MA261 **Calculus I: Notes for October 2 & 4**
Fall 2007 Kallfelz

- *THE NEWTON-RAPHSON METHOD FOR FINDING ROOTS*

Given some n-th degree polynomial $p_n(x) = a_n x^n + a_{n-1}x^{n-1} + ... + a_1 x + a_0$, it's only under quite special circumstances that one can easily find the roots or the value(s) of x (call such values x^*) such that: $p_n(x^*) = 0$. These algebraic root-finding procedures are limited to quadratic and cubic polynomials (n = 2 or 3). A general method does exist in algebra in which one can find the *set of all possible* roots for an n-th degree polynomial, i.e. the *Rational Roots Theorem* (RRT). But this theorem only offers up what set of *possible* roots may obtain. Typically, one uses the RRT as a means to find the set of all *possible* roots. It's a very inefficient and restrictive procedure, since: a.) the RRT can only apply to cases when the leading coefficient a_n and the final coefficient a_0 are *rational numbers*. b.) In such cases, however, success is by no means easily obtainable. The RRT merely presents all possible alternatives for *rational* roots (however the polynomial may have irrational roots as well). One must still devise an even more cumbersome and inefficient trial-and-error procedure usually based on synthetic division, etc., to determine which of those possible roots are the *actual* roots.

Even worse, the RRT can't be extended to searches for roots of any general function $f(x)$, whether rational or otherwise.

The Newton-Raphson Method, on the other hand, is a procedure which given certain more general conditions (differentiability, etc.) the roots of $f(x)$ can be obtained, given certain sufficiency conditions concerning, for instance, the nearness of the initial starting –point x_0 from the actual root c (i.e. the point c such that $f(c) = 0$.)

The idea behind it is simple enough:

1. Start at a point x_0 you think is close to the root c
2. If your guess is correct (i.e. if $x_0 = c$), STOP. Otherwise, extend a tangent line from that point to where it cuts the x-axis.
3. Choose that point as your next guess and repeat procedure in step 2.

Stated mathematically, the tangent line (in point-slope form) formed from your guess in 2. is:

$$y - f(x_0) = f'(x_0)(x - x_0)$$

When the line cuts the x-axis (let's say, at point x_1) then certainly $y = 0$ and the tangent line equation becomes:

$$0 - f(x_0) = f'(x_0)(x_1 - x_0)$$

Solving for x_1 : $$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

Repeating the procedure at step 2 (if x_1 isn't the root…How would you know if x_1 isn't the root? Obvious. If $f(x_1) \neq 0$ then you haven't found the root yet!):

Solving for x_2 : $$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

…Etc.

This is an iterative procedure for obtaining roots, which in general form is expressed by the recursion relation:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

- For example, consider the function $f(x) = x^3 - 3x^2 + 2x$

  It's simple enough to the extent that its roots are easily obtainable using straightforward algebra. $x^3 - 3x^2 + 2x = x(x^2 - 3x + 2) = x(x-2)(x-1)$

  So it has three roots: $c_1 = 0$, $c_2 = 1$, $c_3 = 2$

  Suppose we try the Newton-Raphson procedure and **initially guess: x_0 = -5**

  $f(x) = x^3 - 3x^2 + 2x$ $\qquad$ $f'(x) = 3x^2 - 6x + 2$ $\qquad$ x_{n+1}

Using the Newton-Raphson routine (programmed in EXCEL):

n	x_n	$f(x_n)$	$f'(x_n)$	x_{n+1}
0	-5	-210	107	-3.037383178
1	-3.03738	-61.7738	47.90139	-1.747778989
2	-1.74778	-17.9987	21.65087	-0.916461399
3	-0.91646	-5.12236	10.01847	-0.405169444
4	-0.40517	-1.36934	4.923503	-0.127046505
5	-0.12705	-0.30457	2.810701	-0.018687047
6	-0.01869	-0.03843	2.11317	-0.000501932
7	-0.0005	-0.001	2.003012	-3.77462E-07
8	-3.8E-07	-7.5E-07	2.000002	-2.13716E-13
9	-2.1E-13	-4.3E-13	2	-6.84857E-26
10	-6.8E-26	-1.4E-25	2	0

…You can see that after 10 iterations, the first root $c_1 = 0$ was found**. Note, however, that after 7 iterations the root was effectively found, since x_7 is of the order of magnitude of 10^{-7} (i.e., almost 0).**

What would have happed had we guessed $x_0 = 0.5$ (halfway between $c_1 = 0$ and $c_2 = 1$?)

n	x_n	$f(x_n)$	$f'(x_n)$	x_{n+1}
0	0.5	0.375	-0.25	2
1	2	0	2	2

After 1 iteration, it converged to ***a* root, but not the expected one! (It converged to $c_3 = 2$!** This indicates some of the inherent shortcomings of the N-R procedure—like the **Second Derivative Test,** there are restrictions which determine when it will work, and when it may not work.

Nevertheless, due to the fact that $f(x)$ is everywhere differentiable, the N-R procedure *did* produce a result…it gave us the root $c_3 = 2$ when we started off guessing at a point exactly halfway between $c_1 = 0$ and $c_2 = 1$.

