ANTI-DERIVATIVE FORMULAE FOR DERIVATIVES OF INVERSE TIRGONOMETRIC FUNCTIONS (§ *8.6*)

- **Recall (p. 13, <u>November 13</u> notes)**

$$\frac{d}{dx}\arcsin x = \frac{1}{\sqrt{1-x^2}} \Rightarrow \frac{d}{dx}\arcsin u = \frac{u'}{\sqrt{1-u^2}}$$
$$\frac{d}{dx}\arccos x = \frac{-1}{\sqrt{1-x^2}} \Rightarrow \frac{d}{dx}\arccos u = \frac{-u'}{\sqrt{1-u^2}}$$
$$\frac{d}{dx}\arctan x = \frac{1}{1+x^2} \Rightarrow \frac{d}{dx}\arctan u = \frac{u'}{1+u^2}$$
$$\frac{d}{dx}arc\cot x = \frac{-1}{1+x^2} \Rightarrow \frac{d}{dx}arc\cot u = \frac{-u'}{1+u^2}$$
$$\frac{d}{dx}arc\sec x = \frac{1}{|x|\sqrt{x^2-1}} \Rightarrow \frac{d}{dx}arc\sec u = \frac{u'}{|u|\sqrt{u^2-1}}$$
$$\frac{d}{dx}arc\csc x = \frac{-1}{|x|\sqrt{x^2-1}} \Rightarrow \frac{d}{dx}arc\csc u = \frac{-u'}{|u|\sqrt{u^2-1}}$$

We can exploit this ambiguity in the following way, concerning anti-derivative formulae:
Note that only *three* are required:

$$\int \frac{du}{\sqrt{1-u^2}} = \arcsin u + C = -\arccos u + C$$
$$\int \frac{du}{1+u^2} = \arctan u + C = -arc\cot u + C$$
$$\int \frac{du}{u\sqrt{u^2-1}} = arc\sec u + C = -arc\csc u + C$$

…The question now becomes: what about an integral of the above form whose constant is not equal to 1? (I.e. some general $a > 0$?) (These are the integral formulae summarized in THM 8.8, p. 468) The book indicates how they can be proved by differentiating the right hand side. However, one can also adopt a ***u*-substitution** on the left hand side[1]:

$$\int \frac{du}{\sqrt{a^2-u^2}} = \int \frac{du}{\sqrt{a^2\left(1-\frac{u^2}{a^2}\right)}} = \int \frac{du}{\sqrt{a^2}\sqrt{1-\left(\frac{u}{a}\right)^2}} = \frac{1}{a}\int \frac{du}{\sqrt{1-\left(\frac{u}{a}\right)^2}}$$

[1] Note how in the derivation below, since $a > 0$, we can conclude: $\sqrt{a^2} = a$. Otherwise we'd have to say: $\sqrt{a^2} = |a|$.

$$w=\frac{u}{a}=\tfrac{1}{a}u \Rightarrow \frac{dw}{du}=\tfrac{1}{a}\Rightarrow dw=\frac{du}{a}$$

$$\therefore \tfrac{1}{a}\int\frac{du}{\sqrt{1-\left(\frac{u}{a}\right)^2}}=\int\frac{du/a}{\sqrt{1-\left(\frac{u}{a}\right)^2}}=\int\frac{dw}{\sqrt{1-w^2}}=\arcsin w+C=\arcsin\left(\frac{u}{a}\right)+C$$

Similarly for the second formula:

$$\int\frac{du}{a^2+u^2}=\frac{1}{a^2}\int\frac{du}{1+\left(\frac{u}{a}\right)^2}\Rightarrow w=\frac{u}{a}\Rightarrow dw=\tfrac{1}{a}du$$

$$\therefore \frac{1}{a^2}\int\frac{du}{1+\left(\frac{u}{a}\right)^2}=\frac{1}{a}\int\frac{\frac{1}{a}du}{1+\left(\frac{u}{a}\right)^2}=\frac{1}{a}\int\frac{dw}{1+w^2}=\frac{1}{a}\arctan w+C=\frac{1}{a}\arctan\left(\frac{u}{a}\right)+C$$

The third formula can be differentiated on the left hand side, since it's a little trickier. Notice that the argument for arcsec is stated most generally in terms of |*u*|. (You'll also notice that this is Exercise 52b, part of one of your assigned problems ☺)

$$\frac{d}{dx}\frac{1}{a}arc\sec\left(\frac{|u|}{a}\right)=\frac{1}{a}\frac{d}{dx}arc\sec\left(\frac{|u|}{a}\right)=\begin{cases}\frac{1}{a}\frac{d}{dx}arc\sec\left(\frac{u}{a}\right) & u\ge a\\ \frac{1}{a}\frac{d}{dx}arc\sec\left(\frac{-u}{a}\right) & u\le -a\end{cases}$$

Note a subtlety in terms of the cases defined above. As you recall (November 13 notes) the domain of **arcsec*x*** is $(-\infty,-1]\cup[1,\infty)=\{x:|x|>1\}$. Hence concerning the above two cases, this implies that either $\frac{u}{a}\ge 1\Rightarrow u\ge a$ or $\frac{u}{a}\le -1\Rightarrow u\le -a$.

Differentiating the first case ($u\ge a$):

$$\tfrac{1}{a}\frac{d}{dx}arc\sec\left(\tfrac{u}{a}\right)=\tfrac{1}{a}\frac{\frac{d}{dx}\left(\frac{u}{a}\right)}{\left|\frac{u}{a}\right|\sqrt{\left(\frac{u}{a}\right)^2-1}}=\tfrac{1}{a}\frac{\frac{1}{a}u'}{\frac{|u|}{a}\sqrt{\frac{u^2}{a^2}-1}}=\frac{\frac{1}{a}u'}{u\sqrt{\frac{1}{a^2}\left(u^2-a^2\right)}}=\frac{\frac{1}{a}u'}{\frac{1}{a}u\sqrt{u^2-a^2}}=\frac{u'}{u\sqrt{u^2-a^2}}$$

Note the subtleties: Since $u \ge a$, and $a > 0$, then certainly $|u| = u$, and $|a| = a$.

