MATH 261
NOTES (THURSDAY SEPT 22)

- EXAM I: CHOOSE 5 FROM 6 (@ 20PTS APIECE) + 20 PT BONUS PROBLEM

FROM THE LIST OF 6, THERE WILL BE:
- ONE RELATED-RATES WORD PROBLEM
- ONE PROBLEM INVOLVING THE M.V.T. (Mean Value Theorem)
- TWO PROBLEMS INVOLVING COMPUTATION OF DERIVATIVES (USING ALL THE RULES)
- ONE PROBLEM INVOLVING THE COMPUTATI[ON] OF LIMITS.
- ONE PROBLEM INVOLVING GRAPHICAL ANALYSIS (I.E., ADVANCED CURVE SKETCHING, AS IN §4.6)
- BONUS PROBLEM WILL INVOLVE ~~OP~~ OPTIMIZATI[ON] PROBLEM(S) (AS IN §4.7)

\+ FORMULA SHEET

EXAM REVIEW QUESTIONS

I) CALCULATE THE LIMITS: (OR IF THEY DON'T EXIST, EXPLAIN WHY, IN TERMS OF LEFT AND RIGHT HAND LIMITS

a) $\lim_{x\to 4} \frac{2x-8}{x^2-x-12}$ b) $\lim_{x\to 0} \frac{\sqrt{x+1}-1}{x}$ c) $\lim_{x\to 1} \frac{1}{x-1}$

d) $\lim_{x\to\infty} \frac{2x^2+3x-1}{x^3-x^2+x-1}$ e) $\lim_{x\to\infty} (\sqrt{x-1}-\sqrt{x+1})$

II) a) (#30, p.167): USING THE MVT, SHOW THAT THE VALUE c GUARANTEED BY THE MVT IS HALFWAY ON $[a,b]$ FOR $f(x)=ax^2+bx+c$

b) FIND ALL $c\in(a,b)$ SATISFIED BY THE MVT $f(x)=x(x^2-x-2)$ $[-1,1]$

III) Find the first and second derivative of $f(x)=|x^2-9|$

IV) Given: $f(x)=\left(\frac{x+1}{x^2-2}\right)^2$

a) Calculate $f'(x)$ using the Chain Rule & Quotient Rule (First use Chain Rule, then Quotient Rule)

b) Calculate by using the Quotient Rule & Chain Rule (First use Quotient Rule, then use CHAIN RULE)

V) SUPPOSE A SHIP(1) IS STEAMING NORTH AT 5 knots, and another ship(2) is steaming EAST at 11 knots. How fast are they receding from one another when their distance is 12 miles (when they're 12 miles apart?) Note: 1 knot = 1.6 mph and ship(1) is 6 mi. N from port.

VII) A RECTANGLE IS BOUNDED BY X-AXIS AND SEMICIRCLE BOUNDED ABOVE BY $x^2 + y^2 = 81$. FIND LENGTH AND WIDTH OF RECTANGLE TO ENSURE MAX. AREA (WHAT IS THE MAX. AREA?)

VIII) Given a) $f(x) = x^4 - 2x^2$ find i) ~~Critical~~ Partition pts. for f and sketch a sign chart, and asymptotes
ii) Critical pts. for f' and sketch a sign chart
iii) Inflection pts. and sketch a sign chart (for f'')
iv) Using information from i) – iii), sketch its graph

b) Repeat a) for $f(x) = \dfrac{x}{x^2+1}$

ANSWERS

I a) $\displaystyle \lim_{x\to 4} \frac{2x-8}{x^2-x-12} = \lim_{x\to 4} \frac{2(x-4)}{x^2-x-12} = 2\lim_{x\to 4} \frac{\cancel{(x-4)}}{\cancel{(x-4)}(x+3)} = 2\lim_{x\to 4}\frac{1}{(x+3)} = 2\cdot\frac{1}{7} = \frac{2}{7}$

b) $\displaystyle \lim_{x\to 0} \frac{\sqrt{x+1}-1}{x} = \lim_{x\to 0} \frac{\sqrt{x+1}-1}{x}\cdot\frac{\sqrt{x+1}+1}{\sqrt{x+1}+1} = \lim_{x\to 0}\frac{x+1-1}{x(\sqrt{x+1}+1)} = \lim_{x\to 0}\frac{\cancel{x}}{\cancel{x}(\sqrt{x+1}+1)}$