Automatically the question may come to mind: would the N-R procedure have converged to a local root (i.e. $c_1 = 0$ or $c_2 = 1$) if we'd *not* initially guessed halfway between them, but initially guessed slightly closer to one or the other? You're tempted to say "yes", but again, the devil's in the details. For example, suppose you initially guessed $x_0 = 0.42$:

n	x_n	$f(x_n)$	$f'(x_n)$	x_{n+1}
0	0.42	0.384888	0.0092	-41.41565217
1	-41.4157	-76267.1	5396.263	-27.28234158
2	-27.2823	-22594.5	2398.673	-17.86275494
3	-17.8628	-6692.57	1066.411	-11.58696201
4	-11.587	-1981.59	474.2948	-7.409000184
5	-7.409	-586.202	211.1339	-4.632552
6	-4.63255	-173.064	94.17693	-2.794906797
7	-2.79491	-50.8568	42.20395	-1.589883384
8	-1.58988	-14.7817	19.12249	-0.816879942
9	-0.81688	-4.18074	8.903158	-0.347300894
10	-0.3473	-1.09835	4.445659	-0.100240513
11	-0.10024	-0.23163	2.631588	-0.01222036
12	-0.01222	-0.02489	2.07377	-0.000217797
13	-0.00022	-0.00044	2.001307	-7.11173E-08
14	-7.1E-08	-1.4E-07	2	-7.58651E-15
15	-7.6E-15	-1.5E-14	2	-8.5197E-29
16	-8.5E-29	-1.7E-28	2	0
17	0	0	2	0

After 17 iterations, it converged to the root closest to $x_0 = 0.42$, namely $c_1 = 0$.

But watch what happens if you guessed $x_0 = 0.43$:

n	x_n	$f(x_n)$	$f^{/}(x_n)$	x_{n+1}
0	0.43	0.384807	-0.0253	15.63976285
1	15.63976	3122.993	641.968	10.7750449
2	10.77504	924.2452	285.6545	7.539509812
3	7.53951	273.1239	127.2956	5.393921638
4	5.393922	80.43753	56.91964	3.980744541
5	3.980745	23.50269	25.65451	3.064621638
6	3.064622	6.736164	11.78799	2.493178582
7	2.493179	1.835986	5.688747	2.170438623
8	2.170439	0.432976	3.10978	2.031208071
9	2.031208	0.065368	2.19017	2.001361821
10	2.001362	0.002729	2.008176	2.000002773
11	2.000003	5.55E-06	2.000017	2
12	2	2.31E-11	2	2
13	2	0	2	2

This is *really* surprising! Like in the case of $x_0 = 0.5$ it converged to the *farthest* root $c_3 = 2$ after 13 iterations!

Suppose we initially guessed slightly closer to $c_2 = 1$, for instance $x_0 = 0.56$:

n	x_n	$f(x_n)$	$f^{/}(x_n)$	x_{n+1}
0	0.56	0.354816	-0.4192	1.406412214
1	1.406412	-0.33928	-0.50449	0.73387849
2	0.733878	0.247275	-0.78754	1.047862833
3	1.047863	-0.04775	-0.99313	0.999779189
4	0.999779	0.000221	-1	1
5	1	-2.2E-11	-1	1

As expected, it converged to the closest root $c_2 = 1$ after 5 iterations. However, watch what happens in the case of $x_0 = 0.55$:

n	x_n	$f(x_n)$	$f^{/}(x_n)$	x_{n+1}
0	0.55	0.358875	-0.3925	1.46433121
1	1.464331	-0.36422	-0.35319	0.433100937
2	0.433101	0.384712	-0.03588	11.15637818
3	11.15638	1037.495	308.4561	7.792869789
4	7.79287	306.6511	137.4292	5.561531944
5	5.561532	90.35288	61.42272	4.090531009
6	4.090531	26.42831	27.65415	3.134858351
7	3.134858	7.595015	12.67286	2.53554495
8	2.535545	2.085114	6.073695	2.192242577
9	2.192243	0.502462	3.264327	2.038317583
10	2.038318	0.081096	2.23431	2.002021756

11	2.002022	0.004056	2.012143	2.000006102
12	2.000006	1.22E-05	2.000037	2
13	2	1.12E-10	2	2
14	2	0	2	2

Another unexpected result!

What is going on in all these cases is that the ***sufficiency condition for convergence*** is being violated. Recall the formula on **p. 209** of the text:

If $\left|\dfrac{f(x)f''(x)}{[f'(x)]^2}\right| < 1$, *then* convergence is guaranteed.

It's important to understand that this is a **sufficiency condition, not a necessary condition!** In other words, it's *not* saying that *if* convergence is guaranteed, *then* the above inequality is satisfied. That would make it a necessary condition, something stronger. But in all these weird cases, the above sufficiency condition is violated. (Nevertheless, note that convergence still occurred, just not to the expected points). Using EXCEL (see last column, highlit in red. Obviously all these values far exceed 1:

x	$f(x)$	$f'(x)$	$f''(x)$	$\lvert f(x)f''(x)/[f'(x)]^2\rvert$
0.5	0.375	-0.25	-3	18
0.42	0.384888	0.0092	-3.48	15824.79017
0.43	0.384807	-0.0253	-3.42	2056.023278
0.56	0.354816	-0.4192	-2.64	5.330458598
0.55	0.358875	-0.3925	-2.7	6.289666924

On the other hand, with respect to our initial guess of $\boldsymbol{x_0} = \mathbf{-5}$, which converged to its closest root $c_1 = 0$:

x	$f(x)$	$f'(x)$	$f''(x)$	$\lvert f(x)f''(x)/[f'(x)]^2\rvert$
-5	-210	107	-36	0.660319679

…the sufficiency condition is satisfied.

