MA261 **Calculus I: Notes for November 27**
Fall 2007 Kallfelz

EXAMPLES OF EXERCISES (§ 9.1)

This section in the text doesn't introduce any new material. The integration techniques discussed are a re-capitulation of topics covered in previous chapters. The particular problem solving strategy, however, can get involved (as is evidenced in some of the examples below.) Note the grey shaded table on p. 497, concerning some useful simplification tips. **Note that some of them are sufficient to solve the problem, but not necessary. (Thanks to Ricky Smith's question.) For example, the trick involving adding and subtracting terms in the numerator:**

$$\frac{2x}{x^2+2x+1}=\frac{2x+2-2}{x^2+2x+1}=\frac{2x+2}{x^2+2x+1}-\frac{2}{x^2+2x+1}$$

…is useful insofar as you've reduced the fraction into a difference between one rational expression that can easily be integrated using a u-substitution: $u=x^2+2x+1\Rightarrow du=(2x+2)dx$ and another fraction which is even simpler to integrate: $\frac{2}{x^2+2x+1}=2(x+1)^{-2}\Rightarrow u=x+1\therefore du=dx$. **But this "trick of 0" isn't necessary to solve the problem!** If you factor the denominator: $x^2+2x+1=(x+1)^2$ and then define: $u=x+1\Rightarrow x=u-1$ you could rewrite the above expression solely in terms of u: $\frac{2x}{x^2+2x+1}=\frac{2(u-1)}{u^2}=2u^{-1}-2u^{-2}$, which ends up being just as simple to integrate (in terms of u).

- Example (#7, §9.1)

$$\int\frac{t^2-3}{-t^3+9t+1}dt\Rightarrow u=-t^3+9t+1\Rightarrow du=(-3t^2+9)dt=-3(t^2-3)dt\Rightarrow(t^2-3)=-\tfrac{1}{3}du$$

$$\therefore\int\frac{t^2-3}{-t^3+9t+1}dt=-\tfrac{1}{3}\int\frac{du}{u}=-\tfrac{1}{3}\ln|u|+C=-\tfrac{1}{3}\ln\left|-t^3+9t+1\right|+C$$

- Example (#11, §9.1)

Method 1 (integrate separately)

$$\int\left(\frac{1}{3x-1}-\frac{1}{3x+1}\right)dx=\int\frac{dx}{3x-1}-\int\frac{dx}{3x+1}$$

$$u_1 = 3x-1 \Rightarrow du_1 = 3dx \Rightarrow dx = \tfrac{1}{3}du_1$$
$$u_2 = 3x+1 \Rightarrow du_2 = 3dx \Rightarrow dx = \tfrac{1}{3}du_2$$
$$\tfrac{1}{3}\int \frac{du_1}{u_1} - \tfrac{1}{3}\int \frac{du_2}{u_2} = \tfrac{1}{3}\ln|u_1| - \tfrac{1}{3}\ln|u_2| + C = \tfrac{1}{3}\ln\left|\frac{u_1}{u_2}\right| + C = \tfrac{1}{3}\ln\left|\frac{3x-1}{3x+1}\right| + C$$

Method 2 (cross-multiply and integrate using techniques in ch8)

$$\int\left(\frac{1}{3x-1} - \frac{1}{3x+1}\right)dx = \int \frac{3x+1-(3x-1)}{9x^2-1}dx = 2\int \frac{dx}{9x^2-1}$$
$$u = 3x, a = 1 \Rightarrow du = 3dx \Rightarrow dx = \tfrac{1}{3}du$$
$$\int\left(\frac{1}{3x-1} - \frac{1}{3x+1}\right)dx = \tfrac{2}{3}\int \frac{du}{u^2-1^2} = -\tfrac{2}{3}\int \frac{du}{1^2-u^2} = -\tfrac{2}{3}\left\{\tfrac{1}{2\cdot 1}\ln\left|\frac{1+u}{1-u}\right|\right\} + C$$
$$= \tfrac{2}{3}\cdot\tfrac{1}{2}\ln\left(\left|\frac{1+u}{1-u}\right|\right)^{-1} + C = \tfrac{1}{3}\ln\left|\frac{1-u}{1+u}\right| + C = \tfrac{1}{3}\ln\left|\frac{(-1)(u-1)}{1+u}\right| + C = \tfrac{1}{3}\ln\left|\frac{u-1}{u+1}\right| + C$$
$$= \tfrac{1}{3}\ln\left|\frac{3x-1}{3x+1}\right| + C$$

