MA261 **Calculus I: Notes for October 16**
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ARC-LENGTH, SURFACE AREA, AND CENTROIDS

Arc-Length: $L = \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$

Surface Area: $S = \int_a^b r(x)\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$

(where $r(x)$ is the radius (distance from curve to x-axis, if curve is revolved around x-axis.)

- Example (#7, 327)

Find arclength on [1, 2] for $y = \frac{x^5}{10} + \frac{1}{6x^3}$

$$y = \tfrac{1}{10}x^5 + \tfrac{1}{6}x^{-3} \Rightarrow \tfrac{dy}{dx} = \tfrac{1}{2}x^4 - \tfrac{1}{2}x^{-4}$$
$$\therefore \sqrt{1+\left(\tfrac{dy}{dx}\right)^2} = \sqrt{1+\left(\tfrac{1}{2}x^4 - \tfrac{1}{2}x^{-4}\right)^2} = \sqrt{1+\tfrac{1}{4}x^8 - \tfrac{1}{2} + \tfrac{1}{4}x^{-8}}$$
$$= \sqrt{\tfrac{1}{4}x^8 + \tfrac{1}{2} + \tfrac{1}{4}x^{-8}} = \tfrac{1}{2}\sqrt{x^8 + 2 + x^{-8}} = \tfrac{1}{2}\sqrt{\left(x^4 + x^{-4}\right)^2}$$
$$= \tfrac{1}{2}\left(x^4 + x^{-4}\right)$$
$$\therefore L = \int_a^b \sqrt{1+\left(f'(x)\right)^2}\,dx = \int_1^2 \tfrac{1}{2}\left(x^4 + x^{-4}\right)dx = \tfrac{1}{2}\int_1^2 \left(x^4 + x^{-4}\right)dx$$
$$= \tfrac{1}{2}\left(\tfrac{1}{5}x^5 - \tfrac{1}{3}x^{-3}\right)\Big|_1^2 = \tfrac{1}{2}\left\{\left[\tfrac{1}{5}2^5 - \tfrac{1}{3}2^{-3}\right] - \left[\tfrac{1}{5} - \tfrac{1}{3}\right]\right\}$$
$$= \tfrac{1}{10}(32-1) + \tfrac{1}{6}\left(1 - \tfrac{1}{8}\right) = \tfrac{31}{10} + \tfrac{7}{48} = \tfrac{779}{240}$$

- Example (#21, 327)

From (0,3) to (2,√5) on circle: $x^2 + y^2 = 3^2$

$$x^2 + y^2 = 9 \Rightarrow y = \left(9 - x^2\right)^{1/2} \Rightarrow \tfrac{dy}{dx} = \tfrac{1}{2}\cdot\left(9 - x^2\right)^{-1/2}(-2x)$$
$$= -\frac{x}{\sqrt{9-x^2}} \Rightarrow \sqrt{1+\left(\tfrac{dy}{dx}\right)^2} = \sqrt{1+\tfrac{x^2}{9-x^2}} = \sqrt{\tfrac{9-x^2+x^2}{9-x^2}} = \sqrt{\tfrac{9}{9-x^2}} = 3\left(9-x^2\right)^{-1/2}$$

An integral like this: $3\int_0^2 \frac{dx}{\sqrt{9-x^2}}$ *cannot* be evaluated by exact methods, from the techniques we've learned *so far.*[1] However, one can adopt a numerical procedure like the **trapezoidal rule:**

Suppose we're interested in a 10^{-5} accuracy:

$$E = 10^{-5} \le \frac{(b-a)^3}{12n^3} \max_{a \le x \le b} \left| f''(x) \right|$$

$$f(x) = 3\left(9 - x^2\right)^{-1/2} \Rightarrow f'(x) = -3x\left(9 - x^2\right)^{-3/2} = \frac{-3x}{\left(9-x^2\right)^{3/2}}$$

$$\Rightarrow f''(x) = \frac{-3\left(9 - x^2\right)^{3/2} - (-3x)\frac{3}{2}\left(9 - x^2\right)^{1/2}(-2x)}{\left(9 - x^2\right)^3}$$

$$= \frac{-3\left(9 - x^2\right)^{3/2} - 6x^2\left(9 - x^2\right)^{1/2}}{\left(9 - x^2\right)^3} = \left(9 - x^2\right)^{1/2}\left[\frac{-3\left(9 - x^2\right) - 6x^2}{\left(9 - x^2\right)^3}\right]$$

$$= \frac{-27 - 9x^2}{\left(9 - x^2\right)^{5/2}} = -9\left[\frac{3 + x^2}{\left(9 - x^2\right)^{5/2}}\right]$$

The above function describes the *curvature* of the circular segment. It's obviously complicated and tedious to find its maximum in [0,2] by the ordinary means we learned in chapter 4. We can by-pass the trouble by thinking geometrically: We've got a circle *centered* at the origin. Obviously, then, its curvature is at a minimum at $x = 0$ but at a maximum at $x = 2$.

