MA261 **Calculus I: Notes for September 4**
Fall 2007 Kallfelz

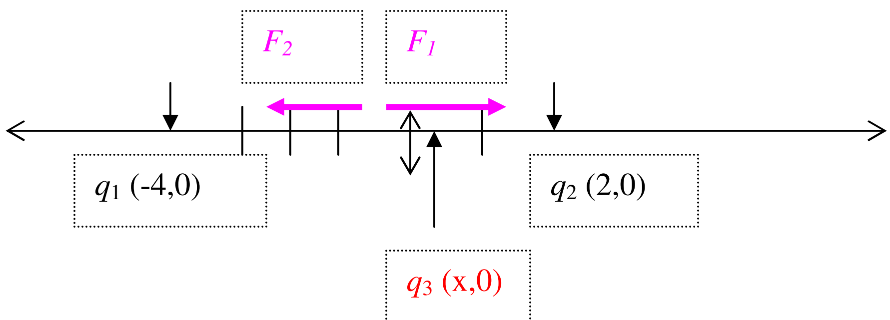
- ***More on Evaluating Limits and Continuity***

- **Example (Similar to Exercise 42, Assignment I)**

Two charges of $q_{1,2}$ = +1.0 C are placed on the x-axis at positions: (-4,0) and (2,0) (for q_1 and q_2 respectively). Suppose a third charge q_3 = 1. 0 C is placed somewhere between them. Fully analyze the function the charge feels (check for asymptotes, draw its graph)

Coulomb's Law states that the electrostatic force felt by two charges q,Q is inversely proportional to the square of their distance r between them and directly proportional to the product of the the magnitudes of their charges. The proportionality constant $k = 9.00 \times 10^{9} \frac{Nm^2}{C^2}$ (in the MKS unit system). I.e.

Coulomb's Law States: $F = k\dfrac{Qq}{r^2}$



The drawing above depicts the physical situation. The purple arrows indicate the force vectors: the force is attractive (since the sign of the charges is opposite). The distance between q_2 and q_3 is: $|2 - x|$ and the distance between q_1 and q_3 is: $|-4 - x| = (4 + x)$. Hence according to Coulomb's Law:

$$F_1 = k\frac{q_3 q_2}{r^2} = \frac{k}{(2-x)^2} \qquad F_2 = k\frac{q_3 q_1}{r^2} = -\frac{k}{(4+x)^2}$$

(where the sign indicates the direction of the experienced force). So the net force q_3 experiences is:

$$F(x) = F_1 + F_2 = \frac{k}{(2-x)^2} - \frac{k}{(4+x)^2} = k\left[\frac{1}{(2-x)^2} - \frac{1}{(4+x)^2}\right]$$

1.) To check for horizontal asymptotes, evaluate:

$$\lim_{x\to\pm\infty} F(x) = \lim_{x\to\pm\infty} k\left[\frac{1}{(2-x)^2} - \frac{1}{(4+x)^2}\right] = k\lim_{x\to\pm\infty}\frac{1}{(2-x)^2} - k\lim_{x\to\pm\infty}\frac{1}{(4+x)^2}$$

(Note how the properties of limits in **Thm2.3** were used to simplify the above expression)

As discussed last week, we know both of these expressions → 0 as $x \to \pm\infty$, since as x gets exceedingly large, the 2 and the 4 in the denominator expression don't matter, so:

$$\lim_{x\to\pm\infty}\frac{1}{(2-x)^2} = \lim_{x\to\pm\infty}\frac{1}{(-x)^2} = \lim_{x\to\pm\infty} x^{-2} = \left(\lim_{x\to\pm\infty} x\right)^2 = 0$$
$$\lim_{x\to\pm\infty}\frac{1}{(4+x)^2} = \lim_{x\to\pm\infty}\frac{1}{(4+x)^2} = \lim_{x\to\pm\infty} x^{-2} = \left(\lim_{x\to\pm\infty} x\right)^2 = 0$$

…Working it out rigorously using **Thm2.3 (.5)**[1]

So: $\lim_{x\to\pm\infty} F(x) = k\lim_{x\to\pm\infty}\frac{1}{(2-x)^2} - k\lim_{x\to\pm\infty}\frac{1}{(4+x)^2} = 0$

Or the horizontal asymptote is $y = 0$ (the x-axis).

