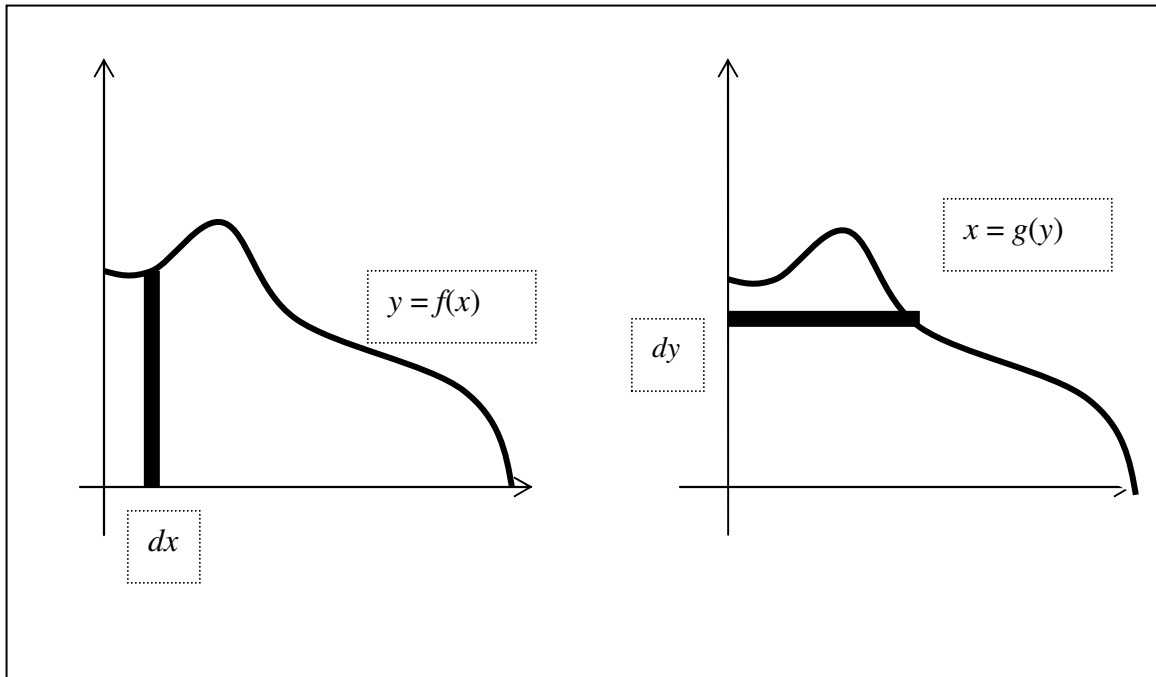


***APPLICATIONS OF THE FUNDAMENTAL THM OF CALCULUS: AREA  
BETWEEN TWO CURVES***

As discussed in detail in §6.1, one can calculate the area using two fundamentally different orientations (either integrating with respect to  $x$ , which geometrically corresponds to a vertical test rectangle of width  $dx$ , or integrating with respect to  $y$ , which geometrically corresponds to a horizontal test rectangle of width  $dy$ ). In this chapter on applications, this is a recurrent theme (that one can in principle treat either  $x$  or  $y$  as an independent variable or a variable of integration).



Choose an orientation that makes the computation the simplest—however keep in mind there's no general formula or recipe that will tell you which is the easiest strategy, you just have to set up the integrals yourself (with respect to  $dx$  and  $dy$  and judge for yourself).

However, it's recommended to avoid integrating anything that requires breaking up the integral into several different cases. If the other orientation won't produce such a case, stick with that!

For example, suppose one were asked to find the area between the curves:

$$y_1 = -3x$$

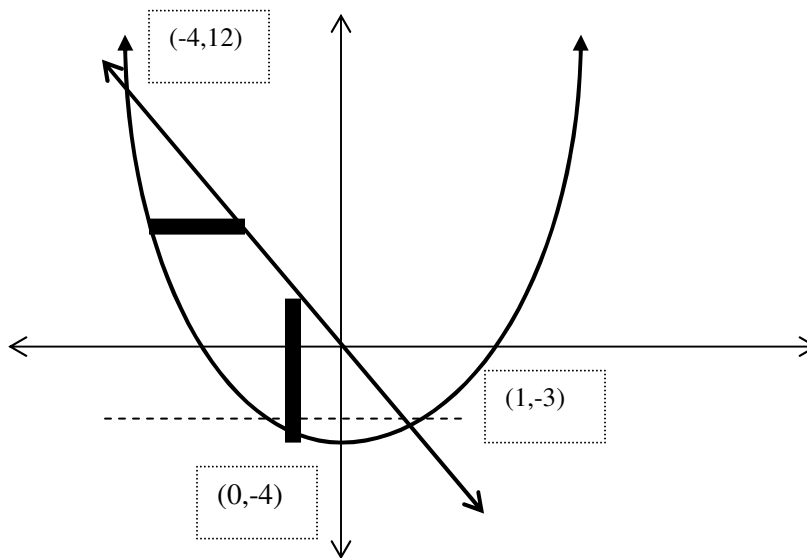
$$y_2 = x^2 - 4$$

The first step would be to determine where they intersect. This is done by setting them equal to each other:

$$y_1 = y_2 \Rightarrow x^2 - 4 = -3x \Rightarrow x^2 + 3x - 4 = 0 \Rightarrow (x-1)(x+4) = 0$$

$$\Rightarrow x_1 = -4, x_2 = 1$$

So the curves intersect at  $(-4, 12)$  and at  $(1, -3)$ . Here is a sketch:



