MA262 **Calculus II: Notes for March 4**
Spring 2008 Kallfelz

- *POWER SERIES*

Defn1: A ***power-series centered at x = c*** is an infinite series of the form:

$$p(x)=\sum_{k=0}^{\infty}a_k(x-c)^k \equiv \lim_{n\to\infty}\sum_{k=0}^{n}a_k(x-c)^k = \lim_{n\to\infty} p_n(x)$$

Remark 1: Recall the Taylor Polynomial (of degree n) from **Feb. 28 Notes**: $P_n(x)=\sum_{k=0}^{n}\frac{1}{k!}f^{(k)}(c)(x-c)^k$. Then in the case in which the Taylor Polynomial becomes a convergent infinite series: $\lim_{n\to\infty}P_n(x)=\sum_{k=0}^{n}\frac{1}{k!}f^{(k)}(c)(x-c)^k=\sum_{k=0}^{\infty}\frac{1}{k!}f^{(k)}(c)(x-c)^k$, i.e. in the case of a *Taylor Series*, this is a special case of a function represented by a power-series centered at $x = c$, where: $a_k=\dfrac{f^{(k)}(c)}{k!}$, for all k. In the even more special case of an infinitely convergent McClaurin polynomial, i.e. a *McClaurin Series* is a special instance of a function represented by a power-series centered at $c = 0$, where: $a_k=\dfrac{f^{(k)}(0)}{k!}$, for all k.

Remark 2: It is always the case that any power series centered at c will also converge at c, since: $p(c)=\sum_{k=0}^{\infty}a_k(c-c)^k = a_0+a_1(c-c)+a_2(c-c)^2+\ldots=a_0<\infty$

Thm (10.20, text): For any power series centered at c $p(x)=\sum_{k=0}^{\infty}a_k(x-c)^k$, there exists an R such that for all x: $|x-c|<R$, then $p(x)=\sum_{k=0}^{\infty}a_k(x-c)^k$ converges *absolutely*, and conversely for all x: $|x-c|>R$, $p(x)$ diverges.

Sketch of Proof: The proof of the above *existence theorem* is not supplied, and usually involves advanced methods in analysis. Nevertheless, a more heuristic approach is feasible using the Simple Comparison test (SCT[1]) and a Geometric

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[1] Recall pp. 7-8 in Feb. 21 posting http://www.glue.umd.edu/%7Ewkallfel/MA261-2/Feb21.pdf

Series: Let $N = \max_k \{|a_k|\}$ (i.e. N is the largest possible value of all the coefficients a_k of the power-series.[2] Then certainly, by the SCT:

$$p(x) = \sum_{k=0}^{\infty} a_k (x-c)^k \leq \sum_{k=0}^{\infty} N(x-c)^k = N\sum_{k=0}^{\infty}(x-c)^k$$

Now, let u = (x – c). The series on the right becomes: $N\sum_{k=0}^{\infty} u^k$ which is a geometric series, converging whenever: $|u| < 1 \Rightarrow |x-c| < 1$

Hence such an R always exists. Though obviously it's not *necessarily* equal to 1. We've just bound the power-series with respect to a simple geometric series, acting an upper bound, though more complicated cases can occur, regarding more precise comparisons to other kinds of geometric series with different radii of convergence, as we'll see in the next section below (the 'Geometric Series Trick').

Remark 3: The above number R is the series' *radius of convergence*. The condition where $|x-c| < R$ in which $p(x)$ absolutely converges is of course fixing the (open) interval of absolute convergence (centered at c, with radius R), since: $|x-c| < R \Rightarrow -R < (x-c) < R \Rightarrow c-R < x < c+R$ Sometimes it's convenient to denote this interval with notation: $I_c^R = (c-R, c+R) = \{x \mid c-R < x < x+R\}$

Establishing $I_c^R = (c-R, c+R) = \{x \mid c-R < x < x+R\}$ for any power-series usually involves the *Ratio Test* (RaT[3]) since many of the coefficients (as evidenced of course in the case of Taylor Series) involve factorials. In the case of no factorials however, one can in principle adopt the Root Test (RoT) as well.