2.) The vertical asymptotes are of course (x = -4) and (x = 2). We can ask, however the question as to whether there *is* an *infinity* limit (positive or negative) at these points, or does a limit not exist at all? Recall from last week that these are two separate questions! A limit DNE (does not exist) if its right and left hand limits disagree, i.e.: $\lim_{x\to c^+} f(x) \neq \lim_{x\to c^-} f(x)$ (whether finite or infinite).

Let's find out:

[1] Recall: $\lim_{x\to c}[f(x)]^n = [\lim_{x\to c} f(x)]^n$

$$\lim_{x\to -4^-} F(x) = k\lim_{x\to -4^-} \frac{1}{(2-x)^2} - k\lim_{x\to -4^-} \frac{1}{(4+x)^2}$$

$$= \tfrac{k}{36} - \infty = -\infty$$

$$\lim_{x\to -4^+} F(x) = k\lim_{x\to -4^+} \frac{1}{(2-x)^2} - k\lim_{x\to -4^+} \frac{1}{(4+x)^2}$$

$$= \tfrac{k}{36} - \infty = -\infty$$

So since they agree, then: $\lim_{x\to -4} F(x) = -\infty$

$$\lim_{x\to 2^-} F(x) = k\lim_{x\to 2^-} \frac{1}{(2-x)^2} - k\lim_{x\to 2^-} \frac{1}{(4+x)^2}$$

$$= \infty - \tfrac{k}{36} = \infty$$

$$\lim_{x\to 2^+} F(x) = k\lim_{x\to 2^+} \frac{1}{(2-x)^2} - k\lim_{x\to 2^+} \frac{1}{(4+x)^2}$$

$$= \infty - \tfrac{k}{36} = \infty$$

So since they agree, then: $\lim_{x\to 2} F(x) = \infty$

3.) To construct the graph of $F(x)$ it's convenient to sketch a sign chart. We need one more critical point (we already have the vertical asymptotes), namely, where the function crosses the x – axis:

$$F(x) = k\left[\frac{1}{(2-x)^2} - \frac{1}{(4+x)^2}\right] = 0 \Rightarrow \frac{1}{(2-x)^2} = \frac{1}{(4+x)^2} \Rightarrow (2-x)^2 = (4+x)^2$$

There are two ways to solve the equation. One way (the easier way) is simply to take the square root of both sides (however makes sure to insert a "±" on one or the other side):

Case 1: $(2 - x) = (4 + x) \Rightarrow 2x = -2 \Rightarrow x = -1$
Case 2: $(2 - x) = -(4 + x) \Rightarrow 2 = -4$ no solutions[2]

Note how this answer (in Case 1) conforms to your intuitions about the physics of the situation: Since all the charges have the same magnitude, but the middle charge is opposite in sign to the charges at the endpoints, then the point in which q_3 would feel zero force would be located halfway between the two endpoint charges.

[2] Recall from algebra: whenever a contradictory results is derived like that, we know automatically the solution set is empty. On the other hand whenever an identity is generated from an equation (i.e. a result like "1 = 1" or "0 = 0", then we know the solution set is infinite, i.e. the solution set is (-∞, ∞)

$x = -1$ is the midpoint between (-4,0) and (2,0)

Another more cumbersome way to solve this is to square both sides and cancel:

$$(2-x)^2 = (4+x)^2 \Rightarrow 4-4x+x^2 = 14+8x+x^2 \Rightarrow 12x = -12 \Rightarrow x = -1$$