$\displaystyle = \lim_{x\to 0}\frac{1}{\sqrt{x+1}+1} = \frac{1}{2}$

c) $\displaystyle \lim_{x\to 1}\frac{1}{x-1}$ $\underline{\underline{\text{DNE}}}$, because: a) $\displaystyle \lim_{x\to 1^+}\frac{1}{x-1} = +\infty$ b) $\displaystyle \lim_{x\to 1^-}\frac{1}{x-1} = -\infty$

$\displaystyle \therefore \lim_{x\to 1^+}\frac{1}{x-1} \neq \lim_{x\to 1^-}\frac{1}{x-1}$

d) $\displaystyle \lim_{x\to\infty}\frac{2x^2+3x-1}{\underbrace{x^3}_{\text{leading term}}-x^2+x-1} = \lim_{x\to\infty}\frac{(2x^2+3x-1)/x^3}{(x^3-x^2+x-1)/x^3} = \lim_{x\to\infty}\frac{2/x+3/x^2-1/x^3}{1-1/x+1/x^2-1/x^3}$

$\displaystyle = \frac{0+0+0}{1-0+0-0} = \frac{0}{1} = 0$

e) $\displaystyle \lim_{x\to\infty}(\sqrt{x-1}-\sqrt{x+1}) = \lim_{x\to\infty}\left(\frac{\sqrt{x-1}-\sqrt{x+1}}{1}\cdot\frac{\sqrt{x-1}+\sqrt{x+1}}{\sqrt{x-1}+\sqrt{x+1}}\right)$

$\displaystyle = \lim_{x\to\infty}\left(\frac{x-1-(x+1)}{\sqrt{x-1}+\sqrt{x+1}}\right) = \lim_{x\to\infty}\frac{-2}{\sqrt{x-1}+\sqrt{x+1}} \approx \lim_{x\to\infty}\frac{-2}{\sqrt{x}} = \underline{0}$

II) $f(x) = ax^2+bx+c$ on $[a, b]$

SINCE $f(x)$ is continuous and differentiable on $[a, b]$, as it's <u>everywhere</u> differentiable (it's a polynomial) then the MVT holds.

According to the MVT: $f'(c) = \frac{f(b) - f(a)}{b - a}$

$$f(b) = ab^2 + bb + c \qquad f(a) = aa^2 + ba + c$$

$$\therefore \frac{f(b) - f(a)}{b - a} = \frac{(ab^2 + bb + c) - (aa^2 + ba + c)}{(b - a)} = \frac{a(b^2 - a^2) + b(b - a)}{(b - a)}$$

$$= \frac{a(b-a)(b+a) + b(b-a)}{(b-a)} = a(b+a) + b \quad (A)$$

on the other hand: $f'(c) = \frac{d}{dx} f(x) \Big|_{x=c} \qquad \frac{d}{dx} f(x) = 2ax + b = f'(x)$

$$f'(c) = 2ac + b \quad (B)$$

According to M.V.T., $(B) = (A) \Rightarrow 2ac + \cancel{b} = a(b+a) + \cancel{b}$

$$\therefore 2ac = a(b+a) \Rightarrow \boxed{c = \frac{1}{2}(b+a)}$$

(midpoint between b and a in $[a, b]$)

b) $f(x) = x(x^2 - x - 2) = x^3 - x^2 - 2x \Rightarrow f'(x) = 3x^2 - 2x - 2$

on $[-1, 1]$, according to MVT, there are $c \in [-1, 1]$ such that $f'(c) = \frac{f(1) - f(-1)}{1 - (-1)}$

i.e. there are $c \in (-1, 1)$ such that: $f'(c) = \frac{1}{2}(f(1) - f(-1))$

$$f(1) = -2 \quad f(-1) = 0 \quad \therefore f'(c) = 3c^2 - 2c - 2 = \frac{1}{2}(-2) = -1$$

$$\therefore 3c^2 - 2c - 2 = -1 \Rightarrow 3c^2 - 2c - 1 = 0$$

$$(3c + 1)(c - 1) = 0$$

$$\therefore c_1 = -\frac{1}{3} \quad c_2 = 1$$

Note: $c_1 \in [-1, 1]$, since $-1 \leq -\frac{1}{3} \leq 1$

$c_2 \in [-1, 1]$ since $-1 \leq +1 \leq 1$

III) $f(x) = |x^2-9| = \begin{cases} x^2-9 & x^2-9 \geq 0 \Rightarrow x^2 \geq 9 \Rightarrow x \leq -3 \text{ or } x \geq 3 \\ 9-x^2 & \text{otherwise} \end{cases}$

