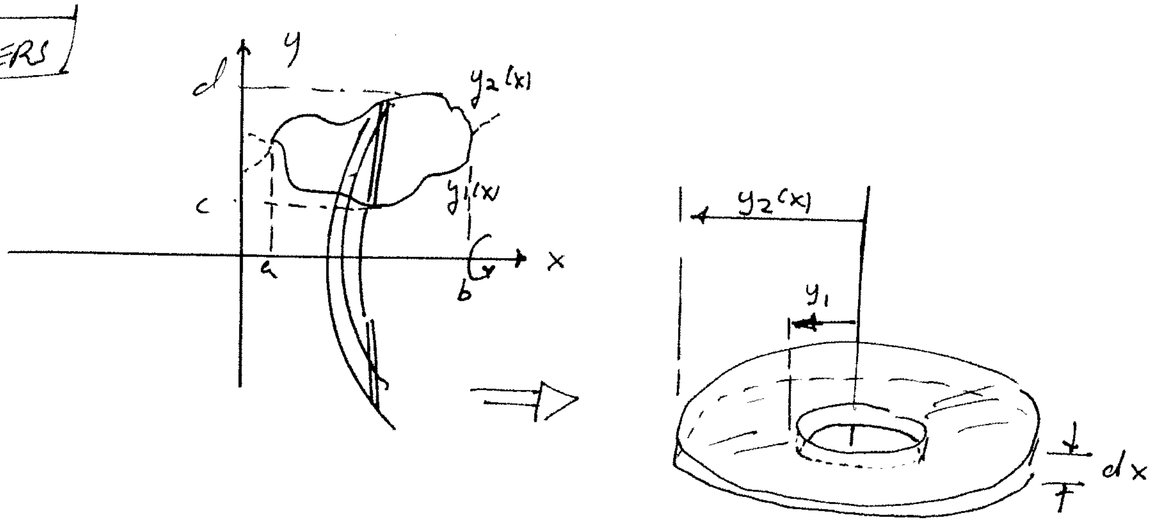
NOTES - MATH 261 OCT 11 & OCT 16

VOLUME - METHOD OF WASHERS & SHELLS

WASHERS



VOLUME OF INFINITESIMAL WASHER: $dV = \pi(y_2^2 - y_1^2)dx$

y_2: outer radius; y_1: inner radius

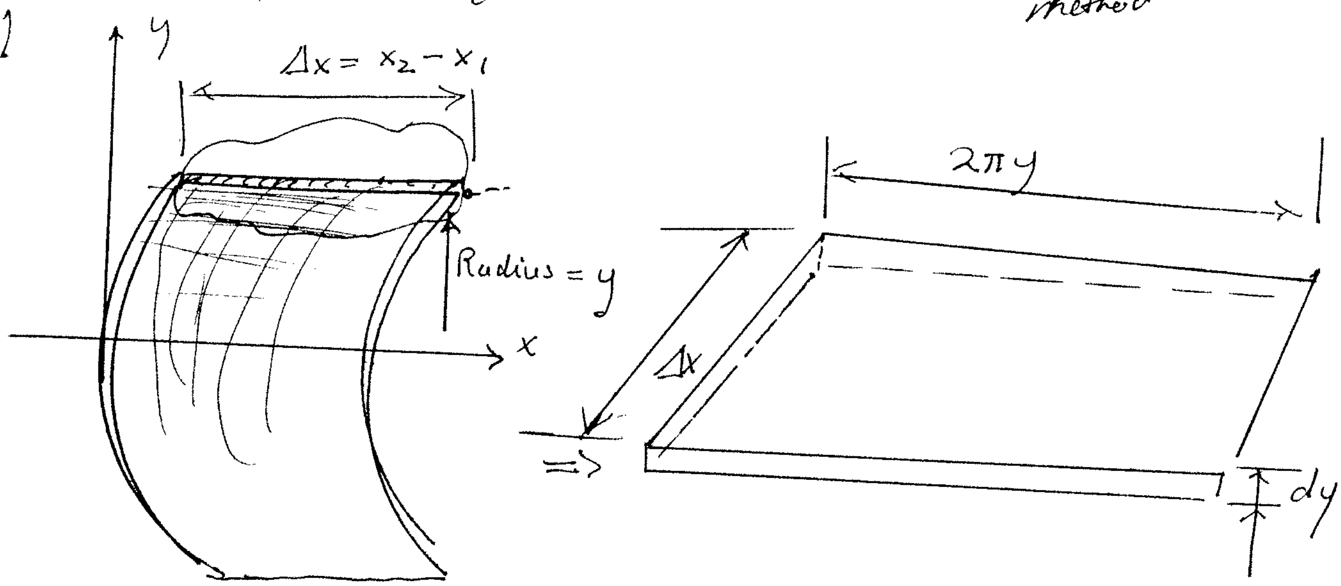
VOLUME OF REGION: $V = \int_a^b dV = \int_a^b \pi(y_2^2(x) - y_1^2(x))dx$