To sketch a sign chart, simply pick some easy to evaluate points in the four regions:

a) (-∞, -4) (i..e $\{x|\, x < -4\}$) *for instance:* $x = -5$

$$\Rightarrow F(-5) = k\left[\frac{1}{(2+5)^2} - \frac{1}{(4-5)^2}\right] = k\left(\tfrac{1}{49} - 1\right) < 0$$

b) (-4, -1): (i..e $\{x|\, -4 < x < -1\}$) *for instance:* $x = -2$

$$\Rightarrow F(-2) = k\left[\frac{1}{(2+2)^2} - \frac{1}{(4-2)^2}\right] = k\left(\tfrac{1}{16} - \tfrac{1}{4}\right) < 0$$

c) (-1, 2): (i..e $\{x|\, -1 < x < 2\}$) *for instance:* $x = 0$

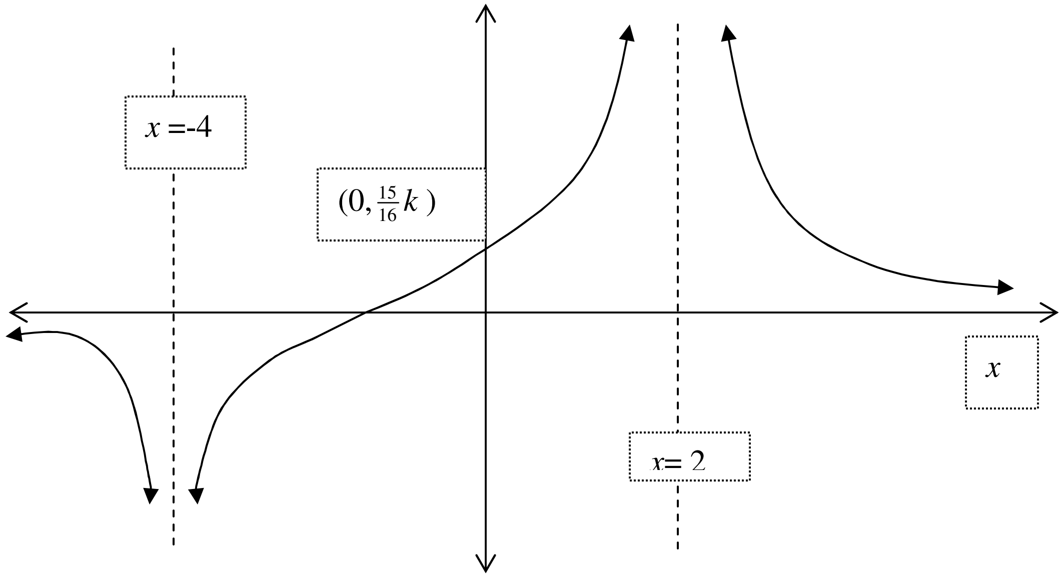
$$\Rightarrow F(0) = k\left[\frac{1}{(2-0)^2} - \frac{1}{(4+0)^2}\right] = k\left(\tfrac{1}{4} - \tfrac{1}{16}\right) > 0$$

d) (2, ∞): (i..e $\{x|\, 2 < x\}$) *for instance:* $x = 3$

$$\Rightarrow F(3) = k\left[\frac{1}{(2-3)^2} - \frac{1}{(4+3)^2}\right] = k\left(1 - \tfrac{1}{49}\right) > 0$$

Note: Admittedly the sign chart procedure is redudandant here, since we analyzed the right and left hand side limits at the points -4 and -2, so we know automatically what the sign of the function is in these four regions covered by cases a), b), c), d). Still, I show this simple algebraic sign chart procedure for purposes of review, and it also functions as a useful check!