- Example (#24, §9.1)

Method1

$$\int \frac{1}{\sec x - 1}dx = \int \frac{1}{\sec x - 1}\cdot\frac{\sec x + 1}{\sec x - 1}dx = \int \frac{\sec x + 1}{\sec^2 x - 1}dx = \int \frac{\sec x + 1}{(\tan^2 x + 1) - 1}dx = \int \frac{\sec x - 1}{\tan^2 x}dx$$
$$= \int \frac{\cos^2 x}{\sin^2 x}\left(\frac{1}{\cos x} - 1\right)dx = \int \sin^{-2} x \cos x dx - \int \cot^2 x dx$$

The first can be integrated using a simple u-substitution ($u = \sin x$) the second can be integrated using Pythagorean identity: $\cot^2 x + 1 = \csc^2 x$

$$\int \frac{\cos^2 x}{\sin^2 x}\left(\frac{1}{\cos x} - 1\right)dx = \int \sin^{-2} x \cos x dx - \int \cot^2 x dx = \int u^{-2}du - \int(\csc^2 x - 1)dx$$
$$= -u^{-1} + \int(-\csc^2 x dx) + \int dx = -(\sin x)^{-1} + \cot x + x + C = -\csc x + \cot x + C$$
$$= -\frac{1}{\sin x} + \frac{\cos x}{\sin x} + C = \frac{\cos x - 1}{\sin x} + C$$

Method 2

$$\int \frac{1}{\sec x - 1}dx = \int \frac{1}{\frac{1}{\cos x}-1}dx = \int \frac{1}{\frac{1-\cos x}{\cos x}}dx = \int \frac{\cos x}{1-\cos x}dx = \int \frac{\cos x}{1-\cos x}\cdot\frac{1+\cos x}{1+\cos x}dx$$

$$= \int \frac{\cos x + \cos^2 x}{1-\cos^2 x}dx = \int \frac{\cos x+\cos^2 x}{\sin^2 x}dx = \int \frac{\cos x}{\sin^2 x}dx + \int \cot^2 x dx$$

The last expression is identical to the previous one which is integrated according to the procedures demonstrated above.

- Example (#48, §9.1)

Method 1: complete square first, then u-substitute

$$\int \frac{1}{(x-1)\sqrt{4x^2-8x+3}}dx = \int \frac{dx}{(x-1)\sqrt{4\left(x^2-2x+\frac{3}{4}\right)}} = \frac{1}{2}\int \frac{dx}{(x-1)\sqrt{x^2-2x+1-1+\frac{3}{4}}}$$

$$= \frac{1}{2}\int \frac{dx}{(x-1)\sqrt{(x-1)^2-\left(\frac{1}{2}\right)^2}} = \frac{1}{2}\int \frac{du}{u\sqrt{u^2-\left(\frac{1}{2}\right)^2}} = \frac{1}{2}\left\{\frac{1}{\frac{1}{2}} arc\sec\left(\frac{|u|}{\frac{1}{2}}\right)\right\} + C = arc\sec\left(2|x-1|\right) + C$$

Method 2: u-substitute then complete the square

$$\int \frac{1}{(x-1)\sqrt{4x^2-8x+3}}dx = \int \frac{du}{u\sqrt{4(u+1)^2-8(u+1)+3}} = \frac{1}{2}\int \frac{du}{u\sqrt{(u+1)^2-2(u+1)+\frac{3}{4}}}$$

$$= \frac{1}{2}\int \frac{dx}{u\sqrt{u^2+2u+1-2u-2+\frac{3}{4}}} = \frac{1}{2}\int \frac{du}{u\sqrt{u^2-\left(\frac{1}{2}\right)^2}} = \frac{1}{2}\left\{\frac{1}{\frac{1}{2}} arc\sec\left(\frac{|u|}{\frac{1}{2}}\right)\right\} + C = arc\sec\left(2|x-1|\right) + C$$

INTEGRATING BY PARTS

The integration by parts procedure is just a reduction strategy (simplifying a difficult integral). It's based on the product rule for derivatives:

$$\frac{d}{dx}(uv) = u'v + uv' \Rightarrow \int \frac{d}{dx}(uv)dx = \int (u'v + v'u)dx$$

$$\Rightarrow uv = \int vdu + \int udv \Rightarrow \int udv = uv - \int vdu$$

In definite integral form: $\int_a^b udv = uv\Big|_a^b - \int_a^b vdu$

Powers of trigonometric functions:

	Integrating:		Use:
I.)	**$\sin^n x\cos^m x$**	***n* or *m* odd**	**Reduce one to first power, use:** **$\sin^2 x + \cos^2 x = 1$to express in terms** **of powers of sine or cosine.** **the first power term is a "du" term or:** ***Use Integ. By Parts, If Necessary***
II.)	**$\sin^n x\cos^m x$**	***n* and *m* even**	**Use**: $\sin 2x = 2\sin x\cos x$ $\sin^2 x = {}^1/_2\,(1 - \cos 2x)$ $\cos^2 x = {}^1/_2\,(1 + \cos 2x)$ ***This procedure may have to be repeated*** ***more than once!***
III.)	**sin(mx)cos(nx)**		**Use:** $\sin A\cos B = {}^1/_2[\sin(A-B)+\sin(A+B)]$ $\sin A\sin B = {}^1/_2[\cos(A-B) - \cos(A+B)]$ $\cos A\cos B = {}^1/_2[\cos(A-B)+\cos(A+B)]$
IV.)	**$\tan^n x$, $\cot^n x$,** **$\sec^n x$, $\csc^n x$,**		**Use: $\tan^2 x + 1 = \sec^2 x$ or $\cot^2 x + 1 = \csc^2 x$** **to express in terms of powers of just** **one of the associated trig functions.** ***Important: Keep residual terms to*** ***express as "du" or Integrate by Parts***

Table IX. 1

Remark: The above reduction formulae can be alternatively derived via expressing sin and cos by the Euler Formulae

$$\sin x = {}^1/_{2i}(e^{ix} - e^{-ix}) \quad \text{(VI.1a)}$$

$$\cos(x) = {}^1/_2\,(e^{ix} + e^{-ix}) \quad \text{(VI.1b)}$$

Find: $\int \cos^5 x \sin^5 x\,dx$

***Answer**: (We recognize this as I.), Table IX.1)*

$$\cos^5 x = \cos^4 x\cos x = (1 - \sin^2 x)^2\cos x = (1 - 2\sin^2 x + \sin^4 x)\cos x$$

Hence: $\int \cos^5 x \sin^5 x dx = \int (\underset{\substack{\uparrow \\ \text{let } u = \sin x}}{\sin^5 x - 2\sin^7 x + \sin^9 x})\underset{\substack{\uparrow \\ du}}{\cos x\, dx}$

$\int (u^5 - 2u^7 + u^9)du = {}^1/_6\, u^6 - {}^1/_4\, u^8 - {}^1/_{10}\, u^{10} + C = \mathbf{{}^1/_6 \sin^6 x - {}^1/_4 \sin^8 x - {}^1/_{10} \sin^{10} x + C}$

<u>Exercise:</u> Derive a reduction formula for any power of cosine, cosecant, cotangent

...For cotangent: $\int \cot^n x dx = \int \cot^{(n-2)} x \cot^2 x dx = \int \cot^{(n-2)} x\, \underset{\substack{\uparrow \\ \textbf{(Case IV, Table)}}}{(\mathbf{\csc^2 x - 1})}dx$

$= \int \cot^{(n-2)} x\, \underset{\substack{\uparrow \\ \mathbf{u = \cot x \text{ then } du = -\csc^2 x dx}}}{\csc^2 x dx} \mathbf{-} \int \cot^{(n-2)} x\, dx$

$= -\int u^{(n-2)} du - \int \cot^{(n-2)} x\, dx = -\dfrac{u^{(n-1)}}{(n-1)} - \int \cot^{(n-2)} x\, dx$

$$\boxed{\textbf{SO:}\quad \int \cot^n x dx = -\frac{\cot^{(n-1)} x}{(n-1)} - \int \cot^{(n-2)} x\, dx \qquad \begin{array}{r} \textit{Formula IX.2} \\ \textbf{(\#46, Table, text)} \end{array}}$$

...For cosecant: $\int \csc^n x dx = \int \csc^{(n-2)} x \csc^2 x dx = \int \cot^{(n-2)} x\, (\mathbf{\cot^2 x + 1})dx$

$= \int \csc^{(n-2)} x \cot^2 x dx + \int \cot^{(n-2)} x dx = \int \underset{\substack{\uparrow \\ \mathbf{u = \csc x}}}{\csc^{(n-3)} x}\, \underset{\substack{\uparrow \\ \mathbf{-du}}}{(\csc x \cot x dx)}\cot x + \int \csc^{(n-2)} x dx$

