MA261 **Calculus I: Notes for November 8**
Fall 2007 Kallfelz

SOME WORKED OUT SOLUTIONS (§§ 8.1-8.3)

- Exercise 37 (modified: 1 at right had side), § 8.1

$$\tan^2\theta - \tan\theta = 1 \Rightarrow \tan^2\theta - \tan\theta - 1 = 0$$
$$X \equiv \tan\theta \Rightarrow X^2 - X - 1 = 0 \Rightarrow X_{1,2} = \frac{1 \pm \sqrt{1+4}}{2} = \tfrac{1}{2}\left(1 \pm \sqrt{5}\right)$$
$$\therefore \tan\theta_{1,2} = \tfrac{1}{2}\left(1 \pm \sqrt{5}\right)$$

- **Note1:** The range of the tangent function is unrestricted, i.e.: for all θ: $-\infty < \tan\theta < \infty$, so both answers qualify (i.e. are part of the domain of arctan)

$$\theta_1 = \arctan\left(\tfrac{1}{2}\left(1+\sqrt{5}\right)\right) \approx 1.02\,rad$$
$$\theta_2 = \arctan\left(\tfrac{1}{2}\left(1-\sqrt{5}\right)\right) \approx -0.56\,rad = 2\pi - 0.56 = 5.73\,rad$$

- **Note2:** The specified domain of θ is $[0, 2\pi)$, and the tangent function has a period of π, hence:

$$\theta_3 = \theta_1 + \pi = 1.02\,rad + \pi = 4.16\,rad$$
$$\theta_4 = \theta_2 + \pi = 5.73\,rad + \pi = 8.87 = 8.87 - 2\pi = 2.59\,rad$$

…also belong to the solution set.

- Exercise 37 (unmodified: ☺), § 8.1

$$\tan^2\theta - \tan\theta = 0 \Rightarrow X \equiv \tan\theta$$
$$\Rightarrow X^2 - X = 0 \Rightarrow X(X-1) = 0$$
$$\Rightarrow X_{1,2} = \tan\theta_{1,2} = 0,1 \Rightarrow \theta_1 = \arctan(0) = 0, \theta_2 = \arctan(1) = \tfrac{\pi}{4}$$

- However, as in the above case, the specified domain of θ is $[0, 2\pi)$, and the tangent function has a period of π, so:

$$\theta_3 = \theta_1 + \pi = 0 + \pi = \pi$$
$$\theta_4 = \tfrac{\pi}{4} + \pi = \tfrac{5}{4}\pi$$

Also belong to the solution set.

- Exercise 39 § 8.1

$$\sec\theta\csc\theta = 2\csc\theta \Rightarrow \left(\sec\theta - 2\right)\csc\theta = 0 \Rightarrow \sec\theta_1 = 2, \csc\theta_2 = 0$$

$$\Rightarrow \theta_1 = arc\sec 2 = \arccos\left(\tfrac{1}{2}\right) = \tfrac{\pi}{3}$$
$$\Rightarrow \theta_2 = arc\csc 0 \Rightarrow DNE$$

- **Note3:** In the second case, the domain of arccsc = (-∞ , -1] ∪ [1 , ∞) = $\{x \mid x \leq 1 \text{ or } x \geq 1\}$ So 0 is outside its domain.

- **Note4:** the specified domain of θ is [0 , 2π), so $\theta_2 = 2\pi - \tfrac{\pi}{3} = \tfrac{5}{3}\pi$

- Exercise 38 § 8.2

$\lim_{x\to\frac{\pi}{4}} \dfrac{1-\tan x}{\sin x - \cos x}$ produces a 0/0 indeterminacy. So it should be reduced:

$$\lim_{x\to\frac{\pi}{4}} \frac{1-\tan x}{\sin x - \cos x} = \lim_{x\to\frac{\pi}{4}} \frac{1-\dfrac{\sin x}{\cos x}}{\sin x - \cos x} = \lim_{x\to\frac{\pi}{4}} \frac{1-\dfrac{\sin x}{\cos x}}{\cos x\left(\dfrac{\sin x}{\cos x} - 1\right)}$$

$$= -\lim_{x\to\pi/4} \frac{1-\dfrac{\sin x}{\cos x}}{\cos x\left(1-\dfrac{\sin x}{\cos x}\right)} = -\lim_{x\to\frac{\pi}{4}} \frac{1}{\cos x} = -\frac{2}{\sqrt{2}} = -\sqrt{2}$$