Differentiating the second case ($u\le a$):

$$\tfrac{1}{a}\frac{d}{dx}arc\sec\left(-\tfrac{u}{a}\right)=\tfrac{1}{a}\frac{\frac{d}{dx}\left(-\frac{u}{a}\right)}{\left|\frac{u}{a}\right|\sqrt{\left(\frac{u}{a}\right)^2-1}}=\tfrac{1}{a}\frac{-\frac{1}{a}u'}{\frac{|u|}{a}\sqrt{\frac{u^2}{a^2}-1}}=\frac{-\frac{1}{a}u'}{-u\sqrt{\frac{1}{a^2}\left(u^2-a^2\right)}}=\frac{\frac{1}{a}u'}{\frac{1}{a}u\sqrt{u^2-a^2}}=\frac{u'}{u\sqrt{u^2-a^2}}$$

So: $\frac{d}{dx}arc\sec\left(\frac{|u|}{a}\right)=\frac{u'}{u\sqrt{u^2-a^2}}\Rightarrow\int\frac{du}{u\sqrt{u^2-a^2}}=arc\sec\left(\frac{|u|}{a}\right)+C$

- **Evaluating integrals adopting the above formulae involve a combination of *u*-substitutions (sometimes more than one) and sometime (in addition) completing the square.**

As a brief review, recall the procedure of completing the square for the following quadratic expression: ax^2+bx+c:

$a\left(x^2+\frac{b}{a}x+\frac{c}{a}\right)$	1. Factor out leading term
$a\left(x^2+\frac{b}{a}x+\left(\frac{b}{2a}\right)^2-\left(\frac{b}{2a}\right)^2+\frac{c}{a}\right)$ $=a\left(x^2+\frac{b}{a}x+\frac{b^2}{4a^2}+\frac{c}{a}-\frac{b^2}{4a^2}\right)$	2. Divide middle coefficient by 2 and square it and add it and its negative to the expression.[2]
$=a\left(x+\frac{b}{2a}\right)^2+a\left(\frac{c}{a}-\frac{b}{4a^a}\right)$	3. Factor the first three terms (they form a perfect square and group the remaining constant terms together.

…This is the essential procedure: basically completing the square means transforming any quadratic expression into a sum of a perfect square term + its remaining constant terms. For example, if the above expression represented some quadratic function:

$$y=f(x)=ax^2+bx+c=a\left(x+\tfrac{b}{2a}\right)^2+a\left(\tfrac{c}{a}-\tfrac{b}{4a^2}\right)=a\left(x+\tfrac{b}{2a}\right)^2+\frac{4ac-b^2}{4a}$$

Then the above reformulation in terms of completing the square makes it easier to *graph* this function: it's a parabola, represented (on the right) in the $f(x)=a(x-h)^2+k$ form.

[2] I.e., the "trick of 0". One can also, in the case of a quadratic equation, add the same term to both sides,

That is to say, its **vertex** is: $(h,k)=\left(\frac{b}{2a},\frac{4ac-b^2}{4a}\right)$, while its leading coefficient a determines the **curvature** of the parabola. (If $a >> 1$, then it's upward opening and narrow. If $0 < a$, but $a << 1$, it's upward opening and broad. If $a << -1$, it's downward opening and narrow, and if $a < 0$, but $a >> -1$, it's downward opening and broad.)

A special case of course occurs for the x-intercepts ($y = 0$) of such a parabola:

$$y=f(x)=0=a\left(x+\tfrac{b}{2a}\right)^2+\frac{4ac-b^2}{4a}\Rightarrow a\left(x+\tfrac{b}{2a}\right)^2=-\frac{4ac-b^2}{4a}\Rightarrow \left(x+\tfrac{b}{2a}\right)^2=\frac{b^2-4ac}{4a^2}$$

$$\Rightarrow \left(x+\tfrac{b}{2a}\right)=\pm\sqrt{\frac{b^2-4ac}{4a^2}}=\pm\frac{\sqrt{b^2-4ac}}{\sqrt{4a^2}}=\frac{\pm\sqrt{b^2-4ac}}{2a}\Rightarrow x_{1,2}=-\tfrac{b}{2a}\pm\tfrac{\sqrt{b^2-4ac}}{2a}=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

…which of course we recognize as the quadratic formula! In other words, the technique of solving quadratic equations is underwritten by the procedure of completing the square. The quadratic formula is just an abbreviation, or a result thereof.

- Example 5, § 8.6

$\int\frac{dx}{x\sqrt{4x^2-1}}$. This expression resembles most the third formula:

$$\int\frac{du}{u\sqrt{u^2-a^2}}=arc\sec\left(\frac{|u|}{a}\right)+C$$

Method 1: Let $u = (2x)$

$$\int\frac{dx}{x\sqrt{4x^2-1}}=\int\frac{dx}{x\sqrt{(2x)^2-1}}\Rightarrow u=2x\Rightarrow x=\tfrac{1}{2}u\ \&\ du=2dx\Rightarrow dx=\tfrac{1}{2}du$$

$$\therefore\int\frac{dx}{x\sqrt{4x^2-1}}=\int\frac{\frac{1}{2}du}{\frac{1}{2}u\sqrt{u^2-1}}=\int\frac{du}{u\sqrt{u^2-1}}=arc\sec\left(\frac{|u|}{1}\right)+C=arc\sec(2|x|)+C$$

Method 2: Factor out the 4 in the quadratic expression

$$\int\frac{dx}{x\sqrt{4x^2-1}}=\int\frac{dx}{2x\sqrt{x^2-\frac{1}{4}}}=\tfrac{1}{2}\int\frac{dx}{x\sqrt{x^2-\left(\frac{1}{2}\right)^2}}=\tfrac{1}{2}\left\{\frac{1}{\frac{1}{2}}arc\sec\left(\frac{|x|}{\frac{1}{2}}\right)\right\}+C$$

$$=arc\sec(2|x|)+C$$

- Example 13, § 8.6

$$\int \frac{\arctan x}{1+x^2}dx = \int \arctan x\left(\frac{dx}{1+x^2}\right) \Rightarrow u = \arctan x \Rightarrow du = \frac{dx}{1+x^2}$$

$$\Rightarrow \int \arctan x\left(\frac{dx}{1+x^2}\right) = \int udu = \tfrac{1}{2}u^2 + C = \tfrac{1}{2}(\arctan x)^2 + C$$

Example 25, § 8.6

$$\int_{\pi/2}^{\pi} \frac{\sin x}{1+\cos^2 x}dx \Rightarrow u(x) = \cos x \Rightarrow \frac{du}{dx} = -\sin x \Rightarrow du = -\sin xdx$$

$$\int_{\pi/2}^{\pi} \frac{\sin x}{1+\cos^2 x}dx = \int_{u(\pi/2)}^{u(\pi)} \frac{-du}{1+u^2} = -\int_0^{-1} \frac{du}{1+u^2} = \int_{-1}^{0} \frac{du}{1+x^2} = \arctan u\Big|_{-1}^{0} = \arctan(0) - \arctan(-1) = 0 - \left(-\tfrac{\pi}{4}\right) = \tfrac{\pi}{4}$$

Example 37, § 8.6

$$\int \frac{x}{x^4+2x^2+2}dx$$

Step 1: $u = x^2 \Rightarrow du = 2xdx \Rightarrow xdx = \tfrac{1}{2}du$

$$\therefore \int \frac{x}{x^4+2x^2+2}dx = \tfrac{1}{2}\int \frac{du}{u^2+2u+2}$$

