MA262 **Calculus II: Notes for March 6**
Spring 2008 Kallfelz

- *REPRESENTING FUNCTIONS BY TAYLOR SERIES (GENERAL METHODS)*

Recall[1] the presentation of Taylor's Theorem:

Given a function $f(x)$ differentiable at least to order $(n+1)$ on interval $[c, x)$, there exists a $z \in (c, x)$ such that:

$$f(x) = P_n(x) + R_n(x)$$

where: $P_n(x) = \sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(c)(x-c)^k$

and: $R_n(x) = \frac{1}{n!}\int_c^x f^{(n+1)}(t)(x-t)^n\,dt = \frac{f^{(n+1)}(z)}{(n+1)!}(x-c)^{n+1}$

Thm 10.23 (p 623) basically establishes that whenever a function can be represented by a power series (within a certain interval of convergence $I_c^R = (c-R, c+R) = \{x \mid c-R < x < x+R\}$):

$$f(x) = \sum_{k-0}^{\infty} a_k (x-c)^k \qquad \text{(for all } x \text{ in } I_c^R\text{)}$$

…then $a_k = \frac{1}{k!} f^{(k)}(c)$ for all $k \geq 0$, i.e. the Power Series *is* $f(x)$'s Taylor Series.

On the one hand, this may not seem so surprising, since on a point c, the representation $f(x) = \sum_{k-0}^{\infty} a_k (x-c)^k$ must be *unique* there. In other words, changing *any* of the series coefficient a_k for *any* $k \geq 0$, means we've changed the series, obviously! To the objection: "Yes, but isn't it possible to alter values of the coefficients in some way such that they still add up to the same function $f(x)$?" …the response is *every* coefficient a_k is *uniquely* associated with a power of x term in a simple 1-1 correspondence[2]: $a_k \overset{1-1}{\longleftrightarrow} (x-c)^k$. So this really isn't possible, i.e. there's *no way* to change a_{k_1}, a_{k_2} to some other values b_{k_1}, b_{k_2} (where: $a_{k_1} \neq b_{k_1}, a_{k_2} \neq b_{k_2}$) such that:

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[1]Feb 28 notes http://www.glue.umd.edu/%7Ewkallfel/MA261-2/Feb28.pdf

[2] Another way of stating this is that the terms $(x-c)^k$ (for all $k \geq 0$) form a *linearly independent set*, or *basis*

$$a_{k_1}(x-c)^{k_1} + a_{k_2}(x-c)^{k_2} = b_{k_1}(x-c)^{k_1} + b_{k_2}(x-c)^{k_2}$$

..*not even in the trivial case* when $x = c$, since k can take on the value 0 as well. So to say that: $a_0(x-c)^0 = b_0(x-c)^0$ but $a_0 \neq b_0$ is clearly to contradict oneself.

So one should think of **Thm 10.23** as a *uniqueness representation theorem*: A function represented by a power series (within a certain interval of convergence $I_c^R = (c-R, c+R) = \{x \mid c-R < x < x+R\}$) is *uniquely represented by that power series.* So therefore the coefficients a_k of that power series "have no choice" but to be the Taylor coefficients $\frac{1}{k!} f^{(k)}(c)$ for all $k \geq 0$! Of course, you'll recognize the proof of the theorem to consist of the same constructive procedure as to how the Taylor coefficients were first generated heuristically in pp. 2-3 in Feb 28 notes[3], so there's no need to re-hash.

But it's very important to understand the *significance* of this unique representation theorem. It's telling us in effect that *whatever method you have available to expand a function into its power series on its interval of absolute convergence, you've automatically generated its Taylor Series*! Hence, for example, the 'Geometric Series Trick' discussed in pp. 10-13 in March 4 notes[4] and in §10.9 of the textbook is one such powerful approach to easily generate Taylor Series for many classes of functions (rational and otherwise) without having to crank out the Taylor coefficients $\frac{1}{k!} f^{(k)}(c)$ from first principles. As we'll see here, other tricks include the use of the *General Binomial Theorem.*