  - **Note4: By using methods in subsequent section (§ 8.3), the above could be evaluated using L'Hopital's Rule:**

$$\lim_{x\to\frac{\pi}{4}} \frac{1-\tan x}{\sin x - \cos x} = \lim_{x\to\frac{\pi}{4}} \frac{\dfrac{d}{dx}(1-\tan x)}{\dfrac{d}{dx}(\sin x - \cos x)} = \lim_{x\to\frac{\pi}{4}} \frac{-\sec^2 x}{(\cos x + \sin x)} = -\frac{\left(\sqrt{2}\right)^2}{\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\right)}$$

$$= -\frac{2}{\sqrt{2}} = -\sqrt{2}$$

- Exercise 41 § 8.2

$$\lim_{x\to 0} \frac{\sin 2x}{\sin 3x} = \lim_{x\to 0} \frac{3x}{\sin 3x} \cdot \frac{\sin 2x}{3x} = \lim_{x\to 0} \frac{3x}{\sin 3x} \cdot \frac{\sin 2x}{\frac{3}{2}(2x)} = \tfrac{2}{3} \lim_{x\to 0} \frac{3x}{\sin 3x} \cdot \frac{\sin 2x}{2x}$$

$$= \tfrac{2}{3} \frac{\lim_{x\to 0} \dfrac{\sin 2x}{2x}}{\lim_{x\to 0} \dfrac{\sin 3x}{3x}} = \tfrac{2}{3} \frac{\lim_{u\to 0} \dfrac{\sin u}{u}}{\lim_{w\to 0} \dfrac{\sin w}{w}} = \tfrac{2}{3}$$

(where $u = 2x, w = 3x$, respectively)

- Exercise 21 § 8.3

$$y=\frac{1+\csc x}{1-\csc x}\Rightarrow y'=\frac{d}{dx}\frac{1+\csc x}{1-\csc x}=\frac{(1-\csc x)(-\csc x\cot x)-(1+\csc x)(\csc x\cot x)}{(1-\csc x)^2}$$

$$=-\csc x\cot x\frac{1-\csc x+1+\csc x}{(1-\csc x)^2}=\frac{-2\csc x\cot x}{(1-\csc x)^2}$$

- Exercise 37 § 8.3

$$y=\ln\left|\csc x-\cot x\right|\Rightarrow y'=\frac{d}{dx}\ln\left|\csc x-\cot x\right|=\frac{1}{\left|\csc x-\cot x\right|}\frac{d}{dx}\left|\csc x-\cot x\right|$$

$$=\frac{|-\csc x\cot x+\csc^2 x|}{\left|\csc x-\cot x\right|}=-\csc x\cdot\frac{\left|\csc x-\cot x\right|}{\left|\csc x-\cot x\right|}=-\csc x$$

- Exercise 40 § 8.3

$$y=\tan^2 e^x\Rightarrow y'=\frac{d}{dx}\left(\tan\left(e^x\right)\right)^2=2\left(\tan\left(e^x\right)\right)\sec^2\left(e^x\right)e^x=2\tan e^x\sec^2 e^x e^x$$

$$=\frac{2e^x\tan e^x}{\cos^2 e^x}=\frac{2e^x\sin e^x}{\cos^3 e^x}$$

- Exercise 61 § 8.3

$$\lim_{x\to\infty}x\sin\frac{1}{x}=\lim_{x\to\infty}\frac{\sin\frac{1}{x}}{\frac{1}{x}}=\lim_{x\to\infty}\frac{\frac{d}{dx}\sin\frac{1}{x}}{\frac{d}{dx}\frac{1}{x}}=\lim_{x\to\infty}\frac{\cos\frac{1}{x}\left(-\frac{1}{x^2}\right)}{\left(-\frac{1}{x^2}\right)}$$

$$=\lim_{x\to\infty}\cos\frac{1}{x}=\lim_{u\to 0}\cos u=1$$

- Exercise 70 § 8.3

$$y=A\sin\left(\sqrt{\tfrac{k}{m}}t\right)+B\cos\left(\sqrt{\tfrac{k}{m}}t\right)$$

a.)