Step 2: Complete the square in the denominator term:

$$\therefore \tfrac{1}{2}\int \frac{du}{u^2+2u+2} = \tfrac{1}{2}\int \frac{du}{u^2+2u+1-1+2} = \tfrac{1}{2}\int \frac{du}{(u+1)^2+1^2}$$

Step 3: $w = u+1 \Rightarrow dw = du, u = w-1$

$$\therefore \tfrac{1}{2}\int \frac{du}{(u+1)^2+1^2} = \tfrac{1}{2}\int \frac{dw}{w^2+1^2} = \tfrac{1}{2}\cdot\tfrac{1}{1}\arctan\left(\frac{w}{1}\right) + C = \tfrac{1}{2}\arctan(u+1) + C = \tfrac{1}{2}\arctan(x^2+1) + C$$

Example 39, § 8.6

$$\int \frac{dx}{\sqrt{-16x^2+16x-3}} = \int \frac{dx}{\sqrt{16\left(-x^2+x-\tfrac{3}{16}\right)}} = \tfrac{1}{4}\int \frac{dx}{\sqrt{-x^2+x+\tfrac{1}{4}-\tfrac{1}{4}-\tfrac{3}{16}}}$$

$$= \tfrac{1}{4}\int \frac{dx}{\sqrt{-\left(x^2-x+\tfrac{1}{4}\right)+\tfrac{1}{4}-\tfrac{3}{16}}} = \tfrac{1}{4}\int \frac{dx}{\sqrt{-\left(x-\tfrac{1}{2}\right)^2+\tfrac{1}{16}}} = \tfrac{1}{4}\int \frac{dx}{\sqrt{\left(\tfrac{1}{4}\right)^2-\left(x-\tfrac{1}{2}\right)^2}}$$

$$u = x - \tfrac{1}{2} \Rightarrow du = dx, a = \tfrac{1}{4}$$

$$\therefore \tfrac{1}{4}\int \frac{dx}{\sqrt{\left(\tfrac{1}{4}\right)^2 - \left(x-\tfrac{1}{2}\right)^2}} = \tfrac{1}{4}\int \frac{du}{\sqrt{\left(\tfrac{1}{4}\right)^2 - u^2}} = \tfrac{1}{4}\arcsin\left(\frac{u}{\tfrac{1}{4}}\right) + C = \tfrac{1}{4}\arcsin\left(4\left(x-\tfrac{1}{2}\right)\right) + C = \tfrac{1}{4}\arcsin(4x-2) + C$$

Example 44, § 8.6

$$\int \frac{dt}{t\sqrt{t}+\sqrt{t}} = \int \frac{t^{-1/2}dt}{t+1} = \int \frac{t^{-1/2}dt}{\left(\sqrt{t}\right)^2+1} \Rightarrow u = t^{1/2} \Rightarrow du = \tfrac{1}{2}t^{-1/2}dt \Rightarrow t^{-1/2}dt = 2du$$

$$\therefore \int \frac{dt}{t\sqrt{t}+\sqrt{t}} = \int \frac{2du}{u^2+1} = 2\int \frac{du}{u^2+1} = 2\left\{\tfrac{1}{1}\arctan\left(\tfrac{u}{1}\right)\right\} + C = \arctan\left(\sqrt{t}\right) + C$$

Check: $\dfrac{d}{dt}2\arctan\left(t^{1/2}\right) = 2\dfrac{\frac{d}{dt}t^{1/2}}{\left(t^{1/2}\right)^2+1} = 2\dfrac{\tfrac{1}{2}t^{-1/2}}{t+1} = \dfrac{1}{t\sqrt{t}+\sqrt{t}}$

Example 47, § 8.6

$$y = \frac{1}{x^2-2x+5}, y = 0, x = 1, x = 3$$

$$A = \int_1^3 \frac{dx}{x^2-2x+5} = \int_1^2 \frac{dx}{x^2-2x+1-1+5} = \int_1^3 \frac{dx}{(x-1)^2+2^2}$$

$$u = x - 1 \Rightarrow du = dx$$

$$\therefore \int_1^3 \frac{dx}{(x-1)^2+2^2} = \int_{u(1)}^{u(3)} \frac{du}{u^2+2^2} = \int_0^2 \frac{du}{u^2+2^2} = \tfrac{1}{2}\arctan\left(\frac{u}{2}\right)\Bigg|_0^2 = \tfrac{1}{2}\left(\arctan 1 - \arctan 0\right) = \tfrac{1}{2}\cdot\tfrac{\pi}{4} = \tfrac{\pi}{8}$$

THE HYPERBOLIC TRIGONOMETRIC FUNCTIONS

Definition: The sinh(x) = $^1/_2\,(e^x - e^{-x})$ cosh(x) = $^1/_2\,(e^x + e^{-x})$ are rigorously defined by the observation that the area A(x) under hyperbolic sector ($x^2 - y^2 = 1$) terminating at endpoint (cosh(x),sinh(x)) = $^x/_2$, exactly in the same manner as the area A(x) under the circular sector ($x^2 + y^2 = 1$) terminating at endpoint (cos$x$, sin$x$) = $^x/_2$,

- ***Remark: The formulaic correspondence between sinh, cosh; sin, cos is rendered more apparent by recalling Euler's Formula:***

$$e^{ix} = \cos x + i\sin x \qquad \text{(where } i^2 = -1\text{)}. \qquad \textbf{(VI.1)}$$

From Euler's Theorem we obtain:

$$\sin x = \tfrac{1}{2i}(e^{ix} - e^{-ix})$$

$$\cos x = \tfrac{1}{2}(e^{ix} + e^{-ix})$$

The absence of i in the definition of sinh and cosh facilitates the calculation of their antiderivatives, as we shall investigate shortly.

First, let's derive the result for the circle.[3] We do this indirectly by showing:

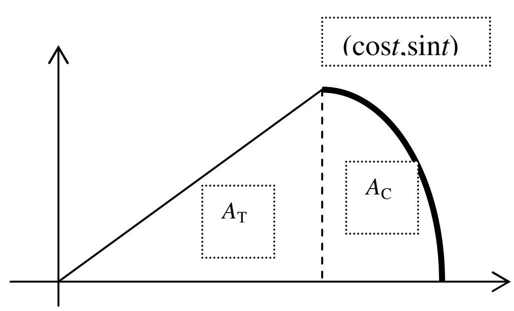
$A'(t) = {}^1/_2$, (for angle(s) of *radian* measure t)

Derivation:

We need to derive the area of the "pie-piece," the circular sector subtended by angle t. As mentioned, this region is composed of two parts: A_T and A_C, the area of the right triangle subtended by (x=cost, y=sint) and the area of the sliver, derived:

$$A_C = \int_{\cos t}^{1} [\,1 - x^2]^{1/2}\,dx$$

And: $A_T = {}^1/_2(\cos t)(\sin t)$



[3] We'll later see this result can be much more simply established by a change of coordinates into Polar.