Thm 10.24 further substantiates this uniqueness representation theorem by specifying the conditions of necessity and sufficiency for such a representation to be possible. Clearly, not only must $f(x)$ be differentiable for *any* order, but the residue term $R_n(x)$ must converge to zero, i.e.:

$$\lim_{n\to\infty} R_n(x) = \lim_{n\to\infty} \frac{1}{n!}\int_c^x f^{(n+1)}(t)(x-t)^n\,dt = \lim_{n\to\infty} \frac{f^{(n+1)}(z)}{(n+1)!}(x-c)^{n+1} = 0$$

…which seems straightforward enough. For (to prove by contradiction) were it *not* the case that the above limit holds, i.e. that there would exist some nonzero N such that:

$$\lim_{n\to\infty} R_n(x) = \lim_{n\to\infty} \frac{1}{n!}\int_c^x f^{(n+1)}(t)(x-t)^n\,dt = \lim_{n\to\infty} \frac{f^{(n+1)}(z)}{(n+1)!}(x-c)^{n+1} = N$$

Then in the interval of convergence: $I_c^R = (c-R, c+R) = \{x \mid c-R < x < x+R\}$:

[3] http://www.glue.umd.edu/%7Ewkallfel/MA261-2/Feb28.pdf
[4] http://www.glue.umd.edu/%7Ewkallfel/MA261-2/Mar4.pdf

$$f(x)=\sum_{k=0}^{\infty}\tfrac{1}{k!}f^{(k)}(c)(x-c)^k+N$$

..according to Taylor's Thm (in the $n\to\infty$ limit). However, as was just shown in **Thm 10.23**, in the interval of convergence, $I_c^R=(c-R,c+R)=\{x\,|\,c-R<x<x+R\}$:

$$f(x)=\sum_{k=0}^{\infty}a_k(x-c)^k=\sum_{k=0}^{\infty}\tfrac{1}{k!}f^{(k)}(c)(x-c)^k$$

…thus we're contradicting ourselves; unless of course $N=0$, i.e. unless:

$$\lim_{n\to\infty}R_n(x)=\lim_{n\to\infty}\tfrac{1}{n!}\int_c^x f^{(n+1)}(t)(x-t)^n\,dt=\lim_{n\to\infty}\frac{f^{(n+1)}(z)}{(n+1)!}(x-c)^{n+1}=0$$

- *THE BINOMIAL SERIES TRICK*

Recall that for any non-negative integer *n*, the binomial series expansion for $(x\pm y)^n$ is[5]

$$(x\pm y)^n=\sum_{k=0}^{n}\binom{n}{k}(\pm1)^k x^{n-k}y^k$$

where: $\binom{n}{k}=\dfrac{n!}{k!(n-k)!}$ are the *Binomial Coefficients,* defined in terms of a *Combination*, i.e. the number of ways to select subsets of *k* objects from a set of *n* objects (where: $0\le k\le n$), to recall footnote 7, p. 5, March 4th notes.[6]

For example:

$$(x-y)^3=\sum_{k=0}^{n}\binom{3}{k}(-1)^k x^{3-k}y^k=(-1)^0\binom{3}{0}x^{3-0}y^0+(-1)^1\binom{3}{1}x^{3-1}y^1+(-1)^2\binom{3}{2}x^{3-2}y^2+(-1)^3\binom{3}{3}x^{3-3}y^3$$
$$=\frac{3!}{0!(3-0)!}x^3-\frac{3!}{1!(3-1)!}x^2y+\frac{3!}{2!(3-2)!}xy^2-\frac{3!}{3!(3-3)!}y^3$$
$$=x^3-3x^2y+3xy^2-y^3$$

…as can easily be verified through the 'foil' method of expansion of $(x-y)^3$.

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[5] See p.7 of http://www.glue.umd.edu/%7Ewkallfel/MA261-2/Notes3.pdf for an application and description of the Binomial expansion, as applied in the proof of the elementary formula: $\frac{d}{dx}x^n=nx^{n-1}$.