IF OBJECT/REGION WAS GENERATED BY ROTATING AROUND y-AXIS, THEN:

$$V_w = \int_c^d \pi(x_2^2(y) - x_1^2(y))dy$$

(V_w: Volume by washer method)

SHELLS



VOLUME OF INFINITESIMAL SLAB / SHELL (RECTANGULAR SOLID):

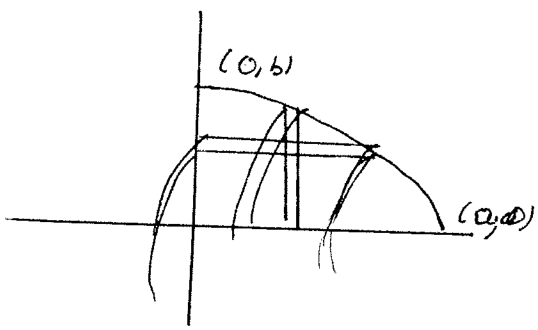
$$dV = 2\pi y (\Delta x) dy \quad \underline{\text{where}} \quad \Delta x = x_2(y) - x_1(y)$$

$$\therefore \quad V_S = \int_c^d 2\pi y (\Delta x(y)) dy \qquad (V_s: \text{Volume by shell method})$$

IF OBJECT/REGION WAS GENERATED BY ROTATING AROUND y-AXIS, THEN:

$$V_S = \int_a^b 2\pi x (\Delta y(x)) dx$$

<u>EXAMPLE</u>: VOLUME OF AN ELLIPSOID:



• By Disks: $V_D = 2 \cdot \int_0^a \pi y^2 dx$

where $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$$\Rightarrow y^2 = b^2(1 - x^2/a^2)$$

$$\therefore V = 2\pi b^2 \int_0^a (1 - x^2/a^2) dx$$

$$= 2\pi b^2 \int_0^a dx - \frac{2\pi b^2}{a^2} \int_0^a x^2 dx$$

$$= 2\pi b^2 x \Big|_0^a - 2\pi \left(\frac{b}{a}\right)^2 \cdot \frac{1}{3} x^3 \Big|_0^a$$

$$= 2\pi a b^2 - \frac{2}{3}\pi \frac{b^2}{a^2} a^3$$

$$= 2\pi a b^2 - \frac{2}{3}\pi a b^2 = \frac{4}{3}\pi a b^2$$

• <u>BY SHELLS</u>: $V_S = 2 \cdot 2\pi \int_0^b x(y)\, y\, dy$ where $\boxed{x(y) = a(1 - y^2/b^2)^{1/2}}$

By symmetry (other side of x-axis)

(Solving for x in $x^2/a^2 + y^2/b^2 = 1$)

$$\therefore V_s = 2\pi \int_0^b a\sqrt{1 - y^2/b^2}\, y\,dy = 2\pi a \int_0^b \sqrt{1 - y^2/b^2}\, y\,dy$$

$$u(y) = 1 - y^2/b^2 \Rightarrow \frac{du}{dy} = -\frac{2}{b^2}y \Rightarrow du = -\frac{2}{b^2}y\,dy$$