Also, two extreme cases can exist: R = 0, i.e. where $p(x)$ *only* converges at the point $x = c$, i.e. $I_c^0 = \{c\}$ or $R = \infty$, i.e. $I_c^\infty = (-\infty, \infty)$. Here are a few illustrations:

- Example 1: Calculate the McClaurin coefficients for e^x, cos$x$, sin$x$, and examine the intervals of convergence for their associated McClaurin Series

k	$f^{(k)}(x) = \frac{d^k}{dx^k} e^x$	$a_k = \frac{1}{k!} f^{(k)}(0)$	$f^{(k)}(x) = \frac{d^k}{dx^k} \sin x$	$a_k = \frac{1}{k!} f^{(k)}(0)$	$f^{(k)}(x) = \frac{d^k}{dx^k} \cos x$	$a_k = \frac{1}{k!} f^{(k)}(0)$
0	e^x	1	sin$x$	0	cos$x$	1

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[2] Subtle problems of course arise when taking into consideration that this set can be infinite, i.e., maximizing over an open-ended set. We ignore these subtleties here in this sketch, whose overall framework is how the more rigorous proofs go.

[3] Recall Feb. 26 postings: http://www.glue.umd.edu/%7Ewkallfel/MA261-2/Feb26.pdf

1	e^x	1	cos$x$	1	-sin$x$	0
2	e^x	$\frac{1}{2!}$	-sin$x$	0	-cos$x$	-$\frac{1}{2!}$
3	e^x	$\frac{1}{3!}$	-cos$x$	-$\frac{1}{3!}$	sin$x$	0
4	e^x	$\frac{1}{4!}$	sin$x$	0	cos$x$	-$\frac{1}{4!}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
k	e^x	$\frac{1}{k!}$	$\pm\sin x, \pm\cos x$	$(-1)^j \frac{1}{j!}, j = 2k+1$	$\pm\sin x, \pm\cos x$	$(-1)^j \frac{1}{j!}, j = 2k$

Hence according to the above table, the McClaurin Polynomials for e^x, sin$x$, cos$x$ are:

$$e^x \approx P_n(x) = \sum_{k=0}^{n} \tfrac{1}{k!} x^k \quad , \quad \sin x \approx P_n(x) = \sum_{k=0}^{n} (-1)^k \tfrac{x^{2k+1}}{(2k+1)!}, \qquad \cos x \approx P_n(x) = \sum_{k=0}^{n} (-1)^k \tfrac{x^{2k}}{(2k)!}$$

We wish to find the above intervals of absolute convergence for the above three McClaurin Polynomials. In this case, according to Taylor's Theorem[4], in their appropriate intervals of convergence we may write:

$$e^x = \sum_{k=0}^{\infty} \tfrac{1}{k!} x^k \ , \qquad \sin x = \sum_{k=0}^{\infty} (-1)^k \tfrac{x^{2k+1}}{(2k+1)!} \ , \qquad \cos x = \sum_{k=0}^{\infty} (-1)^k \tfrac{x^{2k}}{(2k)!}$$

…In other words, the convergence to the McCluarin series establishes the *exact* equality between the function and the associated (infinite) McCluarin series in the appropriate intervals of convergence.

Using the Ratio Test in the case of e^x:

$$r(x) = \lim_{k\to\infty} \left| \frac{a_{k+1}(x)}{a_k(x)} \right| = \lim_{k\to\infty} \frac{|x|^{k+1}}{(k+1)!} \cdot \frac{k!}{|x|^k} = |x| \lim_{k\to\infty} \frac{k!}{(k+1)!} = |x| \lim_{k\to\infty} \left(\frac{1}{k+1} \right)$$
$$= |x| \cdot 0 \Rightarrow r(x) = 0\_for\_all\_x : -\infty < x < \infty$$

Obviously since 0 <1, according to the RaT, $R = \infty$, i.e. $I_c^\infty = (-\infty, \infty)$, hence we may write:

$$e^x = \sum_{k=0}^{\infty} \tfrac{1}{k!} x^k \quad \text{for } all\ x\text{: } -\infty < x < \infty$$

Using the Ratio Test in the case of sin$x$:

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[4] Recall page 4, Feb. 28 notes http://www.glue.umd.edu/%7Ewkallfel/MA261-2/Feb28.pdf

$$r(x)=\lim_{k\to\infty}\left|\frac{a_{k+1}(x)}{a_k(x)}\right|=\lim_{k\to\infty}\frac{|x|^{2k+3}}{(2k+3)!}\cdot\frac{(2k+1)!}{|x|^{2k+1}}=|x|\lim_{k\to\infty}\frac{(2k+1)!}{(2k+3)!}=|x|\lim_{k\to\infty}\left(\frac{1}{(2k+3)(2k+2)}\right)$$

$$=|x|\cdot 0\Rightarrow r(x)=0\_for\_all\_x:-\infty<x<\infty$$

Hence: $\sin x=\sum_{k=0}^{\infty}(-1)^k\frac{x^{2k+1}}{(2k+1)!}$ for *all x*: $-\infty<x<\infty$

-Using the Ratio Test in the case of cos*x*:

$$r(x)=\lim_{k\to\infty}\left|\frac{a_{k+1}(x)}{a_k(x)}\right|=\lim_{k\to\infty}\frac{|x|^{2k+2}}{(2k+2)!}\cdot\frac{(2k)!}{|x|^{2k}}=|x|\lim_{k\to\infty}\frac{(2k)!}{(2k+2)!}=|x|\lim_{k\to\infty}\left(\frac{1}{(2k+2)(2k+1)}\right)$$

$$=|x|\cdot 0\Rightarrow r(x)=0\_for\_all\_x:-\infty<x<\infty$$

Hence: $\cos x=\sum_{k=0}^{\infty}(-1)^k\frac{x^{2k}}{(2k)!}$ for *all x*: $-\infty<x<\infty$

In the generally non-trivial case of *finite R*, one must test separately for the case of convergence of *p*(*x*) at the endpoints: {c–*R*, c+*R*} for the interval of absolute converge $I_c^R=(c-R,c+R)=\{x\mid c-R<x<x+R\}$; i.e. examine the convergence of the series: $p(c-R)=\sum_{k=0}^{\infty}a_k((c-R)-c)^k=\sum_{k=0}^{\infty}(-1)^k a_k R^k$

$$p(c+R)=\sum_{k=0}^{\infty}a_k((c+R)-c)^k=\sum_{k=0}^{\infty}a_k R^k$$

…to see whether essentially three cases may occur: divergence, conditional convergence, or absolute convergence.

Thm (10.21, text): For any power series centered at *c*: $p(x)=\sum_{k=0}^{\infty}a_k(x-c)^k$, on its interval of absolute convergence: $I_c^R=(c-R,c+R)=\{x\mid c-R<x<x+R\}$, *p*(*x*) is differentiable (therefore continuous[5]) and integrable, with derivative and integral on $I_c^R=(c-R,c+R)=\{x\mid c-R<x<x+R\}$ expressed as:

$$p'(x)=\sum_{k=0}^{\infty}a_k(x-c)^k=\frac{d}{dx}\sum_{k=0}^{\infty}a_k(x-c)^k=\sum_{k=0}^{\infty}\frac{d}{dx}a_k(x-c)^k=\sum_{k=1}^{\infty}a_k k(x-c)^{k-1}$$

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[5] See pp. 1-2, Sept 6th notes CalcI: http://www.glue.umd.edu/%7Ewkallfel/MA261-2/Notes3.pdf

$$\therefore\ p^{(n)}(x)=\frac{d^n}{dx^n}\sum_{k=0}^{\infty}a_k(x-c)^k=\sum_{k=n}^{\infty}a_k k\cdot(k-1)\cdot\ldots\cdot(k-n+1)(x-c)^{k-n}=\sum_{k=n}^{\infty}\frac{k!}{(k-n)!}a_k(x-c)^{k-n}$$

$$\int p(x)dx=\int\ \sum_{k=0}^{\infty}a_k(x-c)^k dx=\sum_{k=0}^{\infty}a_k\int(x-c)^k dx=\sum_{k=0}^{\infty}\frac{a_k}{(k+1)}(x-c)^{k+1}+C$$