[6] http://www.glue.umd.edu/%7Ewkallfel/MA261-2/Mar4.pdf

Note that one may simplify the above:

$$(x \pm y)^n = x^n \left(1 \pm \left(\tfrac{y}{x}\right)\right)^n = x^n (1 + u)^n = x^n \sum_{k=0}^{n} \binom{n}{k} (\pm 1)^k u^k$$

…where $u = \frac{y}{x}$. Hence: $(1 \pm u)^n = \sum_{k=0}^{n} \binom{n}{k} (\pm 1)^k u^k$

Interestingly, as illustrated in **Example 4** (pp. 637-638, text) the binomial expansion has a useful generalization, for *any* $\alpha > 0$, into the *General Binomial Series*:

$$(1 \pm u)^\alpha = \sum_{k=0}^{\infty} \binom{\alpha}{k} (\pm 1)^k u^k$$

where the *general binomial coefficients* take on the form: $\binom{\alpha}{k} = \dfrac{\alpha(\alpha - 1)\ldots(\alpha - k + 1)}{k!}$

Remark 1: In the special case when $\alpha = n$ (where n is some positive integer) we see how the General Binomial Series Reduces to the accustomed result:

$$(1 \pm u)^n = \sum_{k=0}^{\infty} \binom{n}{k} (\pm 1)^k u^k = \sum_{k=0}^{n} \binom{n}{k} (\pm 1)^k u^k$$

since in this special case, the coefficients of the first n of the series are:

$$\binom{n}{0} = \frac{1}{0!} = 1 = \frac{n!}{0!(n-0)!}$$

$$\binom{n}{1} = \frac{n-1+1}{1!} = n = \frac{n!}{1!(n-1)!}$$

$$\binom{n}{2} = \frac{n(n-2+1)}{2!} = \tfrac{1}{2} n(n-1) = \frac{n!}{2!(n-2)!}$$

$$\vdots$$

$$\binom{n}{n-1} = \frac{n(n-1)(n-2)\ldots(n-n+1+1)}{(n-1)!} = \frac{n(n-1)\ldots 2}{(n-1)!} = \frac{n!}{(n-1)!(n-(n-1))!}$$

$$\binom{n}{n} = \frac{n(n-1)(n-2)\ldots(n-n+1)}{n!} = \frac{n(n-1)\ldots 2 \cdot 1}{n!} = \frac{n!}{n!(n-n)!}$$

…which exhibit the formal equivalence to the binomial coefficients presented in the previous page: $\binom{n}{k} = \dfrac{n!}{k!(n-k)!}$, for $0 \le k \le n$..

What about the terms $k > n$ of the series? Observe in the case $k = n + 1$

$$\binom{n}{n+1} = \frac{n(n-1)\ldots(n-n-1+1)}{(n+1)!} = \frac{n(n-1)(n-2)\ldots.2\cdot 1\cdot 0}{(n+1)!} = 0$$

Hence: $\binom{n}{k} = 0$ for *all* $k > n$, since the product string in the numerator will contain a 0 term. Hence:

$$(1 \pm u)^n = \sum_{k=0}^{\infty}\binom{n}{k}(\pm 1)^k u^k = \sum_{k=0}^{n}\binom{n}{k}(\pm 1)^k u^k + \sum_{k=n+1}^{\infty}\binom{n}{k}(\pm 1)^k u^k$$
$$= \sum_{k=0}^{n}\binom{n}{k}(\pm 1)^k u^k + 0 + 0 + 0 + \ldots = \sum_{k=0}^{n}\binom{n}{k}(\pm 1)^k u^k$$

As shown in p. 637 (text) the General Binomial Coefficients $\binom{\alpha}{k} = \frac{\alpha(\alpha-1)\ldots(\alpha-k+1)}{k!}$ were generating by calculating (from first principles) the McClaurin coefficients for the function $f(x) = (1+x)^\alpha$. To establish the interval of absolute convergence, using the **RaT:**