- ***Observe: We must perform the first integral by parts:***

  Let: $dV = \mathbf{\cot^{(n-3)} x\, (\csc x \cot x)dx = -u^{n-3} du}$ $\qquad U = \mathbf{\cot x}$
  Then: $V = \mathbf{\dfrac{-u^{n-2}}{(n-2)}}$ $\qquad dU = \mathbf{-\csc^2 x}$

Hence: $\int \cot^{(n-3)}x\,(\csc x\cot x dx)\cot x = \underset{\substack{\uparrow\ \ \uparrow\\ V\ \ U}}{\frac{-\csc^{n-2}x\cot x}{(n-2)}} - \underset{\substack{\uparrow\\ V}}{\frac{1}{(n-2)}}\int \csc^{(n-2)}x\,\underset{\substack{\uparrow\\ dU}}{(\csc^2x)dx}$

Or: $\int \csc^n x dx = \frac{-\csc^{n-2}x\cot x}{(n-2)} - \frac{1}{(n-2)}\int \csc^n x dx + \int \csc^{(n-2)}x dx$

Note: The above result has the same term on either side, which means it can be carried across and algebraically isolated:

$$[\,1 + (n-2)^{-1}\,]\int \csc^n x dx = \frac{-\csc^{n-2}x\cot x}{(n-2)} + \int \csc^{(n-2)}x dx$$

or: $\frac{n-1}{(n-2)}\int \csc^n x dx = \frac{-\csc^{n-2}x\cot x}{(n-2)} + \int \csc^{(n-2)}x dx$

Hence: $\int \csc^n x dx = \frac{-\csc^{n-2}x\cot x}{(n-1)} + \frac{(n-2)}{(n-1)}\int \csc^{(n-2)}x dx$ Formula IX.3

Int. Table #47, text

<u>Exercise:</u> ***Find*** $\int_0^{\pi/3} \tan x \sec^{3/2}x dx$

)

Answer: Focus on the half-integer power of secx. We know: (secx)' = secxtanx

Hence: $\int \tan x\sec^{3/2}x dx = \int \underset{\substack{\uparrow\\ \textbf{let } u=\sec x}}{\sec^{1/2}x}\underset{\substack{\uparrow\\ \textbf{du}}}{(\sec x\tan x)dx} = \int u^{1/2}du = \frac{2}{3}u^{3/2} + C = \mathbf{\frac{2}{3}\sec^{3/2}x + C}$

...evaluating at the limits of integration: $\frac{2}{3}\sec^{3/2}x\Big|_0^{\pi/3} = \frac{2}{3}\left[\frac{2\sqrt{3}}{3} - 1\right]$

<u>Exercise:</u> $\int \sec^4 2x dx = \frac{1}{2}\int \sec^4 u du = \frac{1}{2}\int \sec^2 u\sec^2 u du = \frac{1}{2}\int \sec^2 u[1 + \tan^2 u]du$

$$^{1}/_{2}\int \sec^{2}u\,du \;+\; ^{1}/_{2}\int \underset{\substack{\uparrow\\ \mathbf{U\,=\,\tan u}}}{\tan^{2}u}\,\underset{\substack{\uparrow\\ \mathbf{dU}}}{(\sec^{2}u\,du)} \;=\; ^{1}/_{2}\,\{\tan(2x) + {}^{1}/_{3}\tan^{3}(2x)\} + C$$

- ***Note:, table could also be used to solve the problem***

$$\int \sec^{n}u\,du = \frac{\sec^{(n-1)}u\,\sin u}{(n-1)} + \frac{(n-2)}{(n-1)}\int \sec^{(n-2)}u\,du$$

Int. Table #38, text

Hence, for n=4, our answer according to #38 is: $\mathbf{{}^{1}/_{3}\sec^{3}u\,\sin u + {}^{2}/_{3}\tan u}$

But: $\mathbf{\sec^{3}u\,\sin u = \sec^{2}u\,\tan u}$, ***hence*****:**

$$^{1}/_{3}\sec^{2}u\,\tan u + {}^{2}/_{3}\tan u = {}^{1}/_{3}(\tan^{2}u + 1)\tan u + {}^{2}/_{3}\tan u \;\mathbf{= \tan u + {}^{1}/_{3}\tan^{3}u}$$