So now we have all the information necessary to sketch the graph:



- ***The Squeeze Theorem*** **(Thm 2.14, p. 91)**

Recall : For all x in an open interval containing a point c, if: $f(x) \le h(x) \le g(x)$, and: $\lim_{x \to c} f(x) = L = \lim_{x \to c} g(x)$, then: $\lim_{x \to c} h(x) = L$

We can apply this theorem to interesting cases involving sinusoidal functions (in anticipation to later on when we get to material in chapter 8)

Though the function sin$x$ is obviously continuous (hence $\lim_{x \to c} \sin x = \sin c$ for all c) let's practice using the Squeeze Theorem to show:

$$\lim_{x \to 0} \sin x = \sin 0 = 0$$

Proof: For all x in the interval (0 , ∞), certainly $0 \le \sin x \le x$

Now $\lim_{x \to 0} 0 = 0 = \lim_{x \to 0} x$, so according to the Squeeze Theorem: $\lim_{x \to 0} \sin x = 0$

Similarly: $\lim_{x \to 0} \cos x = \cos 0 = 1$

Proof: For all x in the interval (0 ,δ) (where δ is an arbitrarily small number) certainly $1 - x \leq \cos x \leq 1$

Now $\lim_{x\to 0} 1 = 1 = \lim_{x\to 0}(1-x)$, so according to the Squeeze Theorem: $\lim_{x\to 0} \cos x = 0$

We can use the above results to prove the continuity of cos$x$ and sin$x$:

$$\lim_{x\to c} \sin x = \sin c$$

Proof:

$$\lim_{x\to c} \sin x = \lim_{h\to 0} \sin(c+h) = \lim_{h\to 0}(\sin c \cosh + \cos c \sinh)$$
$$= \lim_{h\to 0} \sin c \cosh + \lim_{h\to 0} \cos c \sinh = \sin c \lim_{h\to 0} \cosh + \cos c \lim_{h\to 0} \sinh$$
$$= \sin c \cdot 1 + \cos c \cdot 0 = \sin c$$

$$\lim_{x\to c} \cos x = \cos c$$

Proof:

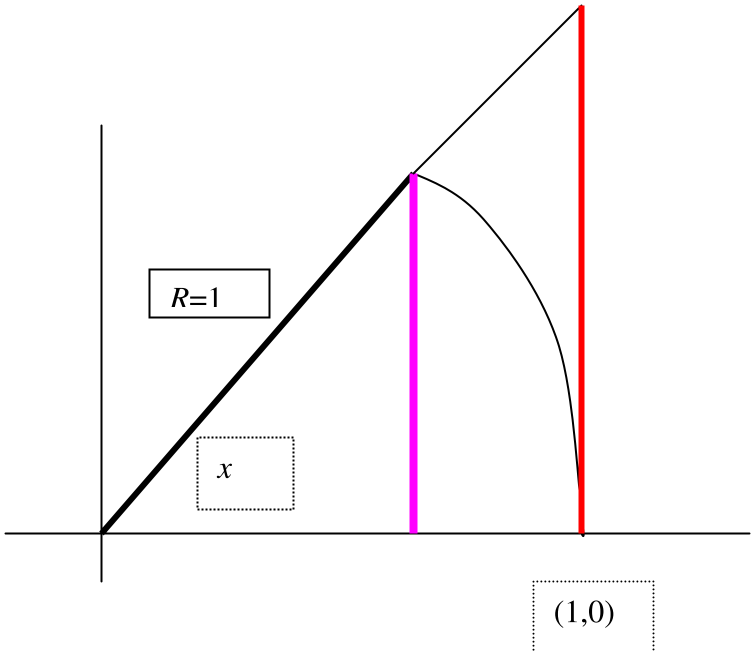
$$\lim_{x\to c} \cos x = \lim_{h\to 0} \cos(c+h) = \lim_{h\to 0}(\cos c \cosh - \sin c \sinh)$$
$$= \lim_{h\to 0} \cos c \cosh - \lim_{h\to 0} \sin c \sinh = \cos c \lim_{h\to 0} \cosh - \sin c \lim_{h\to 0} \sinh$$
$$= \cos c \cdot 1 - \sin c \cdot 0 = \cos c$$

The above results weren't very interesting, since they established results that are obvious (and known already form the continuity properties of sine and cosine). However, they give one practice in using the Squeeze Theorem, and using trigonometric identities.