- **Derive: A'(t) = 1/2**

$$A'(t) = \frac{d}{dt}\left(\tfrac{1}{2}\cos t \sin t + \int_{\cos t}^{1} \sqrt{1-x^2}\,dx \right) = \tfrac{1}{2}\cos^2 t - \tfrac{1}{2}\sin^2 t - \sqrt{1-\cos^2 t}(-\sin t)$$
$$= \tfrac{1}{2}\cos^2 t - \tfrac{1}{2}\sin^2 t - (\sin t)(-\sin t) = \tfrac{1}{2}\left(\cos^2 t + \sin^2 t\right) = \tfrac{1}{2}$$

It's easy to see A(0) = 0,[4] hence $A(t) = \int_0^t A'(t)dt = (A(t) - A(0)) = \tfrac{1}{2}t$

- **Sum/difference identities:**

$\cosh^2(x) - \sinh^2(x) = 1$
$\sinh(x + y) = \sinh x \cosh y + \cosh x \sinh y$
$\cosh(x + y) = \cosh x \cosh y + \sinh x \sinh y$

Verifications:

$$\cosh^2 x - \sinh^2 x = \left[\tfrac{1}{2}\left(e^x + e^{-x}\right)\right]^2 - \left[\tfrac{1}{2}\left(e^x - e^{-x}\right)\right]^2 = \frac{e^{2x} + 2e^0 + e^{-2x}}{4} - \frac{e^{2x} - 2e^0 + e^{-2x}}{4}$$
$$= \tfrac{1}{4}\left(e^{2x} + 2 + e^{-2x} - e^{2x} + 2 - e^{-2x}\right) = \tfrac{1}{4}(2+2) = 1$$

$$\sinh(x + y) = \tfrac{1}{2}(e^{x+y} - e^{-x-y}) = \tfrac{1}{2}(e^{x+y} - e^{x-y} + e^{x-y} - e^{-x+y} + e^{-x+y} - e^{-x-y})$$
$$= \tfrac{1}{4}(2e^{x+y} - 2e^{x-y} + 2e^{x-y} - 2e^{-x+y} + 2e^{-x+y} - 2e^{-x-y})$$
$$= \tfrac{1}{4}(e^{x+y} - e^{-x+y} + e^{x-y} - e^{-x-y}) + \tfrac{1}{4}(e^{x+y} + e^{-x+y} - e^{x-y} - e^{-x-y})$$
$$= \tfrac{1}{2}(e^{x} - e^{-x})\,\tfrac{1}{2}(e^{y} + e^{y}) + \tfrac{1}{2}(e^{x} + e^{-x})\,\tfrac{1}{2}(e^{y} - e^{-y})$$
$$= \sinh x \cosh y + \cosh x \sinh y$$

It's easy to see from the above identities that (see p. 478, grey boxed formulae):

(a) $\tanh^2 x + \sec h^2 x = 1$

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[4] $A(0) = \tfrac{1}{2}\cos 0 \sin 0 + \int_{\cos 0}^{1} \sqrt{1-x^2}\,dx = \tfrac{1}{2}\cdot 0 + \int_{1}^{1}\sqrt{1-x^2}\,dx = 0 + 0 = 0$

…Since: $\dfrac{\cosh^2 x - \sinh^2 x}{\cosh^2 x} = \dfrac{1}{\cosh^2 x} \Rightarrow 1 - \tanh^2 x = \sec h^2 x \Rightarrow \sec h^2 x + \tanh^2 x = 1$

(b) $\coth^2 x - \csc h^2 x = 1$

…Since: $\dfrac{\cosh^2 x - \sinh^2 x}{\sinh^2 x} = \dfrac{1}{\sinh^2 x} \Rightarrow \coth^2 x - 1 = \csc h^2 x \Rightarrow \sec h^2 x - \csc h^2 x = 1$

(c) $\sinh 2x = 2\sinh x \cosh x$

…*Since* sinh(x + x) = sinh$x$cosh$x$ + cosh$x$sinh$y$ = 2sinh$x$cosh$x$

(d) $\cosh 2x = \cosh^2 x + \sinh^2 x$

…*Since* cosh(x + x) = cosh$x$cosh$x$ + sinh$x$sinh$x$ = $(\cosh x)^2 + (\sinh x)^2$

- **Differentiation Formulae:**

  From definitions: sinh(x) = $^1/_2\,(e^x - e^{-x})$ cosh(x) = $^1/_2\,(e^x + e^{-x})$

  tanh(x) = $\dfrac{(e^x - e^{-x})}{(e^x + e^{-x})}$ coth(x) = $\dfrac{(e^x + e^{-x})}{(e^x - e^{-x})}$

  sech(x) = $2(e^x + e^{-x})^{-1}$ sech(x) = $2(e^x - e^{-x})^{-1}$

- **It's simple to see: (sinhx)' = coshx (coshx)' = sinhx**

  (tanhx)' = sech^{2}x (cothx)'= - csch^{2}x

  (sechx) ' = -sechxtanhx (cschx)' = -cschxcothx

Proof:

$$\frac{d}{dx}\sinh x = \frac{d}{dx}\tfrac{1}{2}\left(e^x - e^{-x}\right) = \tfrac{1}{2}\left(\frac{d}{dx}e^x - \frac{d}{dx}e^{-x}\right) = \tfrac{1}{2}\left(e^x - (-1)e^{-x}\right) = \tfrac{1}{2}\left(e^x + e^{-x}\right) = \cosh x$$

$$\frac{d}{dx}\cosh x = \frac{d}{dx}\tfrac{1}{2}\left(e^x + e^{-x}\right) = \tfrac{1}{2}\left(\frac{d}{dx}e^x + \frac{d}{dx}e^{-x}\right) = \tfrac{1}{2}\left(e^x + (-1)e^{-x}\right) = \tfrac{1}{2}\left(e^x - e^{-x}\right) = \sinh x$$

$$\frac{d}{dx}\tanh x = \frac{d}{dx}\left(\frac{\sinh x}{\cosh x}\right) = \frac{\cosh x\left(\frac{d}{dx}\sinh x\right) - \sinh x\left(\frac{d}{dx}\cosh x\right)}{\cosh^2 x} = \frac{\cosh^2 x - \sinh^2 x}{\cosh^2 x} = \frac{1}{\cosh^2 x} = \sec h^2 x$$

$$\frac{d}{dx}\coth x = \frac{d}{dx}\left(\frac{\cosh x}{\sinh x}\right) = \frac{\sinh x\left(\frac{d}{dx}\cosh x\right) - \cosh x\left(\frac{d}{dx}\sinh x\right)}{\sinh^2 x} = \frac{-(\cosh^2 x - \sinh^2 x)}{\sinh^2 x} = -\frac{1}{\sinh^2 x} = -\csc h^2 x$$

