3-3x+1}dx$$

$$u(x)=x^3-3x+1 \Rightarrow du=(3x^2-3)dx \Rightarrow \left(x^2-1\right)dx=\tfrac{1}{3}du$$

$$\therefore \int_0^2 \left(x^2 - 1\right)e^{x^3-3x+1}dx = \tfrac{1}{3}\int_{u(0)}^{u(2)} e^u du = \tfrac{1}{3}\int_1^3 e^u du = \tfrac{1}{3}e^u\Big|_1^3 = \tfrac{1}{3}\left(e^3 - e\right) = \tfrac{e}{3}\left(e^2 - 1\right)$$

- Exercise 58, § 7.2

$$\int \frac{2e^x - 2e^{-x}}{\left(e^x + e^{-x}\right)^2}dx$$

$$u(x) = e^x + e^{-x} \Rightarrow \frac{du}{dx} = e^x - e^{-x} \Rightarrow 2\left(e^x - e^{-x}\right)dx = (2e^x - 2e^{-x})dx = 2du$$

$$\therefore \int \frac{2e^x - 2e^{-x}}{\left(e^x + e^{-x}\right)^2}dx = \int \frac{2du}{u} = 2\int \frac{du}{u} = 2\ln|u| + C = \ln\left(e^x + e^{-x}\right)^2 + C$$

LOGARITHMIC DIFFERENTIATION

This is a useful shortcut technique when dealing functions with variable base and exponent (i.e. functions of the form: $h(x) = (f(x))^{g(x)}$ where neither f nor g are constant functions) or functions dealing with (powers) of products of other functions **of the same type** which would otherwise be tedious to compute. In the former case (the $(f(x))^{g(x)}$ case) one has no choice, *only* logarithmic differentiation can work. In the latter case, logarithmic differentiation presents itself as a useful labor-saving option.

Logarithmic differentiation is a special kind of implicit differentiation, and a relatively simple procedure:

1. First act on both sides of the expression with ln(…) (you're transforming the equation to its logarithmic representation)
2. Use any algebraic simplifying properties of logarithm[1] to simplify the expression.

[1] **Recall (from Oct. 18 notes, pp8-9): (For any base *a* > 0)**

a.) $a^0 = 1$

b.) $a^x \cdot a^y = a^{(x+y)}$

c.) $a^x / a^y = a^x \cdot a^{-y} = a^{(x-y)}$

d.) $\left(a^x\right)^y = a^{xy}$

From the above, we can derive:

a.) $\log_a(1) = \log_a\left(a^0\right) = 0$

b.) $\log_a\left(a^x \cdot a^y\right) = \log_a\left(a^{x+y}\right) = x + y = \log_a\left(a^x\right) + \log_a\left(a^y\right)$

Hence renaming: $u = a^x, v = a^y$ we get the result:

3. Take the derivative of both sides
4. Isolate $\frac{dy}{dx}$ and re-write expression in terms of *x*.

- Example (48, §7.5)

$$y = (1+x)^{1/x}$$
$$1.)\ \ln y = \ln\left[(1+x)^{1/x}\right] = \tfrac{1}{x}\ln(1+x)$$
$$2.)\ \frac{d}{dx}\ln y = \frac{1}{y}\frac{dy}{dx} = \frac{d}{dx}\left[\tfrac{1}{x}\ln(1+x)\right]$$
$$\Leftrightarrow \frac{1}{y}\frac{dy}{dx} = -\frac{1}{x^2}\ln(1+x) + \frac{1}{x}\cdot\frac{1}{1+x}$$
$$3.)\ \Rightarrow \frac{dy}{dx} = y\left[\tfrac{1}{x}\left(-\tfrac{1}{x}\ln(1+x) + \frac{1}{1+x}\right)\right]$$
$$= \frac{(1+x)^{1/x}}{x}\left[-\frac{\ln(1+x)}{x} + (1+x)^{-1}\right]$$
$$= -\frac{(1+x)^{1/x}\ln(1+x)}{x^2} + \frac{(1+x)^{1/x-1}}{x}$$

- Example (45, §7.5)

$$y = x^{2/x} \Rightarrow \ln y = \ln\left(x^{2/x}\right) = \frac{2}{x}\ln x$$
$$\Rightarrow \frac{1}{y}y' = \frac{d}{dx}\left[\frac{2}{x}\ln x\right] = -\frac{2}{x^2}\ln x + \frac{2}{x}\cdot\frac{1}{x} = 2x^{-2}(1-\ln x)$$
$$\Rightarrow y' = y\left(2x^{-2}(1-\ln x)\right) = 2x^{(x/2)-2}(1-\ln x)$$

$$\log_a(uv) = \log_a u + \log_a v$$

c.) $\log_a\left(\frac{a^x}{a^y}\right) = \log_a\left(a^{x-y}\right) = x - y = \log_a\left(a^x\right) - \log_a\left(a^y\right)$