- *DIFFERENTIALS*

Another interesting application concerns the definition of derivative. Recall:

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x}$$

What if for whatever reasons we halted the limit procedure at some small (but finite) value of Δx ? Then the definition reads:

$$f'(x) \approx \frac{f(x+\Delta x) - f(x)}{\Delta x} \approx \frac{\Delta f}{\Delta x} \Rightarrow \Delta f \approx f'(x)\Delta x$$

…Where the "≈" means "is approximately equal to." Notice from the above we isolated the variation of the dependent variable Δf as a simple formula of its derivative $f'(x)$ and the variation of the independent variable Δx.

It turns out that this is a useful shortcut formula for getting a rather accurate linear estimate of the functional variation, or variation of the dependent variable. Take, for example, the function:

$$f(x) = \sqrt[3]{x^3 + 10x^2 - 1}$$

To compute Δf entails computing:

$$\Delta f(x) = \sqrt[3]{(x+\Delta x)^3 + 10(x+\Delta x) - 1} - \sqrt[3]{x^3 + 10x^2 - 1}$$

…Which is a computationally costly procedure. Using the differential approach, on the other hand:

$$f'(x) = \frac{d}{dx}\sqrt[3]{x^3 + 10x^2 - 1} = \tfrac{1}{3}\left(x^3 + 10x^2 - 1\right)^{-\frac{2}{3}}\left(3x^2 + 20x\right) = \frac{3x^2 + 20x}{3\left(\sqrt[3]{x^3 + 10x^2 - 1}\right)^2}$$

Hence: $$\Delta f \approx \frac{3x^2 + 20x}{3\left(\sqrt[3]{x^3 + 10x^2 - 1}\right)^2}\Delta x$$

The table below summarizes the accuracy of the approach, for varying values of x and Δx., for the case of the above function (Note the % error highlit in red)

x	**Δ*x***	**Δ*f* (exact)**	**Δ*f* (approximate)**	**% error**
1	0.1	0.16208158	0.165173326	**1.907522487**
1	0.05	0.08178562	0.082586663	**0.97944506**
1	0.001	0.0016514	0.001651733	**0.020133351**
2	0.1	0.13230258	0.133091458	**0.596268862**
2	0.05	0.06634552	0.066545729	**0.301763687**
2	0.001	0.00133083	0.001330915	**0.006108679**
-1	0.1	-0.14617301	-0.141666667	**3.082880763**
-1	0.05	-0.07191981	-0.070833333	**1.510681836**
-1	0.001	-0.00141709	-0.001416667	**0.029667484**

- ***INTEGRATION (DEFINITE AND INDEFINITE)***

- **Defn.:** Given a continuous function $f(x)$, its antiderivative is a function $F(x)$ such that $\frac{d}{dx}F(x)=F'(x)=f(x)$

Note1: The antiderivative for a function $f(x)$ gets the notation: $F(x)=\int f(x)dx$
And is denoted (for reasons we'll see shortly) as an **indefinite integral.**

Recall the simple derivative formulae:

1. $\frac{d}{dx}C=0$
2. $\frac{d}{dx}(af(x)\pm bg(x))=af'(x)\pm bg'(x)$
3. $\frac{d}{dx}x^q=qx^{q-1}$

The corresponding anti-derivative formulae are:

1. $F(x)=\int 0\cdot dx=C$, since: $\frac{d}{dx}F(x)=\frac{d}{dx}C=0$
2. $\int[af(x)\pm bg(x)]dx=a\int f(x)dx\pm b\int g(x)dx$,
   Since:
   $$\frac{d}{dx}\left[a\int f(x)dx\pm b\int g(x)dx\right]=\frac{d}{dx}[aF(x)\pm bG(x)]$$
   $$=a\frac{d}{dx}F(x)\pm b\frac{d}{dx}G(x)=af(x)\pm bg(x)$$

3. $\int x^q dx=\frac{x^{q+1}}{(q+1)}+C$, since:
   $$\frac{d}{dx}\left[\frac{x^{q+1}}{q+1}++C\right]=\frac{d}{dx}\left[\frac{x^{q+1}}{(q+1)}\right]+\frac{d}{dx}C$$
   $$=\frac{q+1}{q+1}x^{q+1-1}+0=x^q$$

Note2: It's because of anti-derivative formula 1. that the notion 'indefinite' gets its name. For there are **infinitely many different anti-derivative functions** (differing by an arbitrary additive constant) for any associated function $f(x)$. I.e.:

$F(x) = x^3$ is an anti-derivative for $f(x) = 3x^2$
$F(x) = x^3 + 1$ is an anti-derivative for $f(x) = 3x^2$
$F(x) = x^3 - 1$ is an anti-derivative for $f(x) = 3x^2$