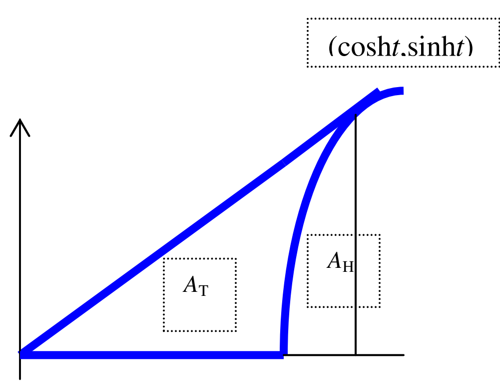
$$\frac{d}{dx}\sec hx = \frac{d}{dx}(\cosh x)^{-1} = -(\cosh x)^{-2}\frac{d}{dx}\cosh x = -\frac{\sinh x}{\cosh^2 x} = -\sec hx \tanh x$$

$$\frac{d}{dx}\csc hx = \frac{d}{dx}(\sinh x)^{-1} = -(\sinh x)^{-2}\frac{d}{dx}\sinh x = -\frac{\cosh x}{\sinh^2 x} = -\csc hx \coth x$$

Hence in their general chain rule form (and anti-derivative form) (Thm 8.9, p. 478):

$$\frac{d}{dx}\sinh u = \cosh u u' \Rightarrow \int \cosh u du = \sinh u + C$$
$$\frac{d}{dx}\cosh u = \sinh u u' \Rightarrow \int \sinh u du = \cosh u + C$$
$$\frac{d}{dx}\tanh u = \sec h^2 u u' \Rightarrow \int \sec h^2 u du = \tanh u + C$$
$$\frac{d}{dx}\coth u = -\csc h u u' \Rightarrow \int \csc h u du = -\coth u + C$$
$$\frac{d}{dx}\sec hu = -\sec hu \tanh u u' \Rightarrow \int \sec hu \tanh u = -\sec hu + C$$
$$\frac{d}{dx}\csc hu = -\csc hu \coth u u' \Rightarrow \int \csc hu \coth u = -\csc hu + C$$

Now that the above formulae were established/constructed, we can derive area formula for hyperbolic sine and cosine:



We need to derive the area of the hyperbolic wedge (etched in blue boundary): this region is composed of two parts: A_T and A_H , the area of the right triangle subtended by (x=cosht, y=sinht) and the area of the hyberbolic sliver, derived:

$$A_H = \int_1^{\cosh t} [\ x^2\ - 1]^{1/2}\, dx$$

And: $A_T = {}^1/_2\,(\cosh t)(\sinh t)$

We *subtract* A_H from A_T

- **Derive: A'(t) = $^1/_2$**

$$A'(t) = \frac{d}{dt}\left(\tfrac{1}{2}\cosh t \sinh t - \int_1^{\cosh t} \sqrt{x^2 - 1}\,dx\right) = \tfrac{1}{2}\cosh^2 t + \tfrac{1}{2}\sinh^2 t - \sqrt{1 - \cosh^2 t}\,(\sinh t)$$

$$= \tfrac{1}{2}\cosh^2 t + \tfrac{1}{2}\sinh^2 t - (\sinh t)(\sinh t) = \tfrac{1}{2}\left(\cosh^2 t - \sinh^2 t\right) = \tfrac{1}{2}$$

It's easy to see A(0) = 0,[5] hence $A(t) = \int_0^t A'(t)dt = (A(t) - A(0)) = \tfrac{1}{2}t$

THE INVERSE HYPERBOLIC FUNCTIONS

Note somewhat unexpected representation of the inverse hyperbolic functions, as summarized in THM8.10 (p. 481). One arrives at these formulae via the procedures of constructing an inverse (see pp. 3-4 of **Notes on Exponentials/Logs (Oct 23 class)**):

For instance: (deriving the expression for inverse sinh*x*, or arcsinh*x*)

$$y = f(x) = \sinh x = \tfrac{1}{2}\left(e^x - e^{-x}\right) \Rightarrow x = \tfrac{1}{2}\left(e^y - e^{-y}\right) \Rightarrow 2x = e^y - e^{-y}$$
$$\Rightarrow 2xe^y = e^{2y} - 1 \Rightarrow e^{2y} - 2xe^y - 1 = 0$$
$$u = e^y \Rightarrow e^{2y} - 2xe^y - 1 = 0 \Rightarrow u^2 - 2xu - 1 = 0$$

Hence the above has been converted to a ‘quadratic equation’, which can be formally solved using the quadratic formula:

[5] $A(0) = \tfrac{1}{2}\cosh 0 \sinh 0 - \int_{\cosh 0}^{1} \sqrt{x^2 - 1}\,dx = \tfrac{1}{2}\cdot 0 - \int_1^1 \sqrt{x^2 - 1}\,dx = 0 + 0 = 0$

$$u_{1,2}=\frac{2x\pm\sqrt{4x^2+4}}{2}=\frac{2x\pm 2\sqrt{x^2+1}}{2}=x\pm\sqrt{x^2+1}\Rightarrow e^{y_{1,2}}=x\pm\sqrt{x^2+1}$$
$$\Rightarrow y_{1,2}=\ln\left(x\pm\sqrt{x^2+1}\right)$$

To avoid the possibility of the argument being negative, select the "+" case:

$$y=\ln\left(x+\sqrt{x^2+1}\right)\Rightarrow f^{-1}(x)=\arcsin hx=\ln\left(x+\sqrt{x^2+1}\right)$$

…Similarly for arccosh*x*:

$$y=f(x)=\cosh x=\tfrac{1}{2}\left(e^x+e^{-x}\right)\Rightarrow x=\tfrac{1}{2}\left(e^y+e^{-y}\right)\Rightarrow 2x=e^y+e^{-y}$$
$$\Rightarrow 2xe^y=e^{2y}+1\Rightarrow e^{2y}-2xe^y+1=0$$
$$u=e^y\Rightarrow e^{2y}-2xe^y+1=0\Rightarrow u^2-2xu+1=0$$

$$u_{1,2}=\frac{2x\pm\sqrt{4x^2-4}}{2}=\frac{2x\pm 2\sqrt{x^2-1}}{2}=x\pm\sqrt{x^2-1}\Rightarrow e^{y_{1,2}}=x\pm\sqrt{x^2-1}$$
$$\Rightarrow y_{1,2}=\ln\left(x\pm\sqrt{x^2-1}\right)$$

To avoid the possibility of the argument being negative, select the "+" case:

$$y=\ln\left(x+\sqrt{x^2-1}\right)\Rightarrow f^{-1}(x)=\arccos hx=\ln\left(x+\sqrt{x^2-1}\right)$$