$\therefore\ f(x) = \begin{cases} x^2-9 & \text{for } x \leq -3 \text{ or } x \geq 3, \text{ i.e. on } (-\infty,-3] \cup [3,\infty) \\ 9-x^2 & \text{for } -3 < x < 3, \text{ i.e. on } (-3,3) \end{cases}$

$\therefore\ f'(x) = \begin{cases} 2x & (-\infty,-3] \cup [3,\infty) \\ -2x & (-3,3) \end{cases}$

$f''(x) = \begin{cases} 2 & (-\infty,-3] \cup [3,\infty) \\ -2 & (-3,3) \end{cases}$

IV) $f(x) = \left(\frac{x+1}{x^2-2}\right)^2 = \frac{(x+1)^2}{(x^2-2)^2}$

a) $f'(x) = 2\left(\frac{x+1}{x^2-2}\right)^1 \frac{d}{dx}\left(\frac{x+1}{x^2-2}\right) = 2\left(\frac{x+1}{x^2-2}\right) \cdot \left[\frac{(x^2-2)-(x+1)(2x)}{(x^2-2)^2}\right]$

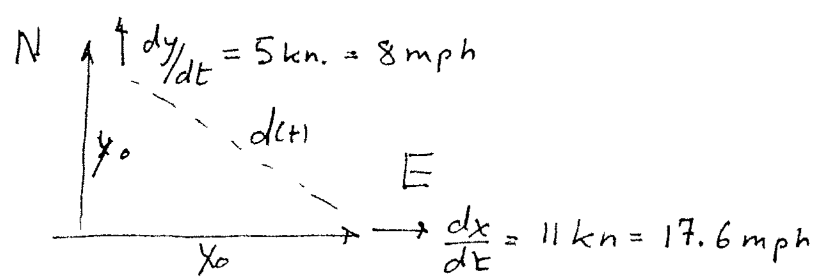
$= \frac{2(x+1)\left[(x^2-2)-(2x^2+2x)\right]}{(x^2-2)^3} = \frac{2(x+1)(-x^2-2x-2)}{(x^2-2)^2} = \frac{-2(x+1)(x^2+2x+2)}{(x^2-2)^2}$

b) $f'(x) = \frac{d}{dx}\left[\frac{(x+1)^2}{(x^2-2)^2}\right] = \frac{(x^2-2)^2 \frac{d}{dx}(x+1)^2 - (x+1)^2 \frac{d}{dx}\left[(x^2-2)^2\right]}{(x^2-2)^4}$

$= \frac{(x^2-2)^2 \cdot 2(x+1) - (x+1)^2 \cdot 2(x^2-2)(2x)}{(x^2-2)^4} = \frac{\cancel{(x^2-2)}\left[2(x^2-2)(x+1) - 2(x+1)^2(2x)\right]}{(x^2-2)^{\cancel{4}}}$

$= \frac{2(x+1)\left[x^2-2x-2-2x^2\right]}{(x^2-2)^3} = \frac{2(x+1)\left[-x^2-2x-2\right]}{(x^2-2)^3} = \frac{-2(x+1)(x^2+2x+2)}{(x^2-2)^3}$

V)



$d = \sqrt{x^2+y^2} \Rightarrow d^2 = x^2+y^2 \Rightarrow \frac{d}{dt}\, d(t)^2 = \frac{d}{dt}\left[x(t)^2 + y(t)^2\right]$

$\therefore 2d(t)\dot{d}(t) = 2x(t)\dot{x}(t) + 2y(t)\dot{y}(t)$ (dot superscript is shorthand for "$\frac{d}{dt}$")

$$d(t)\dot{d}(t) = x(t)\dot{x}(t) + y(t)\dot{y}(t)$$

$$\dot{d}(t) = \frac{x(t)\dot{x}(t) + y(t)\dot{y}(t)}{d(t)}$$

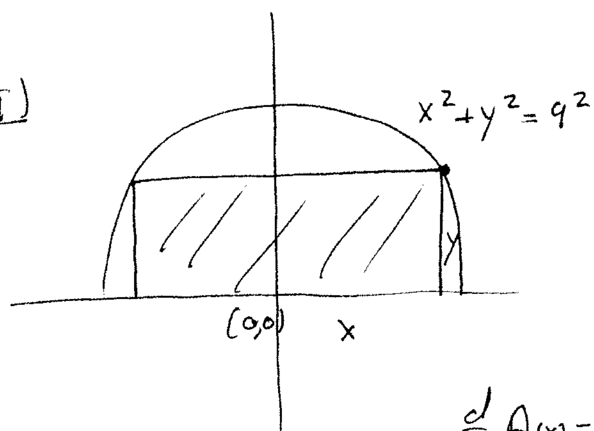
where, at t_0: $d(t_0) = 12mi = \sqrt{x_0^2 + y_0^2}$ and $y_0 = 6mi$.