- If the integration was with respect  $x$ , then notice that the integration would never break down into separate cases: **In the region of interest,  $y_1(x) \geq y_2(x)$  for all  $x: -4 \leq x \leq 1$ .** So:

$$\begin{aligned}
Area &= \int_{-4}^1 [y_1(x) - y_2(x)] dx = \int_{-4}^1 [-3x - (x^2 - 4)] dx = \int_{-4}^1 (-3x - x^2 + 4) dx \\
&= \left(-\frac{3}{2}x^2 - \frac{1}{3}x^3 + 4x\right) \Big|_{-4}^1 = x\left(-\frac{3}{2}x - \frac{1}{3}x^2 + 4\right) \Big|_{-4}^1 = \left[\left(-\frac{3}{2} - \frac{1}{3} + 4\right) - (-4)\left(\frac{3}{2} \cdot 4 - \frac{1}{3} \cdot 4^2 + 4\right)\right] \\
&= \frac{13}{6} + 16\left(\frac{3}{2} - \frac{4}{3} + 1\right) = \frac{13}{6} + 16\left(\frac{9-8+6}{6}\right) = \frac{13+112}{6} = \frac{125}{6}
\end{aligned}$$

- On the other hand, if the integration was with respect to  $y$  instead, note the additional necessity to split up the integral into two cases:

**Case 1:**  $-3 \leq y \leq 12$

Here the horizontal rectangle lying above the line  $y = -3$  is bounded to the **right** by  $x_1(y) = -\frac{1}{3}y$  and to the **left** by  $x_2(y) = -\sqrt{y+4}$  (These equations were obtained by solving for  $x$  in terms of  $y$  for the original curves:

$$\begin{aligned}
y_1 &= -3x \\
y_2 &= x^2 - 4
\end{aligned}$$

described above.

**Case 2:**  $-4 \leq y < -3$

Here the horizontal rectangle lying the line  $y = -3$  is bounded to the **right** by  $x_1(y) = \sqrt{y+4}$  and to the **left** by  $x_2(y) = -\sqrt{y+4}$ .<sup>1</sup>

Hence our integration becomes:

$$Area = \int_{-4}^{-3} [\sqrt{y+4} - (-\sqrt{y+4})] dy + \int_{-3}^{12} \left[-\frac{1}{3}y - (-\sqrt{y+4})\right] dy$$

(after putting **Case 2** and **Case 1** together.) A tad bit more involved!

Nevertheless, solving will yield the same answer:

$$Area = \int_{-4}^{-3} [2\sqrt{y+4}] dy + \int_{-3}^{12} \left[-\frac{1}{3}y + \sqrt{y+4}\right] dy$$

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<sup>1</sup> Note that there are two solutions when solving for  $x$  in  $y_2 = x^2 - 4$ :  $\Rightarrow x = \pm\sqrt{y+4}$

$$= 2 \int_{-4}^{-4} \sqrt{y+4} dy - \frac{1}{3} \int_{-3}^{12} y dy + \int_{-3}^{12} \sqrt{y+4} dy$$

The first and the third integrals can be done with a  $u$ -substitution:

$$u = y + 4 \Rightarrow \frac{du}{dy} = 1 \Rightarrow dy = du$$

So:

$$\begin{aligned} &= 2 \int_{-4}^{-4} \sqrt{y+4} dy - \frac{1}{3} \int_{-3}^{12} y dy + \int_{-3}^{12} \sqrt{y+4} dy = 2 \int_{u(-4)}^{u(-3)} u^{\frac{1}{2}} du - \frac{1}{3} \int_{-3}^{12} y dy + \int_{u(-3)}^{u(12)} u^{\frac{1}{2}} du \\ &= 2 \int_0^1 u^{1/2} du - \frac{1}{3} \int_{-3}^{12} y dy + \int_1^{16} u^{1/2} du = 2 \left[ \frac{2}{3} u^{3/2} \Big|_0^1 \right] - \frac{1}{3} \left[ \frac{1}{2} y^2 \Big|_{-3}^{12} \right] + \frac{2}{3} u^{3/2} \Big|_1^{16} \\ &= \frac{4}{3} u^{3/2} \Big|_0^1 - \frac{1}{6} y^2 \Big|_{-3}^{12} + \frac{2}{3} u^{3/2} \Big|_1^{16} = \frac{4}{3}(1-0) - \frac{1}{6}(144-9) + \frac{2}{3}(64-1) \\ &= \frac{8-135}{6} + \frac{126}{3} = \frac{-127+252}{6} = \frac{125}{6} \end{aligned}$$

...producing of course the same result.

