MA261 **Calculus I: Notes for September 6**
Fall 2007 Kallfelz

- ***THE CONCEPT OF DERIVATIVE (cont.)***

In the September 4 notes, the notion of left- and right- hand derivatives was introduced (useful for evaluating the derivative of a function at its endpoints):

$$f'(c)^{-} = \lim_{\Delta x \to 0^{-}} \frac{f(c-\Delta x) - f(c)}{\Delta x} = \lim_{x \to c^{-}} \frac{f(x) - f(x)}{x - c}$$

$$f'(c)^{+} = \lim_{\Delta x \to 0^{+}} \frac{f(c+\Delta x) - f(c)}{\Delta x} = \lim_{x \to c^{+}} \frac{f(x) - f(x)}{x - c}$$

In the September 4 notes, you'll also see:

Obviously, *f* is **differentiable** (its derivative exists) at $x = c$ provided its left and right hand derivatives agree, i.e.

$$f'(c) = \lim_{\Delta x \to 0} \frac{f(c+\Delta x) - f(c)}{\Delta x} = \lim_{x \to c} \frac{f(x) - f(x)}{x - c}$$ **provided that:**

$$f'(c)^{-} = f'(c)^{+}$$

The discussion (September 4) however, was limited to the case of **continuous functions only.** Recall the last theorem in §3.1, that *if* a function is differentiable in some open interval, *then* it's continuous in that open interval. **Continuity is therefore a necessary condition of differentiability.** In class, I went over the proof of this theorem using the authors' approach. Here in the notes, I use a more precise proof, using ε, δ arguments.

- **Proof (more precise than textbook's)**

Suppose that *f* is differentiable at $c \in (a, b)$. Then the limit:

$$f'(c) = \lim_{\Delta x \to 0} \frac{f(c+\Delta x) - f(c)}{\Delta x} = \lim_{x \to c} \frac{f(x) - f(x)}{x - c}$$

…exists. In other words, for every (arbitrarily small) $\varepsilon > 0$ such that:

$$\left|\frac{f(x)-f(c)}{x-c} - f'(c)\right| < \varepsilon$$

There exists an (arbitrarily small) $\delta > 0$ such that: $|x - c| < \delta$ such that:
$\varepsilon \to 0$ if and only if $\delta \to 0$.

To show that f is continuous at c, we must show (given the above) that:
$\lim_{x\to c} f(x) = f(c)$, which amounts to showing that there are ε', δ (the prime superscript simply mean that this epsilon isn't the same as the one in the above limit, i.e., in the assumption of differentiability) such that whenever:

$$|f(x) - f(c)| < \varepsilon'$$

There exists an (arbitrarily small) $\delta > 0$ such that: $|x - c| < \delta$ such that:
$\varepsilon' \to 0$ if and only if $\delta \to 0$.

Using the Triangle Inequality:

$$\left|\frac{f(x)-f(c)}{x-c}\right| - |f'(c)| \le \left|\frac{f(x)-f(c)}{x-c} - f'(c)\right| < \varepsilon$$

So: $$\left|\frac{f(x)-f(c)}{x-c}\right| < \varepsilon |f'(c)|$$

Note also: $$\left|\frac{f(x)-f(c)}{x-c}\right| = \frac{|f(x)-f(c)|}{|x-c|} < \varepsilon|f'(c)| \Rightarrow |f(x)-f(c)| < \varepsilon|f'(c)||x-c|$$

However, it was given that $|x - c| < \delta$, so:

$$|f(x)-f(c)| < \varepsilon|f'(c)||x-c| < |f'(c)|\varepsilon\delta$$

Now, since $f'(c)$ exists, then $|f'(c)|$ is just some finite non-negative number. Hence we just constructed a *new* $\varepsilon' = |f'(c)|\varepsilon\delta$ which is arbitrarily small (since it's the product of some nonnegative finite number $|f'(c)|$ with two other arbitrarily small

numbers ε, δ. Certainly, then $\varepsilon' \to 0$ if and only if $\delta \to 0$. Therefore:
$|f(x) - f(c)| < |f'(c)|\varepsilon\delta = \varepsilon'$ such that: $|x - c| < \delta$ such that:
$\varepsilon' \to 0$ if and only if $\delta \to 0$. **Therefore**: $\lim_{x\to c} f(x) = f(c)$**, or**
f **is continuous at** ***c*****.**

There can be cases where left and right hand derivatives give the same value at a discontinuity. This does not mean that the derivative exists at that discontinuity! (We just proved above that for *any* function *f*, *f must* be continuous at all points where its derivative exists.)