Hence: $\max_{0 \le x \le 2} \left| f''(x) \right| = 9\frac{3+2^2}{\left(9-2^2\right)^{3/2}} = 9\frac{7}{5\sqrt{5}} = \frac{63}{5\sqrt{5}} = \frac{63\sqrt{5}}{25} \approx 5.63$

So: $$E = 10^{-5} \le \frac{(b-a)^3}{12n^3} \max_{a \le x \le b} \left| f''(x) \right| \Rightarrow 10^{-5} \le \frac{2^3}{12n^3} 5.63$$

$$\Rightarrow 10^{-5} \le \frac{12}{12n^3} \Rightarrow n^3 \ge 10^5 \Rightarrow n \ge \lceil 46.4 \rceil = 47$$

So it will require a regular partition of at least 47 intervals to ensure accuracy to 10^{-5}. The results are summarized in the next page:

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[1] There is a technique known as a *trigonometric substitution* which one typically encounters in Calculus II. However, an *even simpler* technique exists which involves a change of coordinates (to polar system)

n	x_n	f(x)	f(b) - f(a)/2n
1	0.042553191	1.000100614	0.003634476
2	0.085106383	1.000402637	
3	0.127659574	1.000906618	
4	0.170212766	1.001613474	
5	0.212765957	1.002524492	
6	0.255319149	1.003641341	
7	0.29787234	1.004966079	
8	0.340425532	1.00650116	
9	0.382978723	1.008249452	
10	0.425531915	1.010214248	
11	0.468085106	1.012399285	
12	0.510638298	1.014808765	
13	0.553191489	1.017447378	
14	0.595744681	1.020320326	
15	0.638297872	1.023433357	
16	0.680851064	1.026792794	
17	0.723404255	1.030405574	
18	0.765957447	1.034279293	
19	0.808510638	1.038422249	
20	0.85106383	1.042843499	
21	0.893617021	1.047552917	
22	0.936170213	1.052561259	
23	0.978723404	1.057880244	
24	1.021276596	1.063522632	
25	1.063829787	1.069502325	
26	1.106382979	1.075834469	
27	1.14893617	1.082535583	
28	1.191489362	1.08962369	
29	1.234042553	1.097118475	
30	1.276595745	1.105041465	
31	1.319148936	1.113416231	
32	1.361702128	1.12226862	
33	1.404255319	1.131627019	
34	1.446808511	1.141522667	
35	1.489361702	1.151990004	
36	1.531914894	1.163067086	
37	1.574468085	1.17479606	
38	1.617021277	1.187223727	
39	1.659574468	1.200402194	
40	1.70212766	1.214389653	
41	1.744680851	1.229251297	
42	1.787234043	1.245060416	
43	1.829787234	1.261899706	
44	1.872340426	1.27986286	
45	1.914893617	1.299056483	
46	1.957446809	1.319602447	
47	2	1.341640786	
Sum		51.61852295	
Sum*(b-a)/n		2.196532892	
RESULT		**2.200167368**	

We can however obtain an exact result if we calculate the surface area, by revolving the segment about the *y*-axis:

$$S = 2\pi\int_0^2 x\tfrac{3}{\sqrt{9-x^2}}dx = 6\pi\int_0^2 \left(9-x^2\right)^{-1/2} xdx$$

$$u = 9 - x^2 \Rightarrow du = -2xdx \Rightarrow xdx = -\tfrac{1}{2}du$$

$$\therefore S = -3\pi\int_{u(0)}^{u(2)} u^{-1/2}du = -3\pi\cdot 2u^{1/2}\Big|_9^5 = -6\pi\left(\sqrt{5}-3\right) = 6\pi\left(3-\sqrt{5}\right)$$

Note in the special case:

$$S = 2\pi\int_0^3 x\tfrac{3}{\sqrt{9-x^2}}dx = 6\pi\int_0^3 \left(9-x^2\right)^{-1/2} xdx$$

$$u = 9 - x^2 \Rightarrow du = -2xdx \Rightarrow xdx = -\tfrac{1}{2}du$$

$$\therefore S = -3\pi\int_{u(0)}^{u(3)} u^{-1/2}du = -3\pi\cdot 2u^{1/2}\Big|_9^0 = -6\pi(0-3) = 6\pi(3) = 18\pi = 2\pi\cdot 3^2$$

I.e., what we'd expect, which is the surface area of a hemisphere (of radius $r = 3$)

(Recall surface area equation: $S = 4\pi r^2$ for an entire sphere, and obviously $S = 2\pi r^2$ for a hemisphere.