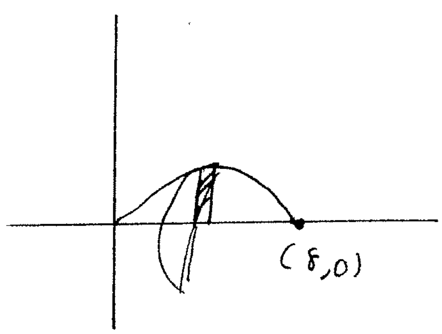
$$\Rightarrow \boxed{y\,dy = -\frac{b^2}{2}du}$$

$$\therefore V = 2\pi a\left(-\frac{b^2}{2}\right)\int_{u(0)}^{u(b)} u^{1/2}du = -2\pi ab^2 \int_1^0 u^{1/2}du$$

$$= 2\pi ab^2 \int_0^1 u^{1/2}du = 2\pi ab^2 \cdot \frac{2}{3}u^{3/2}\Big|_0^1 = \frac{4}{3}\pi ab^2$$

• Example 18, p316

$y = 2x^{2/3} - x \qquad y = 0 \qquad$ (around x-axis)



$$2x^{2/3} - x = 0$$

$$x(2x^{-1/3} - 1) = 0$$

$$\therefore x_1 = 0$$

$$2x^{-1/3} = 1 \Rightarrow x_2 = 8$$

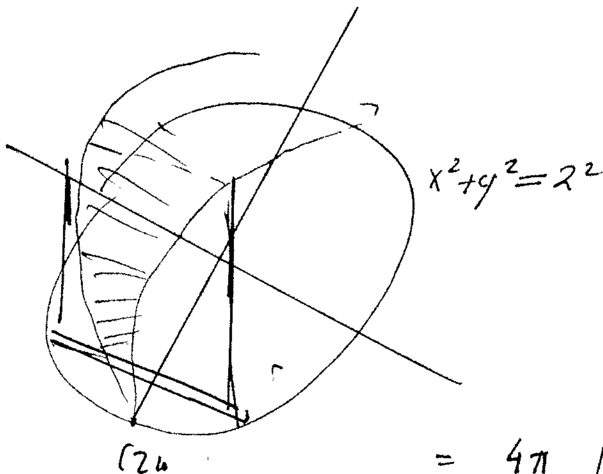
By washers: $\quad V_s = \pi \int_0^8 [y(x)]^2 dx = \pi \int_0^8 (2x^{2/3} - x)^2 dx$

$$= \pi \int_0^8 (4x^{4/3} - 4x^{5/3} + x^2)\,dx$$

$$= 4\pi \cdot \frac{3}{7}x^{7/3}\Big|_0^8 - 4\pi \cdot \frac{3}{8}x^{8/3}\Big|_0^8 + \frac{\pi}{3}x^3\Big|_0^8$$

$$= \pi x^2 \left\{ \frac{12}{7}x^{1/3} - \frac{3}{2}x^{2/3} + \frac{1}{3}x \right\}\Big|_0^8 = 64\pi\left\{ \frac{12}{7}\cdot 2 - \frac{3}{2}\cdot 4 + \frac{8}{3} \right\} = 64\pi\left\{ \frac{72 - \cancel{126} + 56}{21} \right\} = \frac{128\pi}{21}$$