- Example (#20, p. 300)

$$f(x) = x^2 + 5x - 6 \quad g(x) = 6x - 6$$

-Sketching these graphs, note that the  $x$ -intercepts of  $f$  are:

$$f(x) = 0 \Rightarrow (x+6)(x-1) = 0 \Rightarrow x_1 = 1, x_2 = -6$$

-And of course the  $x$ -intercepts of  $g$  is:  $x_3 = 1$

-Completing the square on  $f$  gives the coordinates of the vertex of the parabola:

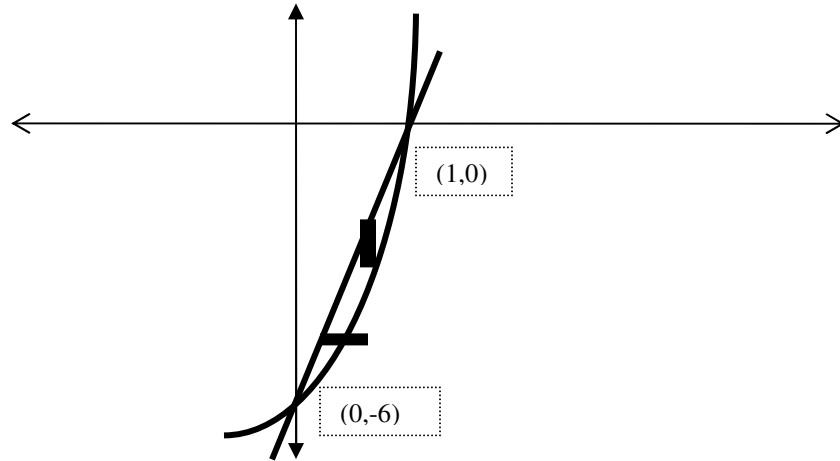
$$f(x) = x^2 + 5x - 6 = x^2 + 5x + \frac{25}{4} - \frac{25}{4} - 6 = \left(x + \frac{5}{2}\right)^2 - \frac{49}{4}$$

(Has vertex located at  $\left(-\frac{5}{2}, \frac{49}{4}\right)$ )

-And setting  $f = g$  give the coordinates of their intersection:

$$x^2 + 5x - 6 = 6x - 6 \Rightarrow x^2 - x = 0 \Rightarrow x(x-1) = 0 \Rightarrow x_4 = 0, x_5 = 1$$

So we have the following picture:



With the rectangle oriented vertically, the area expression becomes:

$$\int_0^1 [g(x) - f(x)] dx = \int_0^1 [(6x - 6) - (x^2 + 5x - 6)] dx = \int_0^1 [-x^2 + x] dx = \left(-\frac{1}{3}x^3 + \frac{1}{2}x^2\right) \Big|_0^1$$

$$= -\frac{1}{3} + \frac{1}{2} = \frac{1}{6}$$

We could have also evaluated this expression according to  $y$ , though like the other example, the process is more involved.

-Solving for  $x$  in terms of  $y$  for  $g(x)$ :

$$y = 6x - 6 \Rightarrow x_2(y) = \frac{1}{6}(y + 6)$$

-Solving for  $x$  in terms of  $y$  for  $f(x)$ :

$$y = x^2 + 5x - 6 = x^2 + 5x + \frac{25}{4} - \frac{25}{4} - 6 = \left(x + \frac{5}{2}\right)^2 - \frac{49}{4} \Rightarrow x_1(y) = \sqrt{y + \frac{49}{4}} - \frac{5}{2}$$

(Note that the positive case was selected, in terms of the square root, since the portion of the graph of interest is located to the right of the  $y$ -axis)

$$\int_{-6}^0 [x_1(y) - x_2(y)] dy = \int_{-6}^0 \left[ \sqrt{y + \frac{49}{4}} - \frac{5}{2} - \frac{1}{6}(y + 6) \right] dy = \int_{-6}^0 \left[ \sqrt{y + \left(\frac{7}{2}\right)^2} - \frac{1}{6}y - \frac{7}{2} \right] dy$$

So:

$$\int_{-6}^0 \left( y + \left(\frac{7}{2}\right)^2 \right)^{1/2} dy - \frac{1}{6} \int_{-6}^0 y dy - \frac{7}{2} \int_{-6}^0 dy = \int_{u(-6)}^{u(0)} u^{1/2} du - \frac{1}{6} \cdot \frac{1}{2} y^2 \Big|_{-6}^0 - \frac{7}{2} y \Big|_{-6}^0$$

$$= \int_{\frac{25}{4}}^{\frac{49}{4}} u^{1/2} du - \frac{1}{12}(0 - 36) - \frac{7}{2}(0 - -6) = \frac{2}{3} u^{3/2} \Big|_{25/4}^{49/4} + 3 - 21 = \frac{2}{3} \left( \frac{49}{4} \cdot \frac{7}{2} - \frac{25}{4} \cdot \frac{5}{2} \right) - 18$$

$$= \frac{1}{12} (7^3 - 5^3) - 18 = \frac{343 - 125 - 216}{12} = \frac{2}{12} = \frac{1}{6}$$