$$\rho(x) < 1 \Rightarrow \lim_{n\to\infty}\frac{|a_{n+1}(x)|}{|a_n(x)|} = \lim_{n\to\infty}\frac{\binom{\alpha}{n+1}|x|^{n+1}}{\binom{\alpha}{n}|x|^n} = |x|\lim_{n\to\infty}\frac{\left|\frac{\alpha(\alpha-1)\ldots(\alpha-n-1+1)}{(n+1)!}\right|}{\left|\frac{\alpha(\alpha-1)\ldots(\alpha-n+1)}{n!}\right|}$$
$$= |x|\lim_{n\to\infty}\left|\frac{\alpha(\alpha-1)\ldots(\alpha-n)}{(n+1)!}\cdot\frac{n!}{\alpha(\alpha-1)\ldots(\alpha-n+1)}\right| = |x|\lim_{n\to\infty}\frac{|(\alpha-n)|}{(n+1)}$$
$$= |x|\lim_{n\to\infty}\frac{\left|\frac{\alpha}{n}-1\right|}{1+\frac{1}{n}} = |x|\cdot 1 = |x| \Rightarrow |x| < 1 \Rightarrow I_0^1 = (-1,1) = \{x \mid -1 < x < 1\}$$

Hence based on **Thm 10.23** & **10.24** we may immediately conclude that the McClaurin Series for any function $f(x) = (1 \pm x)^\alpha$ is the *General Binomial Series* with interval of absolute convergence (-1, 1):

$$f(x) = (1 \pm x)^\alpha = \sum_{k=0}^{\infty}\binom{\alpha}{k}(\pm 1)^k x^k$$

Remark 2: Certainly the above can be generalized to functions of the form: $g(x)=(a+bx)^{\alpha}$, (for any nonzero[7] *a, b*) via the following maneuvers:

$$g(x)=(a+bx)^{\alpha} = a^{\alpha}\left(1+\tfrac{b}{a}x\right)^{\alpha} = a^{\alpha}\sum_{k=0}^{\infty}\binom{\alpha}{k}\tfrac{b^k}{a^k}x^k$$

Whose interval of absolute convergence becomes: $I_0^{a/b} = \left(-\frac{a}{b},\frac{a}{b}\right)=\left\{x \mid \frac{a}{b} < x < \frac{a}{b}\right\}$, as is easily established via **RaT:**

$$\rho(x)<1 \Rightarrow \lim_{n\to\infty}\frac{|a_{n+1}(x)|}{|a_n(x)|} = \lim_{n\to\infty}\frac{\binom{\alpha}{n+1}\left|\frac{b}{a}x\right|^{n+1}}{\binom{\alpha}{n}\left|\frac{b}{a}x\right|^{n}} = \left|\tfrac{b}{a}x\right|\lim_{n\to\infty}\frac{\left|\frac{\alpha(\alpha-1)\ldots(\alpha-n-1+1)}{(n+1)!}\right|}{\left|\frac{\alpha(\alpha-1)\ldots(\alpha-n+1)}{n!}\right|}$$

$$=\left|\tfrac{b}{a}x\right|\cdot 1=\left|\tfrac{b}{a}x\right| \Rightarrow \left|\tfrac{b}{a}x\right|<1 \Rightarrow -1<\tfrac{b}{a}x<1 \Rightarrow -\tfrac{a}{b}<x<\tfrac{a}{b} \Rightarrow I_0^{a/b} = \left(-\tfrac{a}{b},\tfrac{a}{b}\right)=\left\{x \mid -\tfrac{a}{b}<x<\tfrac{a}{b}\right\}$$

Remark 3: As mention (p. 638, text) the particular value of the exponent α determines whether or not convergence occurs at the endpoints of the interval of convergence, which is illustrated in the following example below:

- <u>Example</u>: Determine the McClaurin Series and its Interval of convergence for the function: *f*(*x*) = arcsin*x*

$$f(x)=\arcsin x \Rightarrow f'(x)=\frac{1}{\sqrt{1-x^2}}=\left(1-x^2\right)^{-1/2}$$

Hence using the General Binomial Series Trick:

$$\left(1-x^2\right)^{-1/2} = \sum_{k=0}^{\infty}\binom{-\frac{1}{2}}{k}(-1)^k\left(x^2\right)^k = \sum_{k=0}^{\infty}\binom{-\frac{1}{2}}{k}(-1)^k x^{2k}$$

…with an interval of absolute convergence (-1 , 1):