- **Here is one such example: (Exercise 36, §3.1)**

$$f(x) = \begin{cases} x^2 - 2x & x > 1 \\ x^3 - 3x^2 + 3x & x \le 1 \end{cases}$$

Since $\lim_{x\to 1^+} f(x) = 1^2 - 2\cdot 1 = -1$ but $\lim_{x\to 1^-} f(x) = 1^3 - 3\cdot 1^2 + 3\cdot 1 = 1$**,** then $\lim_{x\to 1^+} f(x) \neq \lim_{x\to 1^-} f(x) \Rightarrow \lim_{x\to 1} f(x)$ **DNE**. So *f* isn't continuous at $x = 1$. (So therefore it's not differentiable there). However, when examining its derivative for all points $x \neq 1$:

<u>For all *x* > 1:</u>

$$f'(x) = \lim_{\Delta x\to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \lim_{\Delta x\to 0} \frac{(x+\Delta x)^2 - 2(x+\Delta x) - \left[x^2 - 2x\right]}{\Delta x}$$

As mentioned in class, this can get cumbersome to evaluate. To avoid excessive writing (increasing the risk of making errors) it's best to simplify the numerator in a separate step first, and then re-insert. Remember to multiply out everything first, *then* factor and cancel!

Simplifying the numerator:

$$(x+\Delta x)^2 - 2(x+\Delta x) - x^2 + 2x = x^2 + 2x\Delta x + \Delta^2 x - 2x - 2\Delta x - x^2 + 2x$$
$$= 2x\Delta x + \Delta^2 x - 2\Delta x$$

Therefore:

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{2x\Delta x + \Delta^2 x - 2\Delta x}{\Delta x}$$
$$= \lim_{\Delta x \to 0} \frac{\Delta x(2x + \Delta x - 2)}{\Delta x} = \lim_{\Delta x \to 0} (2x + \Delta x - 2) = 2x - 2 = 2(x-1)$$

Alternatively, we could have used the second definition: (For all $c > 1$):

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c} = \lim_{x \to c} \frac{x^2 - 2x - (c^2 - 2c)}{x - c}$$
$$= \lim_{x \to c} \frac{x^2 - c^2 - 2x + 2c}{x - c} = \lim_{x \to c} \frac{(x-c)(x+c) - 2(x-c)}{(x-c)}$$
$$= \lim_{x \to c} \frac{(x-c)[(x+c) - 2]}{(x-c)} = \lim_{x \to c} (x + c - 2) = 2c - 2 = 2(c-1)$$

Thus evaluating the right-hand derivative at $x = 1$:

$$f'(1)^+ = \lim_{\Delta x \to 0^+} \frac{f(1+\Delta x) - f(c)}{\Delta x} = \lim_{x \to 1^+} \frac{f(x) - f(1)}{x - 1} = 2(1-1) = 0$$

For all $c < 1$:

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} =$$
$$\lim_{\Delta x \to 0} \frac{(x+\Delta x)^3 - 3(x+\Delta x)^2 + 3(x+\Delta x) - [x^2 - 3x^2 + 3x]}{\Delta x}$$

Simplifying the numerator:

$$(x+\Delta x)^3 - 3(x+\Delta x)^2 + 3(x+\Delta x) - x^3 + 3x^2 - 3x$$
$$= (x^3 + 3x^2\Delta x + 3x\Delta^2 x + \Delta^3 x) - 3(x^2 + 2x\Delta x + \Delta^2 x) + 3(x + \Delta x) - x^3 + 3x^2 - 3x$$
$$= x^3 + 3x^2\Delta x + 3x\Delta^2 x + \Delta^3 x - 3x^2 - 6x\Delta x - 3\Delta^2 x + 3x + 3\Delta x - x^3 + 3x^2 - 3x$$
$$= 3x^2\Delta x + 3x\Delta^2 x + \Delta^3 x - 6x\Delta x - 3\Delta^2 x + 3\Delta x$$

Therefore:

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{3x^2\Delta x + 3x\Delta^2 x + \Delta^3 x - 6x\Delta x - 3\Delta^2 x + 3\Delta x}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\Delta x\left(3x^2 + 3x\Delta x + \Delta^2 x - 6x - 3\Delta x + 3\right)}{\Delta x} = \lim_{\Delta x \to 0}\left(3x^2 + 3x\Delta x + \Delta^2 x - 6x - 3\Delta x + 3\right)$$

$$= 3x^2 - 6x + 3$$

Alternatively, we could have used the second definition:

For all $c < 1$:

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c} = \lim_{x \to c} \frac{x^3 - 3x^2 + 3x - \left(c^3 - 3c^2 + 3c\right)}{x - c}$$

$$= \lim_{x \to c} \frac{x^3 - c^3 - 3x^2 + 3c^2 + 3x - 3c}{x - c} = \lim_{x \to c} \frac{(x-c)\left(x^2 + cx + c^2\right) - 3(x-c)(x+c) + 3(x-c)}{(x-c)}$$

$$= \lim_{x \to c} \frac{(x-c)\left[\left(x^2 + cx + c^2\right) - 3(x+c) + 3\right]}{(x-c)} = \lim_{x \to c}\left[\left(x^2 + cx + c^2\right) - 3(x+c) + 3\right]$$

$$= 3c^2 - 6c + 3$$

Thus evaluating the left-hand derivative at $x = 1$:

$$f'(1)^- = \lim_{\Delta x \to 0^+} \frac{f(1-\Delta x) - f(c)}{\Delta x} = \lim_{x \to 1^-} \frac{f(x) - f(1)}{x - 1} = 3 \cdot 1^2 - 6 \cdot 1 + 3 = 0$$

…So we get and agreement (between left and right hand derivatives at $x = 1$).
However, because f is not continuous at $x = 1$, it has no derivative at $x = 1$

Section 3.2 deals with applications, (nothing new in the way of math). The notion of taking a *time-derivative* of displacement is applied in the concept of *velocity*. The notion of taking a *time-derivative* of velocity is applied in the concept of *acceleration*. Here is one example (# 8)

- Given the velocity $v(t)$ of an object is:

$$v(t) = \frac{100t}{2t + 15}$$

Find its acceleration:

$$a(t)=v'(t)=\frac{dv}{dt}=\lim_{\Delta t\to 0}\frac{v(t+\Delta t)-v(t)}{\Delta t}=\lim_{\Delta t\to 0}\frac{\frac{100(t+\Delta t)}{2(t+\Delta t)+15}-\frac{100t}{2t+15}}{\Delta t}$$

$$=\lim_{\Delta t\to 0}\frac{1}{\Delta t}\cdot\left[\frac{100(t+\Delta t)}{2(t+\Delta t)+15}-\frac{100t}{2t+15}\right]=100\lim_{\Delta t\to 0}\left[\frac{(t+\Delta t)(2t+15)-t(2t+2\Delta t+15)}{\Delta t(2t+2\Delta t+15)(2t+15)}\right]$$

$$=100\lim_{\Delta t\to 0}\frac{\left(2t^2+2t\Delta t+15t+15\Delta t\right)-\left(2t^2+2t\Delta t+15t\right)}{\Delta t(2t+2\Delta t+15)(2t+15)}=100\lim_{\Delta t\to 0}\frac{15\Delta t}{\Delta t(2t+2\Delta t+15)(2t+15)}$$

$$=100\lim_{\Delta t\to 0}\frac{15}{(2t+2\Delta t+15)(2t+15)}=\frac{1500}{(2t+15)^2}$$

Note: We'll see soon how to evaluate this derivative by using the quotient rule [1] instead.

- ***SIMPLE DERIVATIVE FORMULAE***

Naturally, evaluating derivatives just from the definition alone can be quite a tedious process. In §3.2 the following 4 shortcut formulae are offered as theorems that are proved.

- **Formula 1:** $\frac{d}{dx}c=0$ (i.e. if $f(x) = c$, then: $f'(x) = 0$)

- **Formula 2:** $\frac{d}{dx}x^r = rx^{r-1}$ for any rational number r. (I.e., $f(x)=x^r \Rightarrow f'(x)=rx^{r-1}$

- **Formula 3:** $\frac{d}{dx}kf(x)=k\frac{df}{dx}$ (I.e., if $g(x) = kf(x)$, then $g'(x) = kf'(x)$)
- **Formula 4:** $\frac{d}{dx}(f(x)\pm g(x))=\frac{df}{dx}\pm\frac{dg}{dx}=f'(x)\pm g'(x)$

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[1] If $f(x)=\frac{g(x)}{h(x)}$ then: $f'(x)=\frac{h(x)g'(x)-g(x)h'(x)}{(h(x))^2}$

So for instance:

$$a(t)=v'(t)=\frac{(2t+15)\frac{d}{dt}(100t)-100t\frac{d}{dt}(2t+15)}{(2t+15)^2}=\frac{(2t+15)100-100t\cdot 2}{(2t+15)^2}=\frac{1500}{(2t+15)^2}$$