…Similarly for arctanh*x*:

$$y=f(x)=\tanh x=\frac{\sinh x}{\cosh x}=\frac{e^x-e^{-x}}{e^x+e^{-x}}\Rightarrow x=\frac{e^y-e^{-y}}{e^y+e^{-y}}$$
$$\Rightarrow x\left(e^y+e^{-y}\right)=e^y-e^{-y}\Rightarrow x\left(e^{2y}+1\right)=e^{2y}-1$$
$$\Rightarrow (1-x)e^{2y}=1+x\Rightarrow e^{2y}=\frac{1+x}{1-x}\Rightarrow 2y=\ln\left(\frac{1+x}{1-x}\right)\Rightarrow y=\tfrac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$$
$$\Rightarrow f^{-1}(x)=\arctan hx=\tfrac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$$

…Similarly for arctanh*x*:

$$y = f(x) = \coth x = \frac{\cosh x}{\sinh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}} \Rightarrow x = \frac{e^y + e^{-y}}{e^y - e^{-y}}$$
$$\Rightarrow x(e^y - e^{-y}) = e^y + e^{-y} \Rightarrow x(e^{2y} - 1) = e^{2y} + 1$$
$$\Rightarrow (x-1)e^{2y} = 1 + x \Rightarrow e^{2y} = \frac{1+x}{x-1} \Rightarrow 2y = \ln\left(\frac{x+1}{x-1}\right) \Rightarrow y = \tfrac{1}{2}\ln\left(\frac{x+1}{x-1}\right)$$
$$\Rightarrow f^{-1}(x) = arc\coth x = \tfrac{1}{2}\ln\left(\frac{x+1}{x-1}\right)$$

…Similarly for arcsech*x*:

$$y = f(x) = \sec hx = 2(e^x + e^{-x})^{-1} \Rightarrow x = 2(e^y + e^{-y})^{-1} \Rightarrow \frac{2}{x} = e^y + e^{-y}$$
$$\Rightarrow \frac{2}{x}e^y = e^{2y} + 1 \Rightarrow e^{2y} - \left(\tfrac{2}{x}\right)e^y + 1 = 0$$
$$u = e^y \Rightarrow e^{2y} - \left(\tfrac{2}{x}\right)e^y + 1 = 0 \Rightarrow u^2 - \left(\tfrac{2}{x}\right)u + 1 = 0$$

$$u_{1,2} = \frac{\frac{2}{x} \pm \sqrt{\frac{4}{x^2} - 4}}{2} = \frac{\frac{2}{x} \pm \frac{2}{x}\sqrt{1-x^2}}{2} = \frac{2 \pm 2\sqrt{1-x^2}}{2x} \Rightarrow e^{y_{1,2}} = \frac{1 \pm \sqrt{1-x^2}}{x}$$
$$\Rightarrow y_{1,2} = \ln\left(\frac{1 \pm \sqrt{1-x^2}}{x}\right)$$

To avoid the possibility of the argument being negative, select the "+" case:

$$y = \ln\left(\frac{1 + \sqrt{1-x^2}}{x}\right) \Rightarrow f^{-1}(x) = arc\sec hx = \ln\left(\frac{1 + \sqrt{1-x^2}}{x}\right)$$

(And finally) for arccsch*x*:

$$y = f(x) = \csc hx = 2(e^x - e^{-x})^{-1} \Rightarrow x = 2(e^y - e^{-y})^{-1} \Rightarrow \frac{2}{x} = e^y - e^{-y}$$
$$\Rightarrow \frac{2}{x}e^y = e^{2y} - 1 \Rightarrow e^{2y} - \left(\tfrac{2}{x}\right)e^y - 1 = 0$$
$$u = e^y \Rightarrow e^{2y} - \left(\tfrac{2}{x}\right)e^y - 1 = 0 \Rightarrow u^2 - \left(\tfrac{2}{x}\right)u - 1 = 0$$

$$u_{1,2}=\frac{\frac{2}{x}\pm\sqrt{\frac{4}{x^2}+4}}{2}=\frac{\frac{2}{x}\pm\frac{2}{x}\sqrt{1+x^2}}{2}=\frac{2\pm2\sqrt{1+x^2}}{2x}\Rightarrow e^{y_{1,2}}=\frac{1\pm\sqrt{1+x^2}}{x}$$

$$\Rightarrow y_{1,2}=\ln\left(\frac{1\pm\sqrt{1+x^2}}{x}\right)$$

To avoid the possibility of the argument being negative, select the "+" case:

$$y=\ln\left(\frac{1+\sqrt{1+x^2}}{x}\right)\Rightarrow f^{-1}(x)=arc\,\text{csc}\,hx=\ln\left(\frac{1+\sqrt{1+x^2}}{x}\right)=\ln\left(\frac{1}{x}+\frac{\sqrt{1+x^2}}{|x|}\right)$$

Note the subtlety in the last step: The domain for **arccsch*x*** are all the real numbers (except 0) (i.e. $(-\infty,0)\cup(0,\infty)=\{x\mid x\neq 0\}$)hence one must secure the domain in such a manner to avoid the possibility of taking the log of a negative number. This doesn't happen in the case of **arcsech*x*** since its domain is only (0,1] = { x| $0<x\leq 1$}.

The graphs on page 482 supply the reason why the domains are what they are for the respective inverse hyperbolic functions

- **Deriving their derivative formulae:**

$$\frac{d}{dx}\arcsin hx=\frac{d}{dx}\ln\left(x+\sqrt{x^2+1}\right)=\frac{\frac{d}{dx}\left(x+\sqrt{x^2+1}\right)}{\left(x+\sqrt{x^2+1}\right)}=\frac{1+\frac{1}{2}\left(x^2+1\right)^{-1/2}(2x)}{\left(x+\sqrt{x^2+1}\right)}$$

$$=\frac{1+\frac{x}{\sqrt{x^2+1}}}{\left(x+\sqrt{x^2+1}\right)}=\frac{\left(x^2+1\right)^{-1/2}\left(\sqrt{x^2+1}+1\right)}{\left(x+\sqrt{x^2+1}\right)}=\left(x^2+1\right)^{-1/2}=\frac{1}{\sqrt{x^2+1}}$$

$$\therefore\frac{d}{dx}\arcsin hx=\frac{1}{\sqrt{1+x^2}}\Rightarrow\frac{d}{dx}\arcsin hu=\frac{u'}{\sqrt{1+u^2}}\Rightarrow\int\frac{du}{\sqrt{1+u^2}}=\arcsin hu$$