Etc…

- Example (#10, § 5.1) $\int\left(\sqrt{x}+\frac{1}{2\sqrt{x}}\right)dx$

To solve, simply re-write in a way that antiderivative formulae 1., 2., 3. can be applied:

$$\int\left(\sqrt{x}+\frac{1}{2\sqrt{x}}\right)dx = \int\left(x^{\frac{1}{2}}+\tfrac{1}{2}x^{-\frac{1}{2}}\right)dx = \int x^{\frac{1}{2}}dx + \tfrac{1}{2}\int x^{-\frac{1}{2}}dx = \frac{x^{\left(\frac{1}{2}+1\right)}}{\left(\frac{1}{2}+1\right)} + \tfrac{1}{2}\cdot\frac{x^{\left(-\frac{1}{2}+1\right)}}{\left(-\frac{1}{2}+1\right)} + C$$

$$= \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + \tfrac{1}{2}\cdot\frac{x^{\frac{1}{2}}}{\frac{1}{2}} + C = \tfrac{2}{3}x^{\frac{3}{2}} + \tfrac{1}{2}\cdot 2x^{\frac{1}{2}} + C = \tfrac{2}{3}x^{\frac{3}{2}} + x^{\frac{1}{2}} + C = \tfrac{2}{3}x\sqrt{x} + \sqrt{x} + C = \sqrt{x}\left(\tfrac{2}{3}x+1\right) + C$$

Note that you can always check your answer, buy taking the derivative:

$$\frac{d}{dx}\left(\tfrac{2}{3}x\sqrt{x}+\sqrt{x}+C\right) = \frac{d}{dx}\left(\tfrac{2}{3}x^{\frac{3}{2}}+x^{\frac{2}{2}}+C\right) = \frac{d}{dx}\tfrac{2}{3}x^{\frac{3}{2}} + \frac{d}{dx}x^{\frac{1}{2}} + \frac{d}{dx}C$$

$$= \tfrac{2}{3}\cdot\tfrac{3}{2}x^{\left(\frac{3}{2}-1\right)} + \tfrac{1}{2}x^{\left(\frac{1}{2}-1\right)} + 0 = x^{\frac{1}{2}} + \tfrac{1}{2}x^{-\frac{1}{2}} = \sqrt{x} + \frac{1}{2\sqrt{x}}$$

- Example (#18, § 5.1) $\int\left(\frac{x^2+1}{x^2}\right)dx$

$$\int\left(\frac{x^2+1}{x^2}\right)dx = \int\left(1+x^{-2}\right)dx = \int 1\cdot dx + \int x^{-2}dx = \int x^0dx + \int x^{-2}dx$$

$$= \frac{x^{0+1}}{0+1} + \frac{x^{-2+1}}{-2+1} + C = x - x^{-1} + C = x - \frac{1}{x} + C$$

Check: $\frac{d}{dx}\left(x - x^{-1} + C\right) = 1 + x^{-2} = \frac{x^2+1}{x^2}$

- <u>Example (#19, §5.1)</u> $\int (x+1)(3x-2)dx$

$$\int (x+1)(3x-2)dx = \int (3x^2+x-2)dx = 3\int x^2 dx + \int x dx - 2\int dx$$
$$= 3\cdot\tfrac{1}{3}x^3 + \tfrac{1}{2}x^2 - 2\cdot x + C = x^3 + \tfrac{1}{2}x^2 - 2x + C$$

Check:

$$\frac{d}{dx}\left(x^3+\tfrac{1}{2}x^2-2x+C\right) = \frac{d}{dx}x^3 + \tfrac{1}{2}\frac{d}{dx}x^2 - 2\frac{d}{dx}x + \frac{d}{dx}C$$
$$= 3x^2 + \tfrac{1}{2}\cdot 2x - 2 + 0 = 3x^2 + x - 2 = (x+1)(3x-2)$$

- <u>Example (#38, §5.1)</u> Find the function $f(x)$ given: $f''(x) = x^{-\frac{3}{2}}$ and $f'(1) = 2, f(9) = -4$.

Solution: $f'(x) = \int f''(x)dx = \int x^{-\frac{3}{2}}dx = \dfrac{x^{-\frac{3}{2}+1}}{-\frac{3}{2}+1} + C = -2x^{-\frac{1}{2}} + C$

Inserting condition: $f'(1) = 2$:

$$f'(1) = -2\cdot 1^{-\frac{1}{2}} + C = 2 \Rightarrow C - 2 = 2 \Rightarrow C = 4$$

So: $f'(x) = -2x^{-\frac{1}{2}} + 4$, hence:

$$f(x) = \int f'(x)dx = \int\left(-2x^{-\frac{1}{2}}+4\right)dx = -2\int x^{-\frac{1}{2}}dx + 4\int dx = -2\cdot\tfrac{1}{\left(-\frac{1}{2}+1\right)}x^{\left(-\frac{1}{2}+1\right)} + 4x + C$$
$$= -2\cdot 2x^{\frac{1}{2}} + 4x + C = -4x^{\frac{1}{2}} + 4x + C$$

Inserting condition: $f(9) = -4$:

$$f(9) = -4\cdot 9^{\frac{1}{2}} + 4\cdot 9 + C = -12 + 36 + C = 24 + C = -4 \Rightarrow C = 28$$