$$y=A\sin\left(\sqrt{\tfrac{k}{m}}t\right)+B\cos\left(\sqrt{\tfrac{k}{m}}t\right)=\sqrt{A^2+B^2}\left[\frac{A}{\sqrt{A^2+B^2}}\sin\left(\sqrt{\tfrac{k}{m}}t\right)+\frac{B}{\sqrt{A^2+B^2}}\cos\left(\sqrt{\tfrac{k}{m}}t\right)\right]$$

Define: $\cos\phi = \dfrac{A}{\sqrt{A^2+B^2}} \Rightarrow \sin\phi = \dfrac{B}{\sqrt{A^2+B^2}}$

So: $y = \sqrt{A^2+B^2}\left[\cos\phi\sin\left(\sqrt{\tfrac{k}{m}}t\right) + \sin\phi\cos\left(\sqrt{\tfrac{k}{m}}t\right)\right] = \sqrt{A^2+B^2}\,\sin\left(\sqrt{\tfrac{k}{m}}t+\phi\right)$

$y_{\max} = \sqrt{A^2+B^2}\,\max(\sin(...)) = \sqrt{A^2+B^2}$

b.) Angular frequency: $\omega = 2\pi f = \sqrt{\tfrac{k}{m}} \Rightarrow f = \tfrac{1}{2\pi}\sqrt{\tfrac{k}{m}}$

If $k >> m$ then $f >> 1$ (frequency gets very high)
If $m >> f$ then $f << 1$ (frequency gets very low)

ANTIDERIVATIVE FORMULAE FOR SINUSOIDAL FUNCTIONS

In class, it was shown:

$$\frac{d}{dx}\sin x = \cos x \Rightarrow \int \cos x dx = \sin x + C \qquad \frac{d}{dx}\sec x = \sec x \tan x \Rightarrow \int \sec x \tan x dx = \sec x + C$$
$$\frac{d}{dx}\cos x = -\sin x \Rightarrow \int \sin x dx = -\cos x + C \qquad \frac{d}{dx}\csc x = -\csc x \cot x \Rightarrow \int \cot x \csc x dx = -\csc x + C$$
$$\frac{d}{dx}\tan x = \sec^2 x \Rightarrow \int \sec^2 x dx = \tan x + C \qquad \frac{d}{dx}\cot x = -\csc^2 x \Rightarrow \int \csc^2 x dx = -\cot x + C$$

…with their appropriate chain rule and u-substitution generalizations (for any implicit function $u(x)$):

$$\frac{d}{dx}\sin u = \cos u u' \Rightarrow \int \cos u du = \sin u + C$$
$$\frac{d}{dx}\cos u = -\sin u u' \Rightarrow \int \sin u du = -\cos u + C$$
$$\frac{d}{dx}\tan u = \sec^2 u \Rightarrow \int \sec^2 u du = \tan u + C$$

$$\frac{d}{dx}\sec u = \sec u \tan u u' \Rightarrow \int \sec u \tan u du = \sec u + C$$
$$\frac{d}{dx}\csc u = -\csc u \cot u \Rightarrow \int \cot u \csc u du = -\csc u + C$$
$$\frac{d}{dx}\cot u = -\csc^2 u \Rightarrow \int \csc^2 u du = -\cot u + C$$