Hence renaming: $u = a^x, v = a^y$ we get the result:

$$\log_a\left(\frac{u}{v}\right) = \log_a u - \log_a v$$

d.) $\log_a\left(\left(a^x\right)^y\right) = \log_a\left(a^{xy}\right) = xy = yx = y\log_a a^x$

Hence renaming: $u = a^x$ we get the result:

$$\log_a\left(u^y\right) = y\log_a u$$

- Example (41, §7.5)

$$y=\frac{x^2\sqrt{3x-2}}{(x-1)^2}$$

Unlike the previous two examples, logarithmic differentiation is not the *only* way to find the derivative. But it sure is the easiest way!

$$y=\frac{x^2\sqrt{3x-2}}{(x-1)^2}=\frac{x^2(3x-2)^{1/2}}{(x-1)^2}$$

$$\ln y=\ln\left[\frac{x^2(3x-2)^{1/2}}{(x-1)^2}\right]=\ln x^2+\ln\left((3x-2)^{1/2}\right)-\ln\left((x-1)^2\right)$$

$$=2\ln x+\tfrac{1}{2}\ln(3x-2)-2\ln(x-1)$$

$$\Rightarrow \frac{1}{y}y'=\frac{d}{dx}\left[2\ln x+\tfrac{1}{2}\ln(3x-2)-2\ln(x-1)\right]$$

$$=\frac{2}{x}+\tfrac{1}{2}\cdot\frac{3}{3x-2}-\frac{2}{x-1}\Rightarrow y'=y\left(\frac{2}{x}+\frac{3/2}{3x-2}-\frac{2}{x-1}\right)$$

$$=\frac{x^2(3x-2)^{1/2}}{(x-1)^2}\left[\frac{2}{x}+\frac{3/2}{(3x-2)}-\frac{2}{(x-1)}\right]$$

$$=\frac{2x(3x-2)^{1/2}}{(x-1)^2}+\frac{3x^2}{2(3x-2)^{1/2}(x-1)^2}-\frac{2x^2(3x-2)^{1/2}}{(x-1)^3}$$

$$=\frac{x^2(3x-2)^{1/2}}{(x-1)^2}\left[\frac{2(3x-2)(x-1)+\frac{3}{2}x(x-1)-2x(3x-2)}{x(3x-2)(x-1)}\right]$$

$$=\frac{x[4(3x-2)(x-1)+3x(x-1)-4x(3x-2)]}{2(x-1)^3\sqrt{3x-2}}$$

$$=\frac{x\left[4(3x^2-5x+2)+(3x^2-3x)-(12x^2-8x)\right]}{2(x-1)^3\sqrt{3x-2}}$$

$$=\frac{x\left[3x^2-15x+8\right]}{2(x-1)^3\sqrt{3x-2}}=\frac{3x^3-15x^2+8x}{2(x-1)^3\sqrt{3x-2}}$$

Note: The algebraic details (of combining everything into one fraction again, canceling, and multiplying the rest out) were done to show agreement with the text's answer. Though tedious, note how easy the step was in terms of computing the actual derivative (the calculus was a one-step procedure). This would have been a *far* more difficult problem (aside from the algebra) if one had attempted to calculate the derivative by means of the chain rule, quotient rule, etc.

The text derives the differentiation and integration formulae for exponential and logarithmic functions (for bases other than *e*) in manners analogous to the way the remaining formulae were derived in last week's notes. The remaining formulae are:

$$\frac{d}{dx}a^x = a^x \ln a = (\ln a)a^x \qquad \textbf{(I)}$$

$$\frac{d}{dx}a^{u(x)} = (\ln a)a^{u(x)}u'(x) \qquad \textbf{(J)}$$

$$\int a^x dx = \frac{1}{\ln a}a^x + C \qquad \int_c^d a^x dx = \frac{1}{\ln a}a^x\Big|_c^d = \frac{1}{\ln a}\left(a^d - a^c\right) \qquad \textbf{(K)}$$

$$\int a^u du = \frac{1}{\ln a}a^u + C \qquad \int_{u(c)}^{u(d)} a^u du = \frac{1}{\ln a}a^u\Big|_{u(c)}^{u(d)} = \frac{1}{\ln a}\left(a^{u(d)} - a^{u(c)}\right) \qquad \textbf{(L)}$$