So: $f(x) = -4x^{\frac{1}{2}} + 4x + 28 = 4\left(-\sqrt{x} + x + 7\right)$

Check: $f'(x) = \dfrac{d}{dx}f(x) = \dfrac{d}{dx}\left(-4x^{\frac{1}{2}} + 4x + 28\right) = -2x^{-\frac{1}{2}} + 4$

$$f''(x) = \frac{d}{dx}f'(x) = \frac{d}{dx}\left(-2x^{-\frac{1}{2}} + 4\right) = x^{-\frac{3}{2}}$$

- ***THE FUNDAMENTAL THEOREM OF CALCULUS AND DEFINITE INTEGRALS***

The Area Problem

Since the time of Archimedes, there were estimation procedures for obtaining areas and volumes for irregularly shaped objects. The idea was to insert rectangles/regular polygons, in which their areas were easy to determine (for instance if one were interested in calculating area). Then the procedure was to shrink them to the extent that the error/overlaps became relatively insignificant.

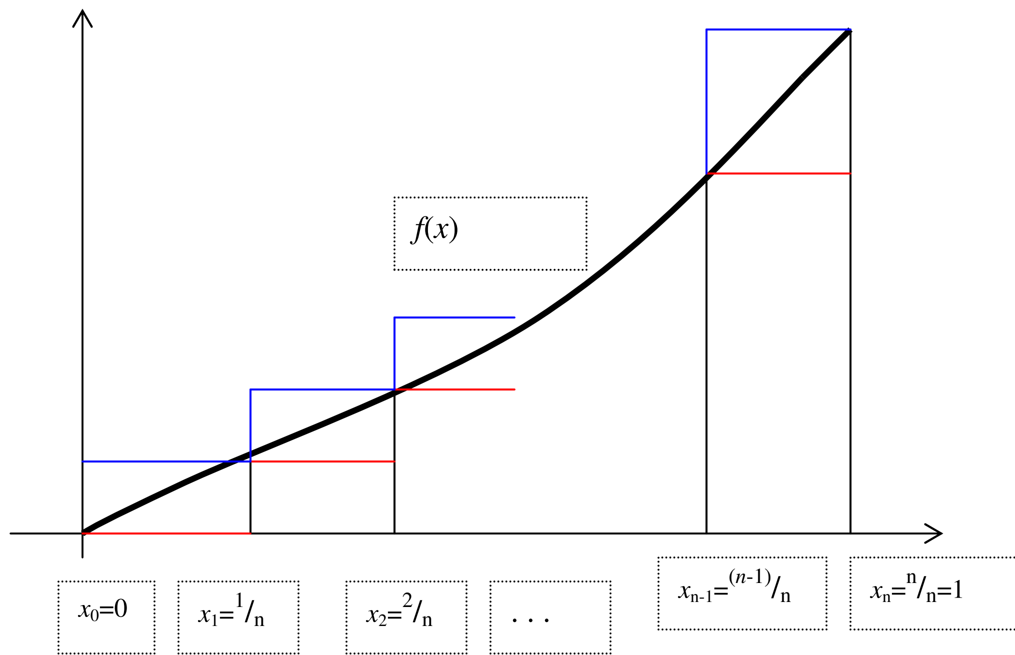
The above procedure, used by engineers and mathematicians since Antiquity, worked fairly well. But it's a fundamentally *approximate* procedure. Armed with the tools of the **limit concept,** (a central concept in Calculus) one can make this procedure exact.

Prior to examining an example, however, the following formulae involving series (i.e. sums of sequences) must be stated:

1. $\sum_{i=1}^{n} c = c + c + ... + c(n\_times) = nc$
2. $\sum_{i=1}^{n} i = 1 + 2 + 3 + ... + (n-1) + n = \frac{n(n-1)}{2}$
3. $\sum_{i=1}^{n} i^2 = 1^2 + 2^2 + 3^2 + ... + n^2 = \frac{n(n-1)(2n-1)}{6}$
4. $\sum_{i=1}^{n} i^3 = 1^3 + 2^3 + ... + n^3 = \left(\sum_{i=1}^{n} i\right)^2 = \frac{n^2(n-1)^2}{4}$

For example, consider the function $f(x) = 2x - x^3$ on [0, 1]. We'll use the above considerations to obtain the *exact* area under *f* on the interval [0, 1].

Consider inserting *n* rectangles, all of equal width $w = \frac{1}{n}$ in the interval [0, 1]. This is an example of a regular partition, which means that each subinterval $[x_{k-1}, x_k]$ in which the *k*th rectangle is located (where $1 \le k \le n$) is of equal width, i.e. $|\Delta x_k| = |x_k - x_{k-1}| = \frac{1}{n}$. We have the following picture:



As indicated in the above picture, there remains an ambiguity with how to determine the *height* of each rectangle. On the one hand, we could choose the height to be the *y*-value of the **leftmost** endpoint of each rectangle (indicated by the red line). So for instance, for **rectangle 1:** $h_1 = f(x_0) = f(^0/_n)$, **for rectangle 2:** $h_2 = f(x_1) = f(^1/_n)$, , for **rectangle 3:** $h_3 = f(x_2) = f(^2/_n)$, , …, for **rectangle k**: $h_k = f(x_{k-1}) = f(^{(k-1)}/_n)$, etc. On the other hand, we could choose the height to be the *y*-value of the **rightmost** endpoint of each rectangle (indicated by the blue line). So for instance, for **rectangle 1:** $h_1 = f(x_1) = f(^1/_n)$, **for rectangle 2:** $h_2 = f(x_2) = f(^2/_n)$, for **rectangle 3:** $h_3 = f(x_3) = f(^3/_n)$, …, for **rectangle k**: $h_k = f(x_k) = f(^k/_n)$, etc.