Using **Thm 10.21:**

$$\arcsin x = \int\left(1-x^2\right)^{1/2}dx = \sum_{k=0}^{\infty}(-1)^k\binom{-\frac{1}{2}}{k}\int x^{2k}dx = \sum_{k=0}^{\infty}(-1)^k\binom{-\frac{1}{2}}{k}\frac{x^{2k+1}}{(2k+1)}$$

[7] In addition, further restrictions would apply for *a* depending on the value of α. I.e., if α were half-integer valued, then $a > 0$.

$$= \sum_{k=0}^{\infty} (-1)^k \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\ldots\left(-\frac{1}{2}-k+1\right)}{k!} \cdot \frac{x^{2k+1}}{(2k+1)} = \sum_{k=0}^{\infty} (-1)^k \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\ldots\left(-\frac{(2k-1)}{2}\right)}{k!} \cdot \frac{x^{2k+1}}{2k+1}$$

$$= \sum_{k=0}^{\infty} (-1)^k \frac{(-1)^k (1\cdot 3\cdot 5\cdot \ldots \cdot (2k-1))}{2^k k!(2k+1)} x^{2k+1} = \sum_{k=0}^{\infty} (-1)^{2k} \frac{(1\cdot 3\cdot 5\cdot \ldots \cdot (2k-1))}{(2k)\cdot (2k-2)\ldots \cdot (2)} \cdot \frac{x^{2k+1}}{2k+1}$$

$$= \sum_{k=0}^{\infty} \frac{(2k)(2k-1)(2k-2)\cdot \ldots \cdot 5\cdot 4\cdot 3\cdot 2\cdot 1}{[(2k)\cdot(2k)]\cdot[(2k-2)\cdot(2k-2)]\cdot \ldots \cdot [2\cdot 2]} \cdot \frac{x^{2k+1}}{(2k+1)} = \sum_{k=0}^{\infty} \frac{(2k)!}{[(2k)(2k-2)\cdot \ldots \cdot 2][(2k)\cdot(2k-2)\cdot \ldots \cdot 2]} \cdot \frac{x^{2k+1}}{(2k+1)}$$

$$= \sum_{k=0}^{\infty} \frac{(2k)!}{[2^k k!][2^k k!]} \cdot \frac{x^{2k+1}}{(2k+1)} = \sum_{k=0}^{\infty} \frac{(2k)!}{(2^k k!)^2} \cdot \frac{x^{2k+1}}{(2k+1)}$$

Note how the "trick of 1" appeared via the creation of even terms in the numerator expression, so that a simpler expression for the coefficients can be obtained in this relatively intricate expansion.

To test convergence at the endpoints:

$x = 1$: $\displaystyle \sum_{k=0}^{\infty} \frac{(2k)!}{(2^k k!)^2 (2k+1)}$

Using **Rat:**

$$\rho = \lim_{k\to\infty} \frac{|a_{k+1}|}{|a_k|} = \lim_{k\to\infty} \frac{(2k+2)!}{(2^{k+1}(k+1)!)^2 (2k+3)} \cdot \frac{(2^k k!)^2 (2k+1)}{(2k)!}$$

$$= \lim_{k\to\infty} \frac{(2k+2)(2k+1)(2k)!(2^k k!)(2^k k!)(2k+1)}{(2\cdot 2^k \cdot (k+1)k!)(2\cdot 2^k \cdot (k+1)k!)(2k+3)(2k)!}$$

$$= \lim_{k\to\infty} \frac{(2k+2)(2k+1)^2}{(2(k+1))^2 (2k+3)} = \lim_{k\to\infty} \frac{(2k+1)^2}{(2k+2)(2k+3)} = \lim_{k\to\infty} \frac{4k^2+4k+1}{4k^2+7k+6}$$

$$= \lim_{k\to\infty} \frac{1+\frac{1}{k}+\frac{1}{4k^2}}{1+\frac{7}{4k}+\frac{3}{2k^2}} = 1$$

…which is inconclusive! (All that work for nothing ☺)

However, note that for all k: $\displaystyle \frac{(2k)!}{(2^k k!)^2 (2k+1)} \le \frac{(2k+1)(2k)!}{(2^k k!)^2 (2k+1)} = \frac{(2k)!}{(2^k k!)^2}$