Consider now a not-so-obvious result:

$$\lim_{x\to c} \frac{\sin x}{x} = 1$$

Proof: See diagram below



We have a segment of the unit circle in which a small triangle is inscribed in the circular segment.

- The length of the opposite side of this small triangle (the purple line) = sin*x*, since the angle is *x* radians, and $\sin x = \frac{O}{H}$. However, for the small triangle, $H = 1$. So the opposite side $O = \sin x$.
- The length of the circular arc = *x*, since the arclength of a circular segment is $s = r\theta$ (where θ is in radians). Here the angle θ is denoted *x*, and the radius $r = 1$. So the arclength is simply = *x*

- The length of the opposite side of this large triangle (the red line) = tan*x*, since $\tan x = \frac{O}{A}$. However, for the large triangle, $A = 1$. So the opposite side $O = \sin x$.

Furthermore: $\sin x < x < \tan x \Rightarrow \frac{\sin x}{\sin x} < \frac{x}{\sin x} < \frac{\tan x}{\sin x} \Rightarrow 1 < \frac{x}{\sin x} < \frac{1}{\cos x}$

Reciprocating this inequality: $1 > \frac{\sin x}{x} > \cos x$. Now: $\lim_{x\to 0} 1 = 1 = \lim_{x\to 0} \cos x$.

Hence according to the Squeeze Theorem: $\lim_{x\to c} \frac{\sin x}{x} = 1$

We can use the above result to show:

$$\lim_{x\to c} \frac{1-\cos x}{x} = 0$$

Proof:

$$\lim_{x\to c} \frac{1-\cos x}{x} = \lim_{x\to 0} \frac{(1-\cos x)}{x} \cdot \frac{(1+\cos x)}{(1+\cos x)} = \lim_{x\to 0} \frac{1-\cos^2 x}{x(1+\cos x)} = \lim_{x\to 0} \frac{\sin^2 x}{x(1+\cos x)}$$

$$= \lim_{x\to 0} \frac{\sin x}{x} \lim_{x\to 0} \frac{\sin x}{(1+\cos x)} = 1 \cdot \frac{\lim_{x\to 0} \sin x}{\lim_{x\to 0}(1+\cos x)} = 1 \cdot \frac{0}{1+1} = 0$$

Note how some of the properties of limits and the previous results were utilized!

- ***The Derivative***

Note the two definitions (pages 98, 100 from text):

$$f'(c) = \lim_{\Delta x\to 0} \frac{f(c+\Delta x) - f(c)}{\Delta x} = \lim_{x\to c} \frac{f(x) - f(x)}{x - c}$$

Which basically represents the slope of a *tangent* line representing the *instantaneous rate of change of f(x)* at the point $x = c$. On the other hand, the *average rate of change of f(x)* at $x = c$ is the slope of a *secant* line intersecting the graph of f(x) at points $P_1(c, f(c))$ $P_2(c + \Delta x, f(c + \Delta c))$:

$$m = \frac{\Delta y}{\Delta x} = \frac{f(c+\Delta x) - f(x)}{c + \Delta x - c} = \frac{f(c+\Delta x) - f(x)}{\Delta x}$$

In addition we can also define *left hand* and *right-hand* derivatives:

$$f'(c)^{-} = \lim_{\Delta x \to 0^{-}} \frac{f(c-\Delta x) - f(c)}{\Delta x} = \lim_{x \to c^{-}} \frac{f(x) - f(x)}{x - c}$$

$$f'(c)^{+} = \lim_{\Delta x \to 0^{+}} \frac{f(c+\Delta x) - f(c)}{\Delta x} = \lim_{x \to c^{+}} \frac{f(x) - f(x)}{x - c}$$

Note: In the case of the the left-hand derivative, we're approaching *c* from the left hand side. This means we're looking at the quantity (c - Δx) in the top expression (since Δx is an arbitrarily small positive deviation)