- **Denote the total area of the rectangles (with red tops) as *Left*-sum (or in this case, since *f* is strictly monotone increasing in this interval, a *Lower* – sum):**

$$LS = h_1w_1 + h_2w_2 + h_3w_3 + \ldots + h_nw_n = \sum_{i=1}^{n} f\left(x_{i-1}\right)\left(\tfrac{1}{n}\right) = \tfrac{1}{n}\sum_{i=1}^{n} f\left(\tfrac{i-1}{n}\right)$$

- **Denote the total area of the rectangles (with blue tops) as *Right*-sum (or in this case, since *f* is strictly monotone increasing in this interval, an *Upper* – sum):**

$$RS = h_1 w_1 + h_2 w_2 + h_3 w_3 + \ldots + h_n w_n = \sum_{i=1}^{n} f(x_i)\left(\tfrac{1}{n}\right) = \tfrac{1}{n}\sum_{i=1}^{n} f\left(\tfrac{i}{n}\right)$$

As you'd expect, in the limit when the rectangles becomes infinitely narrow, both left and right sums should converge to the same result, which of course is an exact expression for the area under the curve:

$$\lim_{n\to\infty} \tfrac{1}{n}\sum_{i=1}^{n} f\left(\tfrac{i-1}{n}\right) = AREA = \lim_{n\to\infty} \tfrac{1}{n}\sum_{i=1}^{n} f\left(\tfrac{i}{n}\right)$$

Start with the left-sum:

$$LS = \tfrac{1}{n}\sum_{i=1}^{n} f\left(\tfrac{i-1}{n}\right) = \tfrac{1}{n}\sum_{i=1}^{n}\left[2\left(\tfrac{i-1}{n}\right) - \left(\tfrac{i-1}{n}\right)^3\right] = \tfrac{2}{n^2}\sum_{i=1}^{n}(i-1) - \tfrac{1}{n^4}\sum_{i=1}^{n}(i-1)^3$$

One can simplify the sum (to avoid having to multiply out the $(i - 1)^3$ terms and distribute across by the sum) by introducing a change of index: $j \to (i - 1)$. Then:

$$LS = \tfrac{2}{n^2}\sum_{i=1}^{n}(i-1) - \tfrac{1}{n^4}\sum_{i=1}^{n}(i-1)^3 = \tfrac{2}{n^2}\sum_{j=0}^{n-1} j - \tfrac{1}{n^4}\sum_{j=0}^{n-1} j^3 = \tfrac{2}{n^2}\sum_{j=1}^{n-1} j - \tfrac{1}{n^4}\sum_{j=1}^{n-1} j^3$$

The last step was justified because:

$$\sum_{j=0}^{n-1} j = 0 + 1 + 2 + \ldots + (n-1) = 1 + 2 + 3 + \ldots + (n-1) = \sum_{j=1}^{n} j$$
$$\sum_{j=0}^{n-1} j^3 = 0^3 + 1^3 + 2^3 + \ldots + (n-1)^3 = 1^3 + 2^3 + 3^3 + \ldots + (n-1)^3 = \sum_{j=1}^{n} j^3$$

Using the summation formulae (note that we substitute now $n - 1$ for n, since the sum ends at $n - 1$):

$$\sum_{j=1}^{n-1} j = \frac{(n-1)(n-2)}{2} \qquad \sum_{j=1}^{n-1} j^3 = \frac{(n-1)^2(n-2)^2}{4}$$

Re-Insert into the above LS to obtain:

$$LS = \tfrac{2}{n^2}\sum_{j=1}^{n-1} j - \tfrac{1}{n^4}\sum_{j=1}^{n-1} j^3 = \tfrac{2}{n^2}\cdot\tfrac{(n-1)(n-2)}{2} - \tfrac{1}{n^4}\cdot\tfrac{(n-1)^2(n-2)^2}{4}$$
$$= \frac{n^2-3n+2}{n^2} - \frac{(n^2-2n+1)(n^2-4n+4)}{4n^4}$$
$$= \frac{n^2-3n+2}{n^2} - \frac{n^4-2n^3+n^2-4n^3+8n^2-4n+4n^2-8n+4}{4n^4}$$
$$= 1-\frac{3}{n}+\frac{2}{n^2}-\frac{1}{4}-\frac{3}{2n}+\frac{13}{4n^2}-\frac{12}{n^3}+\frac{1}{n^4}$$

Taking the ∞ - limit:

$$\lim_{n\to\infty} LS = \lim_{n\to\infty}\left[1-\frac{3}{n}+\frac{2}{n^2}-\frac{1}{4}-\frac{3}{2n}+\frac{13}{4n^2}-\frac{12}{n^3}+\frac{1}{n^4}\right] = 1-\tfrac{1}{4} = \tfrac{3}{4}$$