$$=\ln\left(u+\sqrt{1+u^2}\right)+C$$

$$\frac{d}{dx}\operatorname{arccos}hhx = \frac{d}{dx}\ln\left(x+\sqrt{x^2-1}\right) = \frac{\frac{d}{dx}\left(x+\sqrt{x^2-1}\right)}{\left(x+\sqrt{x^2-1}\right)} = \frac{1+\frac{1}{2}\left(x^2-1\right)^{-1/2}(2x)}{\left(x+\sqrt{x^2-1}\right)}$$

$$= \frac{1+\frac{x}{\sqrt{x^2-1}}}{\left(x+\sqrt{x^2-1}\right)} = \frac{\left(x^2-1\right)^{-1/2}\left(\sqrt{x^2-1}+1\right)}{\left(x+\sqrt{x^2-1}\right)} = \left(x^2-1\right)^{-1/2} = \frac{1}{\sqrt{x^2-1}}$$

$$\therefore \frac{d}{dx}\operatorname{arccos}hhx = \frac{1}{\sqrt{x^2-1}} \Rightarrow \frac{d}{dx}\operatorname{arccos}hhu = \frac{u'}{\sqrt{u^2-1}} \Rightarrow \int\frac{du}{\sqrt{u^2-1}} = \operatorname{arccos}hu$$

$$= \ln\left(u+\sqrt{u^2-1}\right)+C$$

Certainly the above two formulae can be combined, by stating:

$$\int\frac{du}{\sqrt{u^2\pm 1}} = \ln\left(u+\sqrt{u^2\pm 1}\right)+C$$

…And of course the above holds for any positive constant a:

$$\int\frac{du}{\sqrt{u^2\pm a^2}} = \ln\left(u+\sqrt{u^2\pm a^2}\right)+C$$

…Which can be established via the following (w-substitution):

$$\int\frac{du}{\sqrt{u^2\pm a^2}} = \int\frac{du}{a\sqrt{\left(\frac{u}{a}\right)^2\pm 1}} = \int\frac{du/a}{\sqrt{(u/a)^2\pm 1}} \Rightarrow w=\frac{u}{a} \Rightarrow dw = du/a$$

$$\Rightarrow \int\frac{dw}{\sqrt{w^2\pm 1}} = \ln\left(w+\sqrt{w^2\pm 1}\right)+C = \ln\left(\frac{u}{a}+\sqrt{\left(\frac{u}{a}\right)^2\pm 1}\right)+C = \ln\left(\frac{u}{a}+\frac{\sqrt{u^2\pm a^2}}{a}\right)+C$$

$$= \ln\left(\frac{u+\sqrt{u^2\pm a^2}}{a}\right)+C = \ln\left(u+\sqrt{u^2\pm a^2}\right)-\ln a + C = \ln\left(u+\sqrt{u^2\pm a^2}\right)+C$$

Note the last step: ln$a$ is just a constant, so it can be absorbed in the undetermined constant term C.

$$\frac{d}{dx}\operatorname{arctan}hx = \tfrac{1}{2}\frac{d}{dx}\ln\left(\frac{1+x}{1-x}\right) = \tfrac{1}{2}\frac{d}{dx}\left(\ln(1+x)-\ln(1-x)\right) = \tfrac{1}{2}\left(\frac{1}{x+1}-\frac{-1}{1-x}\right) = \tfrac{1}{2}\left(\frac{2}{1-x^2}\right)$$
$$= \frac{1}{1-x^2} \Rightarrow \frac{d}{dx}\operatorname{arctan}hu = \frac{u'}{1-u^2} \Rightarrow \int\frac{du}{1-u^2} = \operatorname{arctan}hu + C = \tfrac{1}{2}\ln\left(\frac{1+u}{1-u}\right)+C$$

$$\frac{d}{dx}arc\coth hx = \tfrac{1}{2}\frac{d}{dx}\ln\left(\frac{1+x}{x-1}\right) = \tfrac{1}{2}\frac{d}{dx}\left(\ln(1+x)-\ln(x-1)\right) = \tfrac{1}{2}\left(\frac{1}{x+1}-\frac{1}{x-1}\right) = \tfrac{1}{2}\left(\frac{-2}{1-x^2}\right)$$
$$= \frac{-1}{x^2-1} \Rightarrow \frac{d}{dx}\operatorname{arctan}hu = \frac{u'}{u^2-1} \Rightarrow \int\frac{du}{u^2-1} = \operatorname{arctan}hu + C = \tfrac{1}{2}\ln\left(\frac{1+u}{u-1}\right)+C$$

Certainly the above two antiderivative formulae can be combined to:

$$\int\frac{du}{1-u^2} = \tfrac{1}{2}\ln\left|\frac{1+u}{1-u}\right|+C$$

And the appropriate generalization with respect to any constant $a > 0$:

$$\int\frac{du}{a^2-u^2} = \tfrac{1}{2a}\ln\left|\frac{a+u}{a-u}\right|+C$$

…Via the appropriate w-substitution:

$$\int\frac{du}{a^2-u^2} = \tfrac{1}{a^2}\int\frac{du}{1-\left(\frac{u}{a}\right)^2} \Rightarrow w = \tfrac{u}{a} \Rightarrow dw = \tfrac{1}{a}du \Rightarrow \tfrac{1}{a^2}\int\frac{du}{1-(u/a)^2}$$
$$= \tfrac{1}{a}\int\frac{du/a}{1-(u/a)^2} = \tfrac{1}{a}\int\frac{dw}{1-w^2} = \tfrac{1}{a}\left\{\tfrac{1}{2}\ln\left|\frac{1+w}{1-w}\right|\right\}+C = \tfrac{1}{2a}\ln\left|\frac{1+\frac{u}{a}}{1-\frac{u}{a}}\right|+C$$
$$= \tfrac{1}{2a}\ln\left|\frac{a+u}{a-u}\right|+C$$

$$\frac{d}{dx}arc\sec hx = \frac{d}{dx}\ln\left(\frac{1+\sqrt{1-x^2}}{x}\right) = \frac{d}{dx}\left(\ln\left(1+\sqrt{1-x^2}\right)-\ln x\right)$$
$$= \frac{\tfrac{1}{2}(-2x)\left(1-x^2\right)^{-1/2}}{1+\left(1-x^2\right)^{1/2}}-\frac{1}{x} = \frac{-x/\sqrt{1-x^2}}{1+\sqrt{1-x^2}}-\frac{1}{x} = \frac{-x}{\sqrt{1-x^2}+1-x^2}-\frac{1}{x}$$

$$= \frac{-x}{\sqrt{1-x^2}+1-x^2} - \frac{1}{x} = \frac{1}{\sqrt{1-x^2}}\left(\frac{-x}{1+\sqrt{1-x^2}} - \frac{\sqrt{1-x^2}}{x}\right)$$

$$= \frac{1}{\sqrt{1-x^2}}\left(\frac{-x^2-\sqrt{1-x^2}-\left(1-x^2\right)}{x\left(1+\sqrt{1-x^2}\right)}\right) = \frac{1}{\sqrt{1-x^2}}\left(\frac{-1-\sqrt{1-x^2}}{x\left(1+\sqrt{1-x^2}\right)}\right)$$