$\therefore$ x_0, y_0, d_0 form a $30°/60°/90°$ triangle: 6, 12, x_0

so $x_0 = 6\sqrt{3}$ miles

$$\therefore \dot{d}(t_0) = \frac{x(t_0)\dot{x} + y(t_0)\dot{y}(t_0)}{d(t_0)} = \frac{6\sqrt{3}(17.6) + 6(8)}{12} \frac{\cancel{mi}\,mph}{\cancel{mi}}$$

VI)



Maximize $A = 2xy$

subject to constraint: $x^2+y^2=9^2$

$$y = \sqrt{81-x^2} \Rightarrow A(x) = 2\sqrt{81-x^2}\,x$$

$$\frac{d}{dx}A(x) = \frac{d}{dx}\left(2x(81-x^2)^{1/2}\right) = 2\frac{d}{dx}\left(x\sqrt{81-x^2}\right)$$

$$= 2\left\{x\cdot\tfrac{1}{2}(81-x^2)^{-1/2}(-2x) + (81-x^2)^{1/2}\right\}$$

$$= 2\left\{-x^2(81-x^2)^{-1/2} + (81-x^2)^{1/2}\right\}$$

$$= \frac{2}{\sqrt{81-x^2}}\left\{-x^2 + (81-x^2)\right\} = \frac{+(162-2x^2)}{\sqrt{81-x^2}} = -2(x^2-81)(81-x^2)^{-1/2}$$

$$A'(x) = 2(81-x^2)^{1/2} \Rightarrow A'(x) = 0 \Rightarrow 2(81-x^2)^{1/2} = 0 \Rightarrow x_1 = +3,\ x_2 = -3$$

$x_2 = -3$ isn't physical, (length is always positive) $\therefore$ $x_1 = 3$ is the critical point.

Note: $A''(x) = \frac{d}{dx}A'(x) = \frac{d}{dx}\left[2(81-x^2)^{1/2}\right] = (81-x^2)^{-1/2}(-2x) = \frac{-2x}{\sqrt{81-x^2}}$

$A''(x_1) = A''(3) = \frac{-6}{\sqrt{72}} < 0 \Rightarrow x_1$ is a local max.

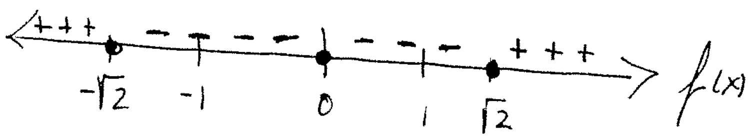
$\therefore$ Max. area: $A_{max} = 2x_1\sqrt{81-x_1^2} = 6\sqrt{72} = 6\sqrt{9\cdot 7} = 18\sqrt{7}$

VII.) a) $f(x) = x^4 - 2x^2$ $\quad \lim\limits_{x\to\pm\infty}(x^4-2x^2) = \lim\limits_{x\to\pm\infty} x^2(x^2-2) = \infty$ No horizontal asymptotes

No vertical asymptotes, either. $\quad f(x) = f(-x) \quad \therefore f$ is an even function

i) Portition points: $f(x) = 0 = x^2(x^2-2) \Rightarrow x_1 = 0, \; x_{2,3} = \pm\sqrt{2}$

Testing sign: 1) $x = 2$ $(x = 2 > \sqrt{2}) \Rightarrow f(2) > 0$
2) $x = 1$ $(0 < 1 < \sqrt{2}) \Rightarrow f(1) < 0$
Not nec. to test sign for other two regions,
since $f(x) = f(-x)$ (f is even)



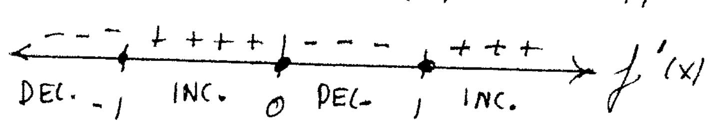
ii) $f'(x) = 4x^3 - 4x \Rightarrow f'(x) = 0 = 4x(x^2-1) = 4x(x-1)(x+1)$

