MA261 **Calculus I: Notes for Week of October 22**
Fall 2007 Kallfelz

CALCULUS OF EXPONENTIAL AND LOGARITHMIC FUNCTIONS (Cont.)

Last week, as summarized in last week's notes, all differentiation and integration formulae (i.e. formulae (**A**) – (**L**)) for exponential and their associated integral formulae (whether definite or indefinite) were obtained by way of the definition of the natural logarithm:

$$\ln x = \int_1^x \frac{dt}{t}$$

using the Fundamental Theorem of Calculus, and the chain rules, and the composition of inverse property: $e^{\ln x} = x = \ln\left(e^x\right)$

Recall Formulae (**A**) – (**L**)

$$\frac{d}{dx}\ln x = \frac{d}{dx}\int_1^x \frac{1}{t}dt = \frac{1}{x} \qquad \textbf{(A)}$$

$$\frac{d}{dx}\ln(u(x)) = \frac{d}{du}\ln u\frac{du}{dx} = \frac{1}{u(x)}u'(x) \qquad \textbf{(B)}$$

$$\int \frac{1}{x}dx = \ln|x| + C \qquad \int_a^b \frac{1}{x}dx = \ln|x|\Big|_a^b = \ln|b| - \ln|a| = \ln\left|\frac{b}{a}\right| \qquad \textbf{(C)}$$

$$\int \frac{1}{u}du = \ln|u| + C \qquad \int_{u(a)}^{u(b)} \frac{1}{u}du = \ln|u|\Big|_{u(a)}^{u(b)} = \ln|u(b)| - \ln|u(a)| = \ln\left|\frac{u(b)}{u(a)}\right| \qquad \textbf{(D)}$$

$$\frac{d}{dx}e^x = e^x \qquad \textbf{(E)} \qquad \frac{d}{dx}e^{u(x)} = e^{u(x)}\frac{d}{dx}u = e^u u' \qquad \textbf{(F)}$$

$$\int e^x dx = e^x + C \qquad \int_a^b e^x dx = e^x \Big|_a^b = e^b - e^a \qquad \textbf{(G)}$$

$$\int e^u du = e^u + C \qquad \int_{u(a)}^{u(b)} e^u du = e^u \Big|_{u(a)}^{u(b)} = e^{u(b)} - e^{u(a)} \qquad \textbf{(H)}$$

$$\frac{d}{dx}a^x = a^x \ln a = (\ln a)a^x \qquad \textbf{(I)}$$

$$\frac{d}{dx}a^{u(x)} = (\ln a)a^{u(x)}u'(x) \qquad \textbf{(J)}$$

$$\int a^x dx = \frac{1}{\ln a}a^x + C \qquad \int_c^d a^x dx = \frac{1}{\ln a}a^x\Big|_c^d = \frac{1}{\ln a}\left(a^d - a^c\right) \qquad \textbf{(K)}$$

$$\int a^u du = \frac{1}{\ln a}a^u + C \qquad \int_{u(c)}^{u(d)} a^u du = \frac{1}{\ln a}a^u\Big|_{u(c)}^{u(d)} = \frac{1}{\ln a}\left(a^{u(d)} - a^{u(c)}\right) \qquad \textbf{(L)}$$

The above formulae can also be derived using more general considerations starting with calculating the derivative of e^x, as done in §7.2, text:

$$\frac{d}{dx}e^x = \lim_{\Delta x\to 0}\frac{e^{x+\Delta x} - e^x}{\Delta x} = \lim_{\Delta x\to 0}\frac{e^x\left(e^{\Delta x} - 1\right)}{\Delta x} = e^x \lim_{\Delta x\to 0}\frac{\left(e^{\Delta x} - 1\right)}{\Delta x}$$

$$= e^x \lim_{\Delta x\to 0}\frac{\left[(1+\Delta x)^{\frac{1}{\Delta x}} - 1\right]}{\Delta x} \approx e^x \lim_{\Delta x\to 0}\frac{(1+\Delta x) - 1}{\Delta x} = e^x \lim_{\Delta x\to 0}\frac{\Delta x}{\Delta x} = e^x$$

…where the definition $e^x = \lim_{\Delta x\to 0}(1+\Delta x)^{\frac{1}{\Delta x}} = \lim_{u\to\infty}\left(1+\frac{1}{u}\right)^u$ was made use of, and the approximation (valid for small Δx) employed: $(1+\Delta x)^{1/\Delta x} \approx (1+\Delta x)$.

Hence, this is an alternate basis for deriving **(E)**, and its associated chain rule and antiderivative formulae **(F) – (H).**

From these above considerations, a more general treatment of inverse functions is possible (§7.3):

Given a function $f: R \to R, f$ *is invertible* (i.e., there exists $f^{-1}: R \to R$ such that $f\left(f^{-1}(x)\right) = x = f^{-1}(f(x))$) provided: *f* is 1-1 (i.e.: $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$ for any x_1, x_2. (Or conversely: $x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$.)