Remark 4: The coefficient $\frac{k!}{(k-n)!}=k\cdot(k-1)\cdot\ldots\cdot(k-n+1)$ for the n-th derivative of the power-series has a special significance in combinatory mathematics: It's known as a *permutation* of n from a set of k objects[6], denoted as: ${}_kP_n=\frac{k!}{(k-n)!}$, and its combinatory significance is to tell one how many different ways to *arrange* (i.e. how many *sequences* one can form) n objects chosen from a set of k objects. In the special case where $n = k$, then: ${}_kP_k=\frac{k!}{(k-k)!}=k!$, which tells us how many ways one can arrange k objects chosen from a set of k objects, or "how many ways to arrange k objects," for short.

On the other hand, if one were just interested in how many different ways to *select* (i.e. how many *sets* one can form) n objects chosen from a set of k objects, this is denoted by a *combination*[7]*:* ${}_kC_n=\binom{k}{n}=\frac{k!}{n!(k-n)!}$. Note the connection: ${}_kC_n=\binom{k}{n}=\frac{k!}{n!(k-n)!}=\frac{{}_kP_n}{n!}=\frac{{}_kP_n}{{}_nP_n}$. This should make intuitive sense. Since in this case one is only concerned with ways to *select irregardless of order,* to collapse all the cases concerning a selection of n objects, one can simply divide by the number of ways to arrange n objects from the previous formula (concerning the number of ways to *arrange* n objects chosen from a set of k objects).

- <u>Example (#15, §10.8)</u>

  $$p(x)=\sum_{k=1}^{\infty}(-1)^{k+1}\frac{x^k}{4^k}=\sum_{k=1}^{\infty}(-1)^{k+1}\left(\frac{x}{4}\right)^k$$

  Using the Ratio Test:

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[6] Obviously, $k \ge n \ge 0$.

[7] Note how this expression comes up in the case of a *Binomial Expansion*:

$$(x\pm y)^N=\sum_{j=0}^{N}(\pm1)^j\binom{N}{j}x^{N-j}y^j=\sum_{j=0}^{N}\frac{N!}{j!(N-j)!}x^j(\pm y)^{N-j}$$

This is due to the fact that when foiling, one is *counting* all the sets of powers of x and y such that they add up to N. For example, in the case of $N = 3$, there are three such sets when the power of $y = 1$: $\{xyx\}$, $\{x^2y\}$, $\{yx^2\}$. Note that if multiplication were *not* commutative (as in the case of some mathematical objects like matrices) one could *not* equate the three cases, i.e. one could *not* write: $3x^2y=x^2y+xyx+yx^2$, but must simple list them separately: $x^2y+xyx+yx^2$

$$r(x) = \lim_{k\to\infty} \frac{|a_{k+1}(x)|}{|a_k(x)|} = \lim_{k\to\infty} \frac{|x|^{k+1}}{4^{k+1}} \cdot \frac{4^k}{|x|^k} = \frac{|x|}{4} \lim_{k\to\infty} 1 = \tfrac{1}{4}|x|$$

(Note how *all* the terms explicitly involving k canceled. Hence we're left with the special case of a limit of a constant function $y = 1$, i.e. $\lim_{x\to\infty} 1$, which of course must equal 1.)

So according to the Ratio Test, absolute convergence occurs when:

$$r(x) < 1 \Rightarrow \tfrac{1}{4}|x| < 1 \Rightarrow |x| < 4 \Rightarrow -4 < x < 4 \Rightarrow I_0^4 = (-4,4) = \{x \mid -4 < x < 4\}$$

> **Remark 5:** Note that the above interval of convergence could have been obtained via the 'Geometric Series Trick' which will be examined in greater detail below. In any case: $\sum_{k=1}^{\infty} |(-1)^{k+1} \frac{x^k}{4^k}| = \sum_{k=1}^{\infty} \left|\frac{x}{4}\right|^k = \sum_{j=0}^{\infty} \left|\frac{x}{4}\right|^{j+1} = \tfrac{1}{4}|x| \sum_{j=0}^{\infty} \left|\left(\tfrac{x}{4}\right)^j\right|$, which of course converges whenever: $\left|\tfrac{x}{4}\right| = \tfrac{1}{4}|x| < 1 \Rightarrow |x| < 4$