And adopting **Rat** for the simpler series on the right:

$$\rho = \lim_{k\to\infty} \frac{(2k+2)!}{\left(2^{k+1}(k+1)!\right)^2} \cdot \frac{\left(2^k k!\right)^2}{(2k)!} = \lim_{k\to\infty} \frac{(2k+2)(2k+1)(2k)!\left(2^k k!\right)\left(2^k k!\right)}{\left(2 \cdot 2^k \cdot (k+1)k!\right)\left(2 \cdot 2^k \cdot (k+1)k!\right)(2k)!}$$

$$= \lim_{k\to\infty} \frac{(2k+2)(2k+1)}{(2k+2)^2} = \lim_{k\to\infty} \frac{4k^2+4k+2}{4k^2+8k+4} = \lim_{k\to\infty} \frac{1+\frac{1}{k}+\frac{1}{2k^2}}{1+\frac{2}{k}+\frac{1}{k^2}} = 1$$

Likewise inconclusive ! ☹

So **SCT & RaT** produces results which are inconclusive.

However, when adopting the **RoT** to the above:

$$r = \lim_{k\to\infty} \sqrt[k]{\frac{(2k)!}{\left(2^k k!\right)}} = \lim_{k\to\infty} \left[\frac{(2k)!}{2^k \cdot 2^k \cdot k! k!}\right]^{1/k} = \tfrac{1}{4} \lim_{k\to\infty} \left[\frac{(2k)!}{k!k!}\right]^{1/k}$$

And taking the logarithm of that expression:

$$\lim_{k\to\infty} \tfrac{1}{k} \ln\left[\frac{(2k)!}{k!k!}\right] = \lim_{k\to\infty} \frac{\ln((2k)!) - 2\ln k!}{k}$$

$$= \lim_{k\to\infty} \frac{\ln 2k + \ln(2k-1) + \ln(2k-2) + \ldots + \ln 2 + 1 - 2[\ln k + \ln(k-1) + \ldots + \ln 2 + 1]}{k}$$

$$\xrightarrow{LHR} \lim_{k\to\infty} \left(\tfrac{1}{2k} + \tfrac{1}{2k-1} + \ldots - 2\left[\tfrac{1}{k} + \tfrac{1}{k-1} + \ldots\right]\right) = 0 \Rightarrow \lim_{k\to\infty} \left[\frac{(2k)!}{k!k!}\right]^{1/k} = e^0 = 1$$

$$\therefore r = \tfrac{1}{4} \lim_{k\to\infty} \left[\frac{(2k)!}{k!k!}\right]^{1/k} = \tfrac{1}{4} < 1$$

And since $\frac{(2k)!}{\left(2^k k!\right)^2 (2k+1)} \le \frac{(2k)!}{\left(2^k k!\right)^2}$, hence by **SCT** and **RoT,** it is established that $\sum_{k=0}^{\infty} \frac{(2k)!}{\left(2^k k!\right)^2 (2k+1)}$ converges. (This is rare instance in which the root test proved advantageous over the ratio test in an expression involving factorials—given the fact that the above was applied to the simpler series, hence a simpler ratio!)

In the case x = -1: $\sum_{k=0}^{\infty} (-1)^k \frac{(2k)!}{\left(2^k k!\right)^2 (2k+1)}$ is an alternating series. However, as shown in the above case, we know immediately that this alternating series converges in absolute value, i.e. absolutely converges, hence we can conclude:

$$\arcsin x = \sum_{k=0}^{\infty} \frac{(2k)!}{(2^k k!)^2 (2k+1)} x^{2k+1} \qquad \text{on} \qquad [-1\,,\,1]$$

The above result establishes the last of the list of Taylor/McClaurin Series listed in the 'toolchest' of functions on p. 638

- Example (# 17, § 10.10)

$$f(x) = e^{x^2/2} \Rightarrow e^u = \sum_{k=0}^{\infty} \tfrac{1}{k!} u^k \xrightarrow{THM\,10.22} e^{x^2/2} = \sum_{k=0}^{\infty} \tfrac{1}{k!} \left(\tfrac{1}{2} x^2\right)^k = \sum_{k=0}^{\infty} \frac{x^{2k}}{2^k k!}$$