45, §6.2



a) Volume of slab = base × height × wi

$= (2y)(2y)\,dx$

$= 4y^2\,dx$

$$V_{ol} = \pi \int_0^2 4y^2\,dx$$

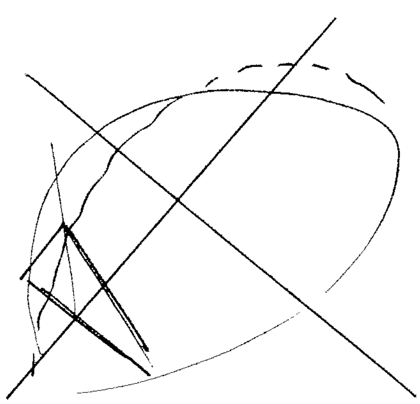
$$= 4\pi \int_0^2 (4-x^2)\,dx$$

$$= \cancel{16}\pi \cdot x \Big|_0^2 - \frac{\cancel{4}\pi}{3}x^3\Big|_0^2$$

$$= 32\pi - \frac{32}{3}\pi = \frac{64}{3}\pi$$

ONLY GENERATED 1/2 (SINCE ~~ingore grau~~ integrated from 0 to 2.) Double the value for the full volume = $\boxed{\frac{128}{3}\pi}$

b)



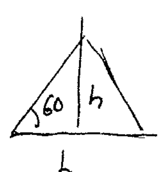
Volume of triangle: $\frac{1}{2}bh\,dx$

$b = 2y = 2\sqrt{4-x^2}$

$h = \frac{\sqrt{3}}{2}b = \frac{\sqrt{3}}{\cancel{2}}\cdot \cancel{2}\sqrt{4-x^2}$

$= \sqrt{3}\sqrt{4-x^2}$

$= \sqrt{12-3x^2}$

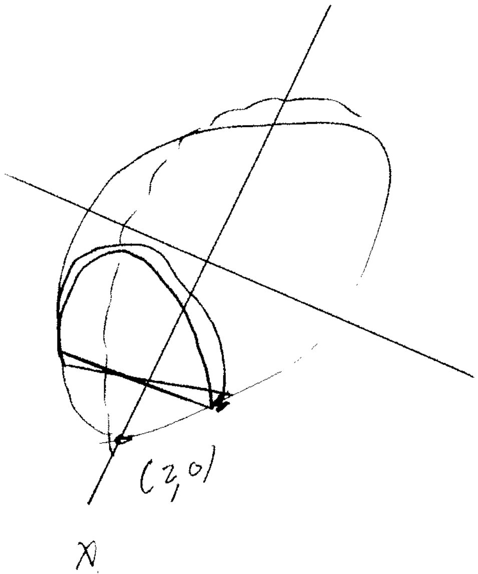


$$V = 2 \int_0^2 \left(2\sqrt{4-x^2}\right)\left(\sqrt{3}\sqrt{4-x^2}\right)dx$$

Symmetry

$$\therefore V = 4\sqrt{3}\int_0^2 (4-x^2)\,dx = 4\sqrt{3}\left\{4x-\frac{x^3}{3}\right\}\Big|_0^2 = 4\sqrt{3}\left\{8-\frac{8}{3}\right\} = \frac{64\sqrt{3}}{3}$$

c)



Volume of semicircular slab:

$dV = \pi r^2 dx$ where $r = y = \sqrt{4-x^2}$

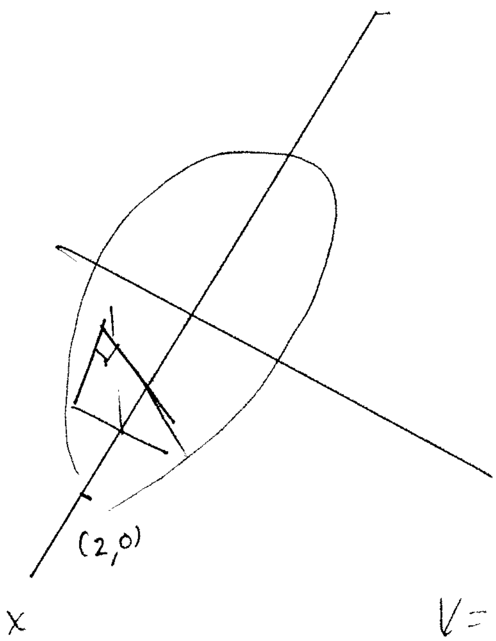
$$\therefore V = 2\pi\int_0^2 (4-x^2)\,dx$$

symmetry

$$= 2\pi\left\{4x-\frac{x^3}{3}\right\}\Big|_0^2 = 2\pi\left\{8-\frac{8}{3}\right\}$$