To test for possible convergence at the endpoints:

$$p(-4) = \sum_{k=1}^{\infty} (-1)^{k+1} \left(\frac{-4}{4}\right)^k = \sum_{k=1}^{\infty} (-1)^{k+1} (-1)^k = \sum_{k=1}^{\infty} (-1)^{2k+1}$$

This series obviously is diverges absolutely, since:

$$\sum_{k=1}^{\infty} \left|(-1)^{2k+1}\right| = \sum_{k=1}^{\infty} 1 = 1 + 1 + 1 + \ldots$$

Note that it also fails the AST [8], as condition i.) is clearly violated :

i.) $\lim_{k\to\infty} |(-1)^{2k+1}| \neq 0$

Hence diverges at -4

Moreover: $p(4) = \sum_{k=1}^{\infty} (-1)^{k+1} \left(\frac{4}{4}\right)^k = \sum_{k=1}^{\infty} (-1)^{k+1}$, which diverges for the same reasons as in the case of the other endpoint.

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[8] See p7., http://www.glue.umd.edu/%7Ewkallfel/MA261-2/Feb26.pdf

Hence $p(x)=\sum_{k=1}^{\infty}(-1)^{k+1}\frac{x^k}{4^k}=\sum_{k=1}^{\infty}(-1)^{k+1}\left(\frac{x}{4}\right)^k$ coverges on (-4 , 4)

- <u>Example (#19, §10.8)</u> $p(x)=\sum_{k=1}^{\infty}(-1)^{k+1}\frac{(x-1)^{k+1}}{(k+1)}$

Using the Ratio Test[9]:

$$r(x)=\lim_{k\to\infty}\frac{|a_{k+1}(x)|}{|a_k(x)|}=\lim_{k\to\infty}\frac{|x-1|^{k+1}}{(k+2)}\cdot\frac{k+1}{|x-1|^k}=|x-1|\lim_{k\to\infty}\frac{k+1}{k+2}$$

$$=|x-1|\lim_{k\to\infty}\frac{1+\frac{1}{k}}{1+\frac{2}{k}}=|x-1|<1\Rightarrow -1<(x-1)<1\Rightarrow 0<x<2\Rightarrow I_1^1=(0,2)=\{x\,|\,0<x<2\}$$

To examine convergence at the endpoints:

$$p(0)=\sum_{k=1}^{\infty}(-1)^{k+1}\frac{(-1)^{k+1}}{(k+1)}=\sum_{k=1}^{\infty}\frac{(-1)^{2k+2}}{k+1}=\sum_{k=1}^{\infty}\frac{1}{k+1}$$

There are quite a few ways to show that this above series diverges. The simplest approach is the SCT: $k+k=2k\geq k+1\Rightarrow\frac{1}{2k}\leq\frac{1}{k+1}$ for all $k\geq 1$.

The series: $\sum_{k=1}^{\infty}\frac{1}{2k}=\frac{1}{2}\sum_{k=1}^{\infty}\frac{1}{k}$ is of course a divergent *p*-series (*p* = 1). Hence $\sum_{k=1}^{\infty}\frac{1}{k+1}$ diverges.

$p(2)=\sum_{k=1}^{\infty}(-1)^{k+1}\frac{(2-1)^{k+1}}{(k+1)}=\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k+1}$, is of course an alternating series which

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[9] Though more complicated in this case, since no factorial expressions were utilized, the Root Test would have worked as well:

$$\rho(x)=\lim_{k\to\infty}\sqrt[k]{|a_k(x)|}=\lim_{k\to\infty}\left(\frac{|x-1|^k}{k}\right)^{1/k}=|x-1|\lim_{k\to\infty}k^{-1/k}$$

Note for ease in computation that can shift the index by 1 since $k\to\infty$. To evaluate the last limit, examine its natural log:

$$\ln\left(\lim_{k\to\infty}k^{-1/k}\right)=-\lim_{k\to\infty}\tfrac{1}{k}\ln k\xrightarrow{LHR}\lim_{x\to\infty}\frac{1/x}{1}=0\Rightarrow\lim_{k\to\infty}k^{-1/k}=e^0=1$$