$$= \frac{-1}{x\sqrt{1-x^2}}$$

And for the arccsch$x$:

$$\frac{d}{dx} arc\sec hx = \frac{d}{dx}\ln\left(\frac{1+\sqrt{1+x^2}}{x}\right) = \frac{d}{dx}\left(\ln\left(1+\sqrt{1+x^2}\right) - \ln x\right)$$

$$= \frac{\frac{1}{2}(2x)\left(1+x^2\right)^{-1/2}}{1+\left(1+x^2\right)^{1/2}} - \frac{1}{x} = \frac{x/\sqrt{1+x^2}}{1+\sqrt{1+x^2}} - \frac{1}{x} = \frac{x}{\sqrt{1+x^2}+1+x^2} - \frac{1}{x}$$

$$= \frac{x}{\sqrt{1+x^2}+1+x^2} - \frac{1}{x} = \frac{1}{\sqrt{1+x^2}}\left(\frac{x}{1+\sqrt{1+x^2}} - \frac{\sqrt{1+x^2}}{x}\right)$$

$$= \frac{1}{\sqrt{1+x^2}}\left(\frac{x^2-\sqrt{1+x^2}-\left(1+x^2\right)}{x\left(1+\sqrt{1+x^2}\right)}\right) = \frac{1}{\sqrt{1+x^2}}\left(\frac{-1-\sqrt{1+x^2}}{x\left(1+\sqrt{1+x^2}\right)}\right)$$

$$= \frac{-1}{x\sqrt{1+x^2}}$$

…Which can be combined accordingly to form appropriate antiderivative formulae:

$$\int \frac{du}{u\sqrt{u^2 \pm 1}} = -\ln\left(\frac{1+\sqrt{u^2 \pm 1}}{|u|}\right) + C$$

(Where the absolute value in the denominator term is to ensure boundedness according to the "+" case: i.e., the **arccsch*x*** whose domain is far broader—recall not on page 15 above.)

And with the appropriate w-substitution ($w = u/a$) (details suppressed here):

$$\int \frac{du}{u\sqrt{u^2 \pm a^2}} = -\tfrac{1}{a}\ln\left(\frac{a+\sqrt{u^2 \pm a^2}}{|u|}\right) + C$$

- <u>Example (17, §8.7)</u>

$$y = \ln\left(\tanh\frac{x}{2}\right) \Rightarrow y' = \frac{d}{dx}\ln\left(\tanh\frac{x}{2}\right) = \frac{\frac{d}{dx}\tanh\left(\frac{x}{2}\right)}{\tanh\left(\frac{x}{2}\right)} = \frac{\frac{1}{2}\sec h^2\left(\frac{x}{2}\right)}{\tanh\left(\frac{x}{2}\right)} = \frac{1}{2}\frac{\cosh(x/2)}{\cosh^2(x/2)\sinh(x/2)}$$

$$= \frac{1/2}{\cosh(x/2)\sinh(x/2)} = \frac{1}{2\cosh(x/2)\sinh(x/2)} = \frac{1}{\sinh\left(2\cdot\frac{x}{2}\right)} = \frac{1}{\sinh x} = \csc hx$$

- <u>Example (33, §8.7)</u>

$$y = 2x\arcsin h(2x) - \sqrt{1+4x^2}$$

$$y' = \frac{d}{dx}\left(2x\arcsin h(2x) - \sqrt{1+4x^2}\right) = 2\arcsin h(2x) + 2x\frac{d}{dx}\arcsin h(2x) - \frac{d}{dx}\sqrt{1+4x^2}$$

$$= 2\ln\left(2x+\sqrt{4x^2+1}\right) + 2x\cdot\frac{2}{\sqrt{4x^2+1}} - \frac{1}{2}(8x)\left(1+4x^2\right)^{-1/2}$$

$$= 2\ln\left(2x+\sqrt{4x^2+1}\right) + \frac{4x}{\sqrt{4x^2+1}} - \frac{4x}{\sqrt{4x^2+1}} = 2\ln\left(2x+\sqrt{4x^2+1}\right) = 2\arcsin h(2x)$$

- <u>Example (43, §8.7)</u>

$$\int\frac{\csc h(1/x)\coth(1/x)}{x^2}dx$$

$$u = \frac{1}{x} \Rightarrow du = -\frac{1}{x^2}dx \Rightarrow \int\csc hu\coth u(-du) = \int(-\csc hu\coth udu) = \csc hu + C = \csc h(1/x) + C$$

- <u>Example (47, §8.7)</u>

$$\int_0^{\sqrt{2}/4}\frac{2}{\sqrt{1-4x^2}}dx = \int_0^{\sqrt{2}/4}\frac{2dx}{\sqrt{1-(2x)^2}} \qquad u = 2x \Rightarrow du = 2dx$$

$$\int_{u(0)}^{u(\sqrt{2}/4)}\frac{du}{\sqrt{1-u^2}} = \int_0^{\sqrt{2}/2}\frac{du}{\sqrt{1-u^2}} = \arcsin u\Big|_0^{\sqrt{2}/2} = \arcsin\left(\frac{\sqrt{2}}{2}\right) - \arcsin(0) = \frac{\pi}{4}$$

Note: Not *all* these problems refer just to topics in this section! (This one obviously refers to material in the previous section)

- Example (60, §8.7)

$$\int \frac{dx}{(x-1)\sqrt{-4x^2+8x-1}} = \tfrac{1}{2}\int \frac{dx}{(x-1)\sqrt{-x^2+2x-\tfrac{1}{4}}} = \tfrac{1}{2}\int \frac{dx}{(x-1)\sqrt{-x^2+2x+1-1-\tfrac{1}{4}}}$$

$$= \tfrac{1}{2}\int \frac{dx}{(x-1)\sqrt{-(x^2-2x+1)+1-\tfrac{1}{4}}} = \tfrac{1}{2}\int \frac{dx}{(x-1)\sqrt{\tfrac{3}{4}-(x-1)^2}}$$

$$u = x-1, du = dx \Rightarrow \tfrac{1}{2}\int \frac{du}{u\sqrt{\left(\tfrac{\sqrt{3}}{2}\right)^2-u^2}} = \tfrac{1}{2}\left\{-\frac{1}{\sqrt{3}/2}\ln\left(\frac{\tfrac{\sqrt{3}}{2}+\sqrt{\tfrac{3}{4}-u^2}}{|u|}\right)\right\}+C$$

$$= -\frac{1}{\sqrt{3}}\ln\left(\frac{\sqrt{3}/2+\sqrt{\tfrac{3}{4}-(x-1)^2}}{|x-1|}\right)+C$$