$\Rightarrow x_4 = 0, \quad x_{5,6} = \pm 1$ (all critical points of type 1)
$\quad\; \overset{\shortparallel}{x_1}$

Note also: $f'(x) = -f'(-x)$ ($f'(x)$ is odd)

Testing sign: $0 < x < 1 \Rightarrow x = 1/2, \; f'(1/2) < 0$
$x > 1 \Rightarrow x = 2, \; f'(2) > 0$

Since $f'(x)$ is odd, sign of $f'(x)$ is opposite to left of origin:



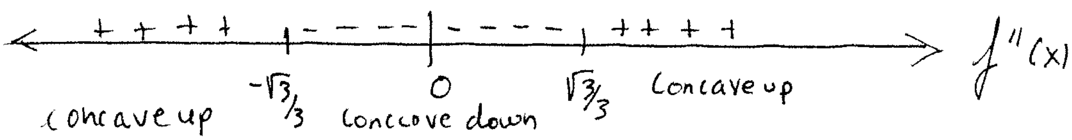
iii) $f''(x) = \frac{d}{dx}[4x^3 - 4x] = 4\frac{d}{dx}(x^3 - x) = 4[3x^2 - 1]$

$f''(x) = 0 \Rightarrow 4[3x^2-1] = 0 \Rightarrow x_{7,8} = \pm \frac{1}{\sqrt{3}} = \pm \frac{\sqrt{3}}{3}$ (INFLECTION PTS)

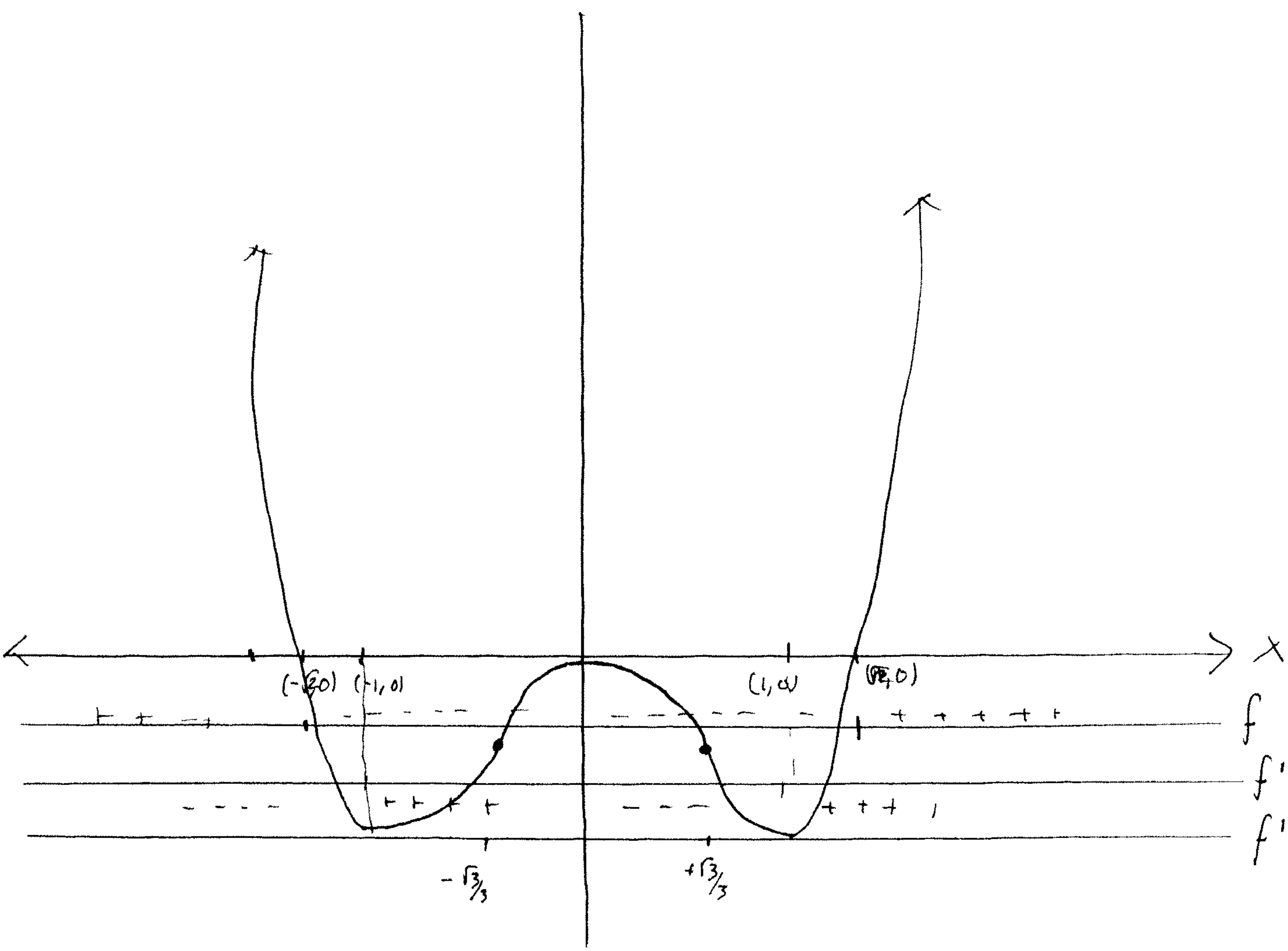
<u>Note</u>: $f''(x) = f''(-x)$ (even)