Which should match the same limit for the Right sum or upper-sum, if this result is indeed the area under the curve:

$$RS = \tfrac{1}{n}\sum_{i=1}^{n} f\left(\tfrac{i}{n}\right) = \tfrac{1}{n}\sum_{i=1}^{n}\left[2\left(\tfrac{i}{n}\right)-\left(\tfrac{i}{n}\right)^3\right] = \tfrac{2}{n^2}\sum_{i=1}^{n}(i) - \tfrac{1}{n^4}\sum_{i=1}^{n}(i)^3 = \tfrac{2}{n^2}\cdot\tfrac{n(n-1)}{2} - \tfrac{1}{n^4}\cdot\tfrac{n^2(n-1)^2}{4}$$
$$= \frac{n^2-n}{n^2} - \frac{n^4-2n^3+n^2}{4n^4} = 1-\frac{1}{n}-\frac{1}{4}+\frac{1}{2n}-\frac{1}{4n^2}$$

Taking the ∞ - limit:

$$\lim_{n\to\infty} RS = \lim_{n\to\infty}\left[1-\frac{1}{n}-\frac{1}{4}+\frac{1}{2n}-\frac{1}{4n^2}\right] = 1-\tfrac{1}{4} = \tfrac{3}{4}$$

Which agrees with the previous result, hence the area under the segment = $\dfrac{3}{4}$

This process is obviously laborious! Not only that, but the summation formulae get forbiddingly complicated if not impossible in more complicated cases of functions. **The Fundamental Theorem of Calculus** enables one to by-pass this limit procedure, and indeed is the central result upon which the field of Calculus is based.

Some notation first. What we computed in the above case is know as a **definite integral.** Stated formally:

$$\text{AREA} = \int_0^1 (2x-x^3)dx = \lim_{n\to\infty} LS = \lim_{n\to\infty} RS$$

Actually, there was no reason why we had to choose **regular partition**, *any* partition would do. A regular partition for LS and RS was chosen above simply for ease in calculation. Stated more generally:

$$\int_0^1 (2x - x^3)dx = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} f(c_i)\Delta x_i$$

…where ||Δ|| stands for the **norm** of the partition (i.e. the width of the largest rectangle). For any rectangle with endpoints $[x_{k-1}, x_k]$, Δx_k stands for the width of the rectangle, i.e. : $\Delta x_k = x_k - x_{k-1}$.

In fact, the above holds for *any* continuous function on *any* interval [*a* , *b*]:

$$\int_a^b f(x)dx = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} f(c_i)\Delta x_i$$

For some arbitrary partition: $a = x_0 < x_1 < x_2 < ... < x_{k-1} < x_k < ... < x_{n-1} < x_n = b$

What the **Fundamental Theorem of Calculus** states is:

$$\int_a^b f(x)dx = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} f(c_i)\Delta x = F(b)_i - F(a)$$, where: $\frac{d}{dx}F(x) = f(x)$, i.e. F is the **antiderivative** or **indefinite integral** of *f*.

Note 3: It's often convenient to adopt the shorthand notation: $F(x)\Big|_a^b = F(b) - F(a)$

Proof of the FTC:

Consider any arbitrary partition:

$a = x_0 < x_1 < x_2 < ... < x_{k-1} < x_k < ... < x_{n-1} < x_n = b$

Hence:

$$F(b) - F(a) = F(x_n) - F(x_0)$$
$$= F(x_n) - F(x_{n-1}) + F(x_{n-1}) - F(x_{n-2}) + F(x_{n-2}) - .... - F(x_1) + F(x_1) - F(x_0)$$
$$= \sum_{i=1}^{n} [F(x_i) - F(x_{i-1})]$$

…obtained by successively applying the "trick of 0." Note also, using the "trick of 1":

$$\sum_{i=1}^{n}\left[F(x_i)-F(x_{i-1})\right]=\sum_{i=1}^{n}\left[F(x_i)-F(x_{i-1})\right]\left[\frac{x_i-x_{i-1}}{x_i-x_{i-1}}\right]=\sum_{i=1}^{n}\left[\frac{F(x_i)-F(x_{i-1})}{x_i-x_{i-1}}\right]\Delta x_i$$

According to the **MVT (Mean Value Theorem)**[1] for each interval $[x_k, x_{k-1}]$, for $1 \le k \le n$, there exists a point $c_k \in (x_{k-1}, x_k)$ such that:

$$F'(c_k)=f(c_k)=\frac{F(x_k)-F(x_{k-1})}{x_k-x_{k-1}}$$

Hence: $F(b)-F(a)=\sum_{i=1}^{n}\left[\frac{F(x_i)-F(x_{i-1})}{x_i-x_{i-1}}\right]\Delta x_i=\sum_{i=1}^{n}f(c_i)\Delta x_i$

This holds for *any* partition. Hence:

$$\int_a^b f(x)dx=\lim_{\|\Delta\|\to 0}\sum_{i=1}^{n}f(c_i)\Delta x=F(b)_i-F(a)$$

Q.E.D.

Corollaries to the **Fundamental Theorem of Calculus** include **Thm 5.12, Thm 5.13** (p 263, 266, text), the Mean Value Theorem for Integrals (MVTI) as well the Leibnitz Rule.