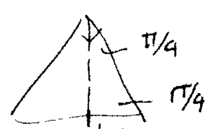
$$= 2\pi\left(\frac{16}{3}\right) = \frac{32\pi}{3}$$

d)



Isocoles: $dV = \frac{1}{2}bh\,dx$

Right



$b = 2y = 2\sqrt{4-x^2}$

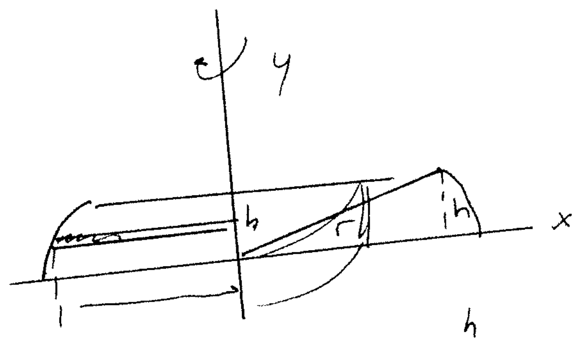
$h = \frac{b}{2} = y = \sqrt{4-x^2}$

$$\therefore dV = \frac{1}{2}(2(4-x^2))dx = (4-x^2)dx$$

$$V = 2\int_0^2 (4-x^2)dx = 8x\Big|_0^2 - \frac{2}{3}x^3\Big|_0^2$$

$$= 16 - \frac{16}{3} = \frac{32}{3}$$

Example #27, p 319



By washer method: $V = \int_0^h \pi [x(y)]^2 dy$

Note: $x^2 + y^2 = r^2 \Rightarrow x(y) = \sqrt{r^2 - y^2}$

$[x(y)]^2 = r^2 - y^2$

$$\therefore V = \pi \int_0^h (r^2 - y^2)\, dy = \pi r^2 \int_0^h dy - \pi \int_0^h y^2 dy$$

$$= \pi r^2 y \Big|_0^h - \frac{\pi}{3} y^3 \Big|_0^h = \pi r^2 h - \frac{\pi}{3} h^3 = \frac{\pi}{3} h (3r^2 - h^2)$$

(Special case: $h = r \Rightarrow V = \frac{\pi}{3} r (3r^2 - r^2) = \frac{2}{3}\pi r^3$ (Vol. of a hemisphere)

By shell method $V = Vol_1 + Vol_2$

$Vol_1 = \pi (r^2 - h^2) h$ (Vol. of middle cylindrical section)

$$Vol_2 = 2\pi \int_{\sqrt{r^2-h^2}}^{r} x\, y(x) dx = 2\pi \int_{\sqrt{r^2-h^2}}^{r} x \sqrt{r^2 - x^2}\, dx$$

$u(x) = r^2 - x^2 \Rightarrow \frac{du}{dx} = -2x \Rightarrow x\, dx = -\frac{1}{2} du$

$u(\sqrt{r^2-h^2}) = h^2 \quad u(r) = 0$

$$\Rightarrow Vol_2 = 2\pi \cdot -\frac{1}{2} \int_{u(\sqrt{r^2-h^2})}^{u(r)} u^{1/2} du = -\pi \int_{h^2}^{0} u^{1/2} du = \pi \int_0^{h^2} u^{1/2} du$$

$$= \frac{2}{3}\pi u^{3/2} \Big|_0^{h^2} = \frac{2}{3}\pi h^3$$

$$\therefore V = Vol_1 + Vol_2 = \pi(r^2 - h^2)h + \frac{2}{3}\pi h^3$$

$$= \pi h r^2 - \pi h^3 + \frac{2}{3}\pi h^3$$

$$= \pi h \left(r^2 + \frac{1}{3}h^2\right)$$

$$= \frac{\pi h}{3}\left(3r^2 + h^2\right)$$

(7)