Testing sign: 1) $0 < x < \frac{\sqrt{3}}{3} \Rightarrow x = \frac{1}{10} \Rightarrow f''(\frac{1}{10}) < 0$

2) $x > \frac{\sqrt{3}}{3} \Rightarrow x = 1 \Rightarrow f''(1) > 0$



<u>Combining</u> all the information:

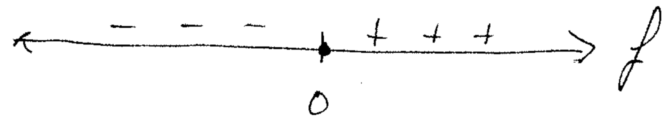


b) $f(x) = \dfrac{x}{x^2+1}$ asymptotes: $\lim_{x\to\pm\infty} \dfrac{x}{x^2+1} = \lim_{x\to\pm\infty} \dfrac{1/x}{1+1/x^2} = \dfrac{0}{1} = 0$

No vertical asymptotes, since x^2+1 is irreducible

Partition pts.: $f(x) = 0 = \dfrac{x}{x^2+1} \Rightarrow x_1 = 0$

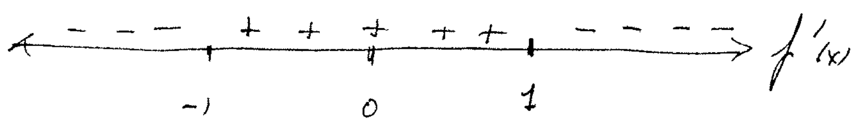
also: $f(x) = -f(-x) \therefore f$ is odd



$$f'(x) = \frac{(x^2+1) - x(2x)}{(x^2+1)^2} = \frac{-x^2+1}{(x^2+1)^2} = -\frac{(x^2-1)}{(x^2+1)^2}$$

$f'(x) = 0 = \dfrac{-(x^2-1)}{(x^2+1)^2} \Rightarrow x^2 - 1 = 0 \Rightarrow x_{2,3} = \pm 1$ (Type-1 critical pts only)

Note: $f'(x) = f'(-x)$ so $f'(x)$ is even

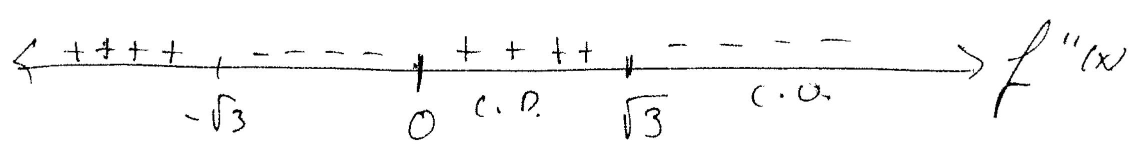


$$f''(x) = \frac{d}{dx}\left\{ \frac{-(x^2-1)}{(x^2+1)^2} \right\} = -\frac{d}{dx} \frac{(x^2-1)}{(x^2+1)^2} = \frac{(x^2+1)^2(2x) - (x^2-1)2(x^2+1)(2x)}{(x^2+1)^4}$$

$$= \frac{2x(x^2+1)\left[(x^2+1) - 2(x^2-1)\right]}{(x^2+1)^4} = \frac{2x(x^2+1)(3-x^2)}{(x^2+1)^4}$$

$f''(x) = 0 = \dfrac{2x(x^2+1)(3-x^2)}{(x^2+1)^4} = 0 \Rightarrow x_4 = 0 \quad x_{5,6} = \pm\sqrt{3}$ (inflection pts)

Note: $f''(-x) = -f(x)$ (f is odd)



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