- The **MVTI** states:

Given continuous $f(x)$ on $[a, b]$ there exists a $c \in (a, b)$ such that:

$$f(c)(b-a)=\int_a^b f(x)dx$$

Proof: According to the **FTC:** $\int_a^b f(x)dx=F(b)-F(a)$, where $F'(x)=f(x)$

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[1] Recall **Sept 18 notes**
If f is differentiable on $[a, b]$ and $f(a)=f(b)$ then there exists (at least one) point $c \in [a, b]$ such that: $f'(c)=\frac{f(b)-f(a)}{b-a}$

Since F is differentiable, according to the **MVT** there exists a a $c \in (a, b)$ such that: $\frac{F(b)-F(a)}{b-a} = F'(c) = f(c)$, hence: $F(b)-F(a) = \int_a^b f(x)dx = f(c)(b-a)$.

Q.E.D.

Note4: The number $f(c)$ is denoted as the **average value of a function** on interval $[a, b]$: $f(c) = \frac{1}{(b-a)}\int_a^b f(x)dx$

THM 5.13 (p. 266) is often denoted as the Leibnitz Rule, and states:

$$\frac{d}{dx}\int_a^x f(t)dt = \frac{d}{dx}[F(x)-F(a)] = F'(x) = f(x)$$

(The above expression is also the proof of the Rule)

- Example (#20, §5.4) $\int_{-8}^{-1} \frac{x-x^2}{2\sqrt[3]{x}}dx$

According to the **FTC**, we need simply find the antiderivative and evaluate it at the endpoints:

$$\int_{-8}^{-1} \frac{x-x^2}{2\sqrt[3]{x}}dx = \frac{1}{2}\int_{-8}^{-1}\left(x^{\frac{2}{3}} - x^{\frac{5}{3}}\right)dx = \left[\frac{1}{2}\cdot\frac{3}{5}x^{\frac{5}{3}} - \frac{1}{2}\cdot\frac{3}{8}x^{\frac{8}{3}}\right]\Big|_{-1}^{-8}$$

$$= \left[\frac{3}{10}x^{\frac{5}{3}} - \frac{3}{16}x^{\frac{8}{3}}\right]_{-1}^{-8} = \left[3x\left(\frac{1}{10}x^{\frac{2}{3}} - \frac{1}{16}x^{\frac{5}{3}}\right)\right]_{-1}^{-8} = \left[-24\left(\frac{2}{5}+2\right)\right] - \left[-3\left(\frac{1}{10}+\frac{1}{16}\right)\right]$$

$$\frac{39}{80} - \frac{12\cdot 24}{5} = \frac{39-12\cdot 24\cdot 16}{80} = \frac{-4569}{80} = -57.1125$$

The negative result means that the function is predominantly negative (its graph sits below the x-axis) in that interval, or the 'area' bound by the curve below the x-axis and the x-axis exceeds the area under the curve for the portions where it's above the x-axis

- Example (#23, § 5.4) $\int_0^4 \left|x^2 - 4x + 3\right|dx$

In the case of differentiation, so also in the case of integrating. First transform:

$$|x^2-4x+3| = \begin{cases} x^2-4x+3 & x^2-4x+3 \geq 0 \Rightarrow (x-3)(x-1) \geq 0 \\ -x^2+4x-3 & x^2-4x+3 < 0 \end{cases}$$

Solving the inequality requires the use of a sign chart/table:

Region	$(x-3)(x-1)$
$x > 3$	+
$1 < x < 3$	-
$x < 1$	+

Hence:

$$|x^2-4x+3| = \begin{cases} x^2-4x+3 & x \geq 3\_OR\_x \leq 1 \\ -x^2+4x-3 & 1 < x < 3 \end{cases}$$

$$\int_0^4 |x^2-4x+3|dx = \int_0^1 (x^2-4x+3)dx + \int_1^3 (-x^2+4x-3)dx + \int_3^4 (x^2-4x-3)dx$$
$$= \left[\tfrac{1}{3}x^3-2x+3x\right]_0^1 + \left[-\tfrac{1}{3}x^3+2x-3x\right]_1^3 + \left[\tfrac{1}{3}x^3-2x+3x\right]_3^4$$
$$= \left[\tfrac{1}{3}-2+3\right]-0+\left[-9+6-9\right]-\left[-\tfrac{1}{3}+2-3\right]+\left[\tfrac{64}{3}-8+12\right]-\left[9-6+9\right]$$
$$= \tfrac{5}{3}-26+\tfrac{5}{3}+\tfrac{76}{3} = \tfrac{86}{3}-26 = \tfrac{86}{3}-\tfrac{78}{3} = \tfrac{8}{3}$$

- <u>Example (#44, §5.4)</u> Verify the Liebnitz Rule for: $\int_0^x t(t^2+1)dt$

Integrating: $\int_0^x t(t^2+1)dt = \int_0^x (t^3+t)dt = \left[\tfrac{1}{4}t^4+\tfrac{1}{2}t^2\right]_0^x = \tfrac{1}{4}x^4+\tfrac{1}{2}x^2$

Differentiating: $\frac{d}{dx}\int_0^x t(t^2+1)dt = \frac{d}{dx}\left(\tfrac{1}{4}x^4+\tfrac{1}{2}x^2\right) = x^3+x = x(x^2+1)$

Which confirms the claim: $\frac{d}{dx}\int_a^x f(t)dt = F'(x) = f(x)$