Except for **Formula2**, the rest of the formulas are easy to prove (consult text). The proof of Formula2 invokes the **Binomial Theorem:**

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k = \binom{n}{0} x^{n-0} y^0 + \binom{n}{1} x^{n-1} y^1 + \ldots + \binom{n}{n-1} x^{n-(n-1)} y^{n-1} + \binom{n}{n} x^{n-n} y^n$$

The Binomial coefficients are defined as: $\binom{n}{k} = \frac{n!}{k!(n-k)!}$

So for instance: $\binom{n}{0} = \frac{n!}{0!(n-k)!} = \frac{n!}{n!} = 1$

$$\binom{n}{1} = \frac{n!}{1!(n-1)!} = \frac{n!}{(n-1)!} = \frac{n(n-1)!}{(n-1)!} = n$$

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$$\binom{n}{n-1} = \frac{n!}{(n-1)!(n-(n-1))!} = \frac{n!}{(n-1)!} = \frac{n(n-1)!}{(n-1)!} = n$$

$$\binom{n}{n} = \frac{n!}{n!(n-n)!} = \frac{n!}{n!} = 1$$

(Note: One can easily prove that: $\binom{n}{k} = \frac{n!}{k!(n-k)!} = \binom{n}{n-k}$)

So the Proof of **Formula 2 is:**

$$\frac{d}{dx} x^n = \lim_{\Delta x \to 0} \frac{(x+\Delta x)^n - x^n}{\Delta x} = \lim_{\Delta x \to 0} \frac{\left[\binom{n}{0} x^n + \binom{n}{1} x^{n-1} \Delta x + \ldots + \binom{n}{n-1} x \Delta^{n-1} x + \binom{n}{n} \Delta^n x\right] - x^n}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{x^n + nx^{n-1}\Delta x + \ldots + nx\Delta^{n-1} x + \Delta^n x - x^n}{\Delta x} = \lim_{\Delta x \to o} \frac{\Delta x \left(nx^{n-1} + \ldots + nx\Delta^{n-2} x + \Delta^{n-1} x\right)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \left(nx^{n-1} + (power\_of\_\Delta x - terms)\right) = nx^{n-1}$$

- **Example (#23)**

$$f(x) = \frac{x^3 - 3x^2 + 4}{x^2}$$

Since you haven't been exposed to the **quotient rule** yet, to use Formulas 1-4 you should first reduce the above fractions to a sum-of-power terms. (Actually because of the simplicity of this function, this is the most efficient procedure anyway. In other words, using the quotient rule wouldn't save you any steps in this case.)

$$f(x) = \frac{x^3 - 3x^2 + 4}{x^2} = \frac{x^3}{x^2} - 3\frac{x^2}{x^2} + \frac{4}{x^2} = x - 3 + 4x^{-2}$$

Hence:

$$f'(x) = \frac{d}{dx}(x - 3 + 4x^{-2}) = \frac{d}{dx}x - \frac{d}{dx}3 + \frac{d}{dx}\left(4x^{-2}\right) \qquad \textbf{Formula 4}$$

$$= 1 - 0 + 4\frac{d}{dx}\left(x^{-2}\right) \qquad \textbf{Formulae 2, 1, 3}$$

$$= 1 - 0 + 4\left(2x^{-3}\right) = 1 - 8x^{-3} \qquad \textbf{Formula 2}$$

- **Example (#39)**

Find the point(s) with a horizontal tangent line (if there are any):

$$y = x^4 - 3x^2 + 2$$

A horizontal tangent line at a point x means a zero derivative there. Hence we must calculate the derivative of y and set it = 0:

$$\frac{d}{dx}y = \frac{d}{dx}(x^4 - 3x^2 + 2) = \frac{d}{dx}x^4 - \frac{d}{dx}3x^2 + \frac{d}{dx}2 = 4x^3 - 3\frac{d}{dx}x^2 + \frac{d}{dx}2$$

$$= 4x^3 - 3 \cdot 2x + 0 = 4x^3 - 6x$$

Setting = 0 and solving:

$$4x^3 - 6x = 0 \Rightarrow x\left(2x^2 - 3\right) = 0 \Rightarrow x_1 = 0 \Rightarrow 2x^2 - 3 = 0 \Rightarrow x_{2,3} = \pm\sqrt{\tfrac{3}{2}} = \pm\tfrac{\sqrt{6}}{2}$$