So: $\rho(x)<1\Rightarrow|x-1|<1$

diverges absolutely (as evidenced in the treatment of the above first case). However, because it's an alternating series, it may still conditionally converge. Adopting the AST:

i.) $\lim_{k\to\infty} a_k = \lim_{k\to\infty} \dfrac{1}{k+1} = 0$

ii.) $\dfrac{a_{k+1}}{a_k} = \dfrac{k+1}{k+2} = \dfrac{1+\frac{1}{k}}{1+\frac{2}{k}} \le 1 \Rightarrow a_{k+1} \le a_k$ for all $k \ge 1$.

Hence $p(2) = \sum_{k=1}^{\infty} \dfrac{(-1)^{k+1}}{k+1}$ conditionally converges.

Hence $p(x) = \sum_{k=1}^{\infty} (-1)^{k+1} \dfrac{(x-1)^{k+1}}{(k+1)}$ converges on (0 , 2] = $\{x \mid 0 < x \le 2\}$

- Example (#33, §10.8)

$$f(x) = \sum_{n=0}^{\infty} (-1)^{n+1} \frac{(x-1)^{n+1}}{n+1}$$

Using the RaT:

$$r(x) = \lim_{k\to\infty} \frac{|a_{k+1}(x)|}{|a_k(x)|} = \lim_{k\to\infty} \frac{|x-1|^{k+1}}{(k+2)} \cdot \frac{k+1}{|x-1|^k} = |x-1| \lim_{k\to\infty} \frac{k+1}{k+2}$$

$$= |x-1| \lim_{k\to\infty} \frac{1+\frac{1}{k}}{1+\frac{2}{k}} = |x-1| < 1 \Rightarrow -1 < (x-1) < 1 \Rightarrow 0 < x < 2 \Rightarrow I_1^1 = (0,2) = \{x \mid 0 < x < 2\}$$

As in the above example, likewise here the interval of convergence of $f(x)$ is (0 , 2].

Hence according to Thm10.21 we may write on the interval of *absolute* converge (0, 2):

$$f'(x) = \sum_{n=0}^{\infty} (-1)^{n+1} \frac{d}{dx} \frac{(x-1)^{n+1}}{n+1} = \sum_{n=0}^{\infty} (-1)^{n+1} (x-1)^n$$

(Note carefully that in this case the derivate still starts at the $n = 0$ term! Why?)

$$f''(x) = \sum_{n=0}^{\infty} (-1)^{n+1} \frac{d}{dx} (x-1) = \sum_{n=1}^{\infty} (-1)^{n+1} n (x-1)^{n-1}$$

$$\int f(x)dx = \sum_{n=0}^{\infty}(-1)^{n+1}\tfrac{1}{n+1}\int(x-1)^{n+1}dx = \sum_{n=0}^{\infty}\tfrac{1}{(n+1)(n+2)}(x-1)^{n+2}$$

Recalling the remark on p. 623 of the text, we know if we can establish the convergence at any endpoint(s) of $f''(x)$, this automatically guarantees convergence for the same endpoint(s) for $f'(x), f(x), \int f(x)dx$, etc.

So start with $f''(x) = \sum_{n=1}^{\infty}(-1)^{n+1}n(x-1)^{n-1}$ first:

$$f''(0) = \sum_{n=1}^{\infty}(-1)^{n+1}n(0-1)^{n-1} = \sum_{n=1}^{\infty}(-1)^{2n}n = \sum_{n=1}^{\infty}n \qquad \text{obviously diverges}$$

$$f''(2) = \sum_{n=1}^{\infty}(-1)^{n+1}n(2-1)^{n-1} = \sum_{n=1}^{\infty}(-1)^{n}n$$

This is an alternating series that clearly diverges absolutely. It also clearly fails the AST (why?).