- Example (18, §7.6)

$$\int \frac{(\ln x)^2}{x}dx = \int (\ln x)^2 \frac{dx}{x}$$

$$\Rightarrow u(x) = \ln x \Rightarrow \frac{du}{dx} = \frac{1}{x} \Rightarrow du = \frac{dx}{x}$$

$$\therefore \int (\ln x)^2 \frac{dx}{x} = \int u^2 du = \tfrac{1}{3}u^3 + C = \tfrac{1}{3}(\ln x)^3 + C$$

- Example (17, §7.6)

$$\int \frac{x^2 + 2x + 3}{x^3 + 3x^2 + 9x}dx$$

$$u(x) = x^3 + 3x^2 + 9x \Rightarrow \frac{du}{dx} = 3x^2 + 6x + 9$$

$$\Rightarrow du = 3(x^2 + 2x + 3)dx \Rightarrow \tfrac{1}{3}du = (x^2 + 2x + 3)dx$$

$$\Rightarrow \int \frac{\frac{1}{3}du}{u} = \tfrac{1}{3}\int \frac{du}{u} = \tfrac{1}{3}\ln|u| + C = \tfrac{1}{3}\ln\left|x^3 + 3x^2 + 9x\right| + C$$

$$\Rightarrow \ln\left|\left(x^3 + 3x^2 + 9x\right)^{1/3}\right| + C$$

- Example (24, §7.6)

$$\int_0^2 \frac{1}{1+\sqrt{2x}}dx = \int_0^2 \left(1 + \sqrt{2}x^{1/2}\right)^{-1} dx$$

$$u(x)=1+\sqrt{2}x^{1/2} \Rightarrow \frac{du}{dx} = \tfrac{\sqrt{2}}{2}x^{-1/2}dx \Rightarrow dx = \tfrac{2}{\sqrt{2}}x^{1/2}du = \left(\frac{u(x)-1}{\sqrt{2}}\right)\frac{2}{\sqrt{2}}du(x)$$

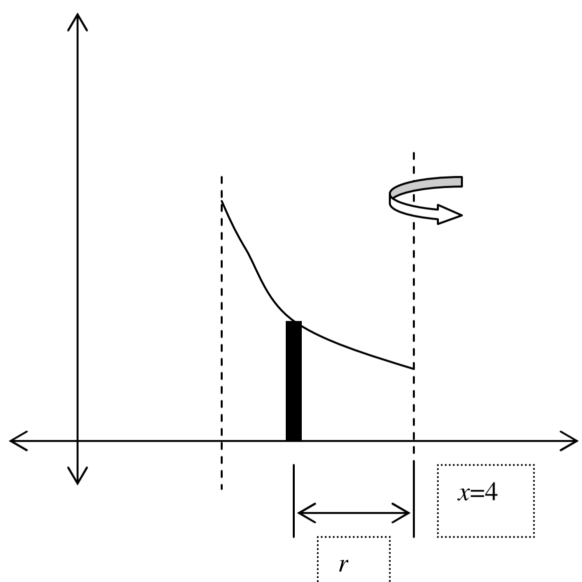
$$\Rightarrow dx = (u \quad -1)du$$

$$\therefore \int_0^2 \frac{1}{1+\sqrt{2x}}dx = \int_{u(0)}^{u(2)} \frac{u-1}{u}du = \int_1^3 \left(1-u^{-1}\right)du = \left(u-\ln|u|\right)|_1^3 = (3-\ln 3)-(1-\ln 1) = 2-\ln 3$$

Note the subtlety in terms of the *u*-substitution in this problem…Look over carefully!

- Example (41, §7.6)

Revolve $xy = 1$, $x = 1$ & $x = 4$ about the line $x = 4$. Find volume



As the above illustration shows, it's easiest to use the method of shells. **Note carefully:** Since the axis is $x = 4$ (not the *y*-axis!) the radius of rotation is: $r(x) = 4 - x$.

$$V = 2\pi\int_1^4 r(x)y(x)dx = 2\pi\int_1^4 (4-x)\frac{dx}{x} = 2\pi\int_1^4\left(\frac{4}{x}-1\right)dx = 8\pi(4\ln x - x)\Big|_1^4$$

$$= 8\pi\left(\ln 4^4 - 4\right) - 8\pi(4\ln 1 - 1) = 8\pi\left(\ln 4^4 - 3\right)$$