Recall also:

$$\int \tan x dx = \int \frac{\sin x dx}{\cos x} \Rightarrow u(x) = \cos x \Rightarrow du = -\sin x dx$$

$$\Rightarrow -\int \frac{1}{u} du = -\ln|u| + C = -\ln|\cos x| + C = \ln\left|(\cos x)^{-1}\right| + C = \ln|\sec x| + C$$

$$\int \cot x dx = \int \frac{\cos x dx}{\sin x} \Rightarrow u(x) = \sin x \Rightarrow du = \cos x dx$$

$$\Rightarrow \int \frac{du}{u} = \ln|u| + C = \ln|\sin x| + C$$

To integrate sec*x*, csc*x*, however, requires the 'trick of 1'

$$\int \sec x dx = \int \sec x\left(\frac{\sec x + \tan x}{\sec x + \tan x}\right)dx = \int \frac{\sec x \tan x + \sec^2 x}{\sec x + \tan x} dx$$

$$u = \sec x + \tan x \Rightarrow du = (\sec x \tan x + \sec^2 x)dx$$

$$\Rightarrow \int \frac{\sec x \tan x + \sec^2 x}{\sec x + \tan x} dx = \int \frac{du}{u} = \ln|u| + C = \ln|\sec x + \tan x| + C$$

A similar trick multiplying by $\left(\frac{\csc x - \cot x}{\csc x - \cot x}\right)$ in the case of the integral $\int \csc x dx$ produces: the result $\int \csc x dx = -\ln|\csc x + \cot x| + C$. Note that this formula was already verified in the case of worked out solution (#37, §8.3)

Hence:

$$\int \sec x dx = \ln|\sec x + \tan x| + C \Rightarrow \frac{d}{dx}\ln|\sec x + \tan x| = \sec x$$

$$\int \csc x dx = \ln|\csc x - \cot x| + C \Rightarrow \frac{d}{dx}\ln|\csc x - \cot x| = \csc x$$

And the chain rule/ *u*-substitution generalization:

$$\int \sec u du = \ln|\sec u + \tan u| + C \Rightarrow \frac{d}{dx}\ln|\sec u + \tan u| = \sec u u'$$

$$\int \csc u du = -\ln|\csc u + \cot u| + C \Rightarrow \frac{d}{dx}\ln|\csc u + \cot u| = -\csc u u'$$

- Example (#34, § 8.4)

$$\int e^{\sec x}\sec x\tan x dx \Rightarrow u=\sec x \Rightarrow du=\sec x\tan x dx$$

$$\Rightarrow \int e^{\sec x}\sec x\tan x dx \Rightarrow \int e^{u}du=e^{u}+C=e^{\sec x}+C$$

- Example (#17, § 8.4)

$$\int \tan^{4} x\sec^{2} xdx \Rightarrow u=\tan x \Rightarrow du=\sec^{2} xdx$$

$$\Rightarrow \int \tan^{4} x\sec^{2} xdx=\int u^{4}du=\tfrac{1}{5}u^{5}+C=\tfrac{1}{5}\tan^{5} x+C$$

- Example (#47, § 8.4)

$$\int_{0}^{\pi}(2\sin x+\sin 2x)dx=2\int_{0}^{\pi}\sin xdx+\int_{0}^{\pi}\sin 2xdx=-2\cos\Big|_{0}^{\pi}+\tfrac{1}{2}\int_{u(0)}^{u(x)}\sin udu$$

$$(u=2x \Rightarrow du=2dx \Rightarrow dx=\tfrac{1}{2}du)$$

$$\Rightarrow -2(\cos\pi-\cos 0)+-\tfrac{1}{2}\cos u\Big|_{0}^{2\pi}=-2(-1-1)-\tfrac{1}{2}(\cos 2\pi-\cos 0)=4$$

- Example (#53, § 8.4) (Using method of washers)

$$V=\pi\int_{0}^{\pi/4}(\sec x)^{2}dx=\pi\int_{0}^{\pi/4}\sec^{2} xdx=\pi\tan x\Big|_{0}^{\frac{\pi}{4}}=\pi\left(\tan\tfrac{\pi}{4}-1\right)=\pi$$