This condition is both a necessary and a sufficient condition. If it's violated, then its inverse 'function' fails its very own definition! To see this, suppose we consider the negation of the converse above:

$$x_1 \neq x_2 \Rightarrow f(x_1) = f(x_2)$$

Let: $y_1 = f(x_1), y_2 = f(x_2)$. Because of the above equality, we can say: $y_1 = y = y_2$. Let's assume there exists an inverse function for *f*.

Then, for any *y*: $\quad f^{-1}(f(x)) = x = f^{-1}(y)$

But the above says: $f^{-1}(y) = x_1, f^{-1}(y) = x_1, \& x_1 \neq x_2$. In other words, the 'function' f^{-1} maps (at least) one domain point to two different range points. **This violates the definition of what it means to *be* a function.**

There are graphical ways of determining is a function is invertible (or shrinking its domain to make it invertible). The *horizontal line test* is a geometric way of determining $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$ (or not) Basically if a horizontal line intersects the graph of *f* **at only one point** we're guaranteed that f^{-1} passes the **vertical line test,** i.e. f^{-1} **is a function.**[1] Also, if a function is **strictly monotone increasing/decreasing** (i.e. $f'(x) > 0$ or $f'(x) < 0$ for all *x* in its domain, then it's obviously 1-1.

- Example (#22, 7.3) $\quad f(x) = \dfrac{x+2}{x}$

There are several ways to show it's 1-1. One could apply the definition:

$$f(x_1) = f(x_2) \Rightarrow \frac{x_1+2}{x_1} = \frac{x_2+2}{x_2} \Rightarrow 1 + \frac{2}{x_1} = 1 + \frac{2}{x_2} \Rightarrow x_1 = x_2$$

One could also use calculus (the First derivative test):

$f(x) = \dfrac{x+2}{x} = 1 + 2x^{-1} \Rightarrow f'(x) = -2x^{-2}$. Now everywhere where the function is differentiable (it has a Type 2 critical point at *x* =0, which is the

[1] Since $f^{-1}(y) = x$, then for every (x, y) lying on the graph of *f*, then (y, x) lies on the graph of f^{-1}. (This is the **mirror property** of the inverse: it's symmetric about the line $y = x$.

function's vertical asymptote) its derivative is negative: $f'(x) < 0$ for all $x \neq 0$. Hence it's strictly **monotone decreasing**

To find its inverse, solve for *x* in terms of *y* and then replace:

$$f(x) = 1 + \frac{2}{x} = y \Rightarrow \frac{2}{x} = y - 1 \Rightarrow x = \frac{2}{y-1} \Rightarrow f^{-1}(x) = \frac{2}{x-1}$$

Check:

$$f^{-1}(f(x)) = \frac{2}{\left(\frac{x+2}{x}\right) - 1} = \frac{2}{\left(1 + \frac{2}{x}\right) - 1} = \frac{2}{\frac{2}{x}} = x$$

$$f\left(f^{-1}(x)\right) = \frac{\left(\frac{2}{x-1}\right) + 2}{\left(\frac{2}{x-1}\right)} = \frac{(x-1)\left[\frac{2}{x-1} + 2\right]}{2} = \frac{2 + 2(x-1)}{2} = 1 + x - 1 = x$$

Hence according to Thm 7.7 (Using the Chain Rule)

$$x = f\left(f^{-1}(x)\right) \Rightarrow \frac{d}{dx} x = 1 = \frac{d}{dx} f\left(f^{-1}(x)\right) = f'\left(f^{-1}(x)\right) \frac{d}{dx} f^{-1}(x)$$
$$\Rightarrow \frac{d}{dx} f^{-1}(x) = \frac{1}{f'\left(f^{-1}(x)\right)}$$

…A quite general result. Consider specifically: $f(x) = e^x, f^{-1}(x) = \ln x$, then:

$$\frac{d}{dx} \ln x = \frac{1}{f'\left(f^{-1}(x)\right)} = \frac{1}{e^{(\ln x)}} = \frac{1}{x}$$

Hence formulae **(A) – (D)** are recovered.

- <u>Example: </u>(#24, p. 392)

$$y = \frac{-\sqrt{x^2+1}}{x} + \ln\left(x + \sqrt{x^2+1}\right)$$

$$y' = \frac{x\left(-\tfrac{1}{2}(x^2+1)^{-1/2}(2x)\right) + \left(x^2+1\right)^{1/2}}{x^2} + \frac{1 + \tfrac{1}{2}\left(x^2+1\right)^{-1/2}(2x)}{x + (x^2+1)^{1/2}}$$

$$= \left(x^2+1\right)^{-1/2}\left\{\frac{-x^2+x^2+1}{x^2}\right\} + \frac{1 + x\left(x^2+1\right)^{-1/2}}{x + \left(x^2+1\right)^{1/2}}$$

$$= \frac{1}{x\left(x^2+1\right)^{1/2}} + \frac{1 + x\left(x^2+1\right)^{-1/2}}{x + \left(x^2+1\right)^{1/2}}$$