Obviously, *f* is **differentiable** (its derivative exists) at $x = c$ provided its left and right hand derivatives agree, i.e.

$$f'(c) = \lim_{\Delta x \to 0} \frac{f(c+\Delta x) - f(c)}{\Delta x} = \lim_{x \to c} \frac{f(x) - f(x)}{x - c} \quad \textbf{provided that:}$$

$$f'(c)^{-} = f'(c)^{+}$$

- **Example (Exercise 24)**

Find the derivative of *f*(x) at $c = 3$, if it exists

Method 1

$$f'(3) = \lim_{\Delta x \to 0} \frac{f(3+\Delta x) - f(3)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\frac{1}{(3+\Delta x)} - \frac{1}{3}}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\frac{3-(3+\Delta x)}{3(3+\Delta x)}}{\Delta x} = \lim_{\Delta x \to 0} \frac{-\Delta x}{3(3+\Delta x)\Delta x} = -\lim_{\Delta x \to 0} \frac{1}{3(3+\Delta x)} = -\tfrac{1}{9}$$

Method 2 (Using the alternate definition of derivative)

$$f'(3) = \lim_{x \to 3} \frac{f(x) - f(3)}{x - 3} = \lim_{x \to 3} \frac{\frac{1}{x} - \frac{1}{3}}{x - 3}$$

$$= \lim_{x \to 3} \frac{\frac{3-x}{3x}}{x - 3} = \lim_{\Delta x \to 0} \frac{3-x}{3x(x-3)} = -\tfrac{1}{3} \lim_{x \to 3} \frac{1}{3x} = -\tfrac{1}{3} \cdot \tfrac{1}{3} = -\tfrac{1}{9}$$

Example (Exercise 41) Is the function differentiable at $x = 1$?

$$f(x) = \begin{cases} (x-1)^3 & x \leq 1 \\ (x-1)^2 & x > 1 \end{cases}$$

Here we have to be careful and use *right* and *left* hand derivatives (since the function is defined by different formulas to the right and left of x =1.

$$f'(1)^- = \lim_{\Delta x \to 0^-} \frac{f(1-\Delta x) - f(1)}{\Delta x} = \lim_{\Delta x \to 0^-} \frac{(1-\Delta x - 1)^3 - (1-1)^3}{\Delta x}$$
$$= \lim_{\Delta x \to 0^-} \frac{-(\Delta x)^3}{\Delta x} = -\lim_{\Delta x \to 0^1} (\Delta x)^2 = 0$$

$$f'(1)^+ = \lim_{\Delta x \to 0^+} \frac{f(1+\Delta x) - f(1)}{\Delta x} = \lim_{\Delta x \to 0^-} \frac{(1+\Delta x - 1)^2 - (1-1)^2}{\Delta x}$$
$$= \lim_{\Delta x \to 0^-} \frac{(\Delta x)^2}{\Delta x} = \lim_{\Delta x \to 0^1} (\Delta x) = 0$$

Since they agree, we can conclude that f is differentiable at $x = 1$, with derivative = 0.

Method 2 (Using the alternate definition of derivative)

$$f'(1)^- = \lim_{x \to 1^-} \frac{f(x) - f(1)}{x-1} = \lim_{x \to 1^-} \frac{(x-1)^3 - (1-1)^3}{x-1}$$
$$= \lim_{x \to 1^-} \frac{(x-1)^3}{x-1} = \lim_{x \to 1^-} (x-1)^2 = 0$$

$$f'(1)^+ = \lim_{x \to 1^+} \frac{f(x) - f(1)}{x-1} = \lim_{x \to 1^-} \frac{(x-1)^2 - (1-1)^2}{x-1}$$
$$= \lim_{x \to 1^-} \frac{(x-1)^2}{x-1} = \lim_{x \to 1^-} (x-1) = 0$$

Which of course gives the same result, as expected