(for all $x \in (-\infty, \infty)$)

- Example (# 24, § 10.10)

$$f(x) = \frac{\arcsin x}{x} = h(x)g(x) = \tfrac{1}{x} \sum_{k=0}^{\infty} \frac{(2k)!}{(2^k k!)^2} \cdot \frac{x^{2k+1}}{(2k+1)} \xrightarrow{THM\,10.22} \sum_{k=0}^{\infty} \frac{(2k)!\, x^{2k}}{(2^k k!)^2 (2k+1)}$$

(for all $x \in [-1,1]$)

- Example: Given the McClaurin Series for arcsin$x$ (derived above) and the McClaurin Series for arctan$x$ (derived in pp. 12-13, March 4 notes[8]) express the transcendental number π in two different ways. Which (finite) series representation is more accurate?

$$\arctan x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{2k+1} \Rightarrow \arctan 1 = \tfrac{\pi}{4} = \sum_{k=0}^{\infty} (-1)^k \frac{1^{2k+1}}{(2k+1)} \Rightarrow \pi = 4 \sum_{k=0}^{\infty} (-1)^k \cdot \frac{1}{2k+1}$$

$$\arcsin x = \sum_{k=0}^{\infty} \frac{(2k)!}{(2^k k!)^2 (2k+1)} x^{2k+1} \Rightarrow \arcsin 1 = \tfrac{\pi}{2} = \sum_{k=0}^{\infty} \frac{(2k)!}{(2^k k!)^2 (2k+1)} \Rightarrow \pi = 2 \sum_{k=0}^{\infty} \frac{(2k)!}{(2^k k!)^2 (2k+1)}$$

To assess their relative accuracies, recall the remainder (error) term in Taylor's Thm:

$$R_n(x) = \tfrac{1}{n!} \int_c^x f^{(n+1)}(t)(x-t)^n \, dt = \frac{f^{(n+1)}(z)}{(n+1)!} (x-c)^{n+1}$$

Now in both cases, arctan$x$ and arcsin$x$ have McClaurin Series which converge in the interval [-1, 1]. The remainder term in the McClaurin case is of course:

[8] http://www.glue.umd.edu/%7Ewkallfel/MA261-2/Mar4.pdf

$$R_n(x) = \tfrac{1}{n!}\int_0^x f^{(n+1)}(t)(x-t)^n\,dt = \frac{f^{(n+1)}(z)}{(n+1)!}x^{n+1}$$

Hence for the respective arcsine and arctan, their remainder terms are bounded above by the expressions:

$$R_n(x)_{\text{arcsin}} \le \tfrac{1}{(n+1)!}\max_{-1\le z\le 1}\left(\tfrac{d^{n+1}}{d^{n+1}}\arcsin z\right)x^{n+1}$$
$$R_n(x)_{\text{arcsin}} \le \tfrac{1}{(n+1)!}\max_{-1\le z\le 1}\left(\tfrac{d^{n+1}}{d^{n+1}}\arctan z\right)x^{n+1}$$

So for instance, in the case n =1 (linear approximation of π):

$$\frac{d^{1+1}}{d^{1+1}z}\arcsin z = \frac{d^2}{dz^2}\arcsin z = \frac{d}{dz}\left(1-z^2\right)^{-1/2} = -\tfrac{1}{2}\left(1-z^2\right)^{-3/2}(-2z) = \frac{z}{\left(1-z^2\right)^{3/2}}$$
$$\frac{d^{1+1}}{d^{1+1}z}\arctan z = \frac{d^2}{dz^2}\arctan z = \frac{d}{dz}\left(1+z^2\right)^{-1} = -\left(1+z^2\right)^{-2}(2z) = \frac{2z}{\left(1+z^2\right)^2}$$

In the interval [-1, 1] it is apparent that:

$$\max_{-1<z<1}\left|\tfrac{d^2}{dz^2}\arcsin z\right| >> \max_{-1<z<1}\left|\tfrac{d^2}{dz^2}\arctan z\right|$$

…for the obvious reason that as $z \to 1$, $\dfrac{d^{1+1}}{d^{1+1}z}\arcsin z = \dfrac{z}{\left(1-z^2\right)^{3/2}} \to \infty$

But $g(z) = \dfrac{d^{1+1}}{d^{1+1}z}\arctan z = \dfrac{2z}{\left(1+z^2\right)^2} \to \tfrac{1}{2}$. Moreover, though $g(z)$ isn't monotone increasing or decreasing in the interval [-1 , 1], it's well-behaved (no asymptotic divergences).

Hence the remainder term for arctan$z$ is strictly bounded above by a relatively smaller amount, than for the arcsin case. So the expansion: $\pi = 4\sum_{k=0}^{n}(-1)^k\dfrac{1}{2k+1}$

…is more accurate. Note the results below (Excel) for the n=10 case

k	**arcsine expansion**		**arctan expansion**	
0	1	**0th term**	1	**0th term**
1	0.166667	**1st term**	-0.33333333	**1st term**
2	0.075	**2nd term**	0.2	**2nd term**
3	0.044643	**3rd term**	-0.14285714	**3rd term**

4		0.030382	**4th term**		0.11111111	**4th term**
5		0.022372			-0.09090909	
6		0.017353			0.07692308	
7		0.013965			-0.06666667	
8		0.011552			0.05882353	
9		0.009762			-0.05263158	
10		0.00839			0.04761905	
SUM	**2.800169964**			**3.232315809**		
PI	**3.141592654**			**3.141592654**		
Error	**0.34142269**			**0.090723156**		

Where the error term is calculated via:

$$Err_{\text{arcsin}} = \left| \pi - 2\sum_{k=0}^{n} \frac{(2k)!}{\left(2^k k!\right)^2 (2k+1)} \right| \qquad Err_{\text{arctan}} = \left| \pi - 2\sum_{k=0}^{n} \frac{(-1)^k}{(2k+1)} \right|$$

For the respective expansions (n = 10)

- <u>Example (# 29, § 10.10)</u>

$$f(x) = \cos^2 x = \tfrac{1}{2}(1+\cos 2x) = \tfrac{1}{2}\left(1 + \sum_{k=0}^{\infty} (-1)^k \frac{(2x)^{2k}}{(2k)!}\right) = \tfrac{1}{2}\left(1 + 1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} - \ldots\right)$$

$$= \tfrac{1}{2}\left(2 + \frac{(2x)^2}{2!} - \frac{(2x)^4}{4!} + \ldots\right) = 1 + \frac{2^1 x^2}{2!} - \frac{2^3 x^4}{4!} + \ldots = 1 + \sum_{k=1}^{\infty} (-1)^{k-1} \cdot 2^{2k-1} \cdot \frac{x^{2k}}{(2k)!}$$

- <u>Example (#26,27 modified)</u>

Using McClaurin Series expansion, prove Euler's Thm: $e^{i\theta} = \cos\theta + i\sin\theta$

$$e^{i\theta} = \sum_{k=0}^{\infty} \frac{(i\theta)^k}{k!} = 1 + i\theta + \tfrac{1}{2!} i^2\theta + \tfrac{1}{3!} i^3\theta^3 + \tfrac{1}{4!} i^4\theta^4 + \tfrac{1}{5!} i^5\theta^5 + \ldots$$

$$= 1 + i\theta - \tfrac{1}{2!}\theta - \tfrac{1}{3!} i\theta + \tfrac{1}{4!}\theta^4 + \tfrac{1}{5!} i\theta^5 + \ldots$$

$$= \left(1 - \tfrac{1}{2!}\theta^2 + \tfrac{1}{4!}\theta^4 - \tfrac{1}{6!}\theta^6 + \ldots\right) + i\left(\theta - \tfrac{1}{3!}\theta^3 + \tfrac{1}{5!}\theta^5 - \tfrac{1}{7!}\theta^7 + \ldots\right)$$

$$= \sum_{k=0}^{\infty} (-1)^k \frac{\theta^{2k}}{(2k)!} + i\sum_{k=0}^{\infty} (-1)^k \frac{\theta^{2k+1}}{(2k+1)!} = \cos\theta + i\sin\theta$$

(since: $i^2 = -1, i^3 = -i, i^4 = 1$, etc.)