Hence: $f''(x) = \sum_{n=1}^{\infty}(-1)^{n+1}n(x-1)^{n-1}$ on (-2, 2)

Since no convergence was established for the endpoints for $f''(x)$, nothing conclusive can be said about the possible convergence at the endpoints for $f'(x)$, so we must test separately:

$$f'(0) = \sum_{n=0}^{\infty}(-1)^{n+1}(0-1)^{n} = \sum_{n=0}^{\infty}(-1)^{2n+1} = -\sum_{n=0}^{\infty}1 \qquad \text{clearly diverges.}$$

$f'(2) = \sum_{n=0}^{\infty}(-1)^{n+1}(2-1)^{n} = -\sum_{n=0}^{\infty}(-1)^{n}$ …an alternating series that clearly diverges absolutely. It also clearly fails the AST (why?).

Hence: $f'(x) = \sum_{n=0}^{\infty}(-1)^{n+1}(x-1)^{n}$ on (-2, 2)

Without further ado, since *f*(*x*) converges at the endpoint 2, so does its integral (recalling again the Remark on p. 623) So what remains is to test for convergence for the integral of the power series at 0:

$$\int f(x)dx \;\Big|_{x=0} = \sum_{n=0}^{\infty} \tfrac{1}{(n+1)(n+2)} (-1)^{n+2} = \sum_{n=0}^{\infty} (-1)^{n} \tfrac{1}{(n+1)(n+2)}$$

This is an example of an alternating series that *absolutely* converges. To establish its absolute convergence, consider for instance applying the SCT:

$$(n+1)(n+2) = n^2 + 3n + 2 > n^2 \Rightarrow \frac{1}{(n+1)(n+2)} < \frac{1}{n^2}$$

But $\sum_{k=1}^{\infty} \frac{1}{n^2}$ is a convergent p-series ($p = 2$). Hence $\sum_{n=0}^{\infty} \tfrac{1}{(n+1)(n+2)} = \tfrac{1}{2} + \sum_{n=1}^{\infty} \tfrac{1}{(n+1)(n+2)}$ converges.

Hence:

$$\int f(x)dx = \sum_{n=0}^{\infty} (-1)^{n+1} \tfrac{1}{n+1} \int (x-1)^{n+1} dx = \sum_{n=0}^{\infty} \tfrac{1}{(n+1)(n+2)} (x-1)^{n+2} \quad \text{converges on } [0\,,\,2]$$

- *THE GEOMETRIC SERIES 'TRICK'*

The underlying idea here is simple, but very powerful. Recall that any geometric series: $\sum_{k=0}^{\infty} r^k = \sum_{k=1}^{\infty} r^{k-1} = \frac{1}{1-r}$ for $|r| < 1 \Rightarrow -1 < r < 1$

Hence for *any* rational function $f(x) = \frac{p(x)}{q(x)}$, one can easily determine its power series representation, centered at *any* point *c*, by simple algebraic transformations via the above representation.

- Example 1: Consider the function $f(x) = \frac{a}{b+kx}$, where *a*, *b*, *k* are positive constants.
  - i.) Find using the above the power series representation centered at 0
  - ii.) Find using the above the power series representation centered at any *c* >0

i.) $f(x) = \frac{a}{b+kx} = \frac{a}{b}\left(\frac{1}{1+\frac{k}{b}x}\right) = \frac{a}{b}\left(\frac{1}{1-\left(-\frac{k}{b}x\right)}\right) = \frac{a}{b}\sum_{j=0}^{\infty}\left(-\frac{k}{b}x\right)^j = \frac{a}{b}\sum_{j=0}^{\infty}(-1)^j \frac{k^j}{b^j}x^j$

The above absolutely converges for all *x* whenever:
$\left|\frac{k}{b}x\right| < 1 \Rightarrow -1 < \frac{k}{b}x < 1 \Rightarrow -\frac{b}{k} < x < \frac{b}{k} \Rightarrow I_0^{b/k} = \left(-\frac{b}{k}, \frac{b}{k}\right) = \left\{x \mid -\frac{b}{k} < x < \frac{b}{k}\right\}$

It's also easily shown that the series diverges at the endpoints.

$$f(x)=\frac{a}{b+kx}=\frac{a}{b+k(x-c)+kc}=\frac{a}{(b+kc)+k(x-c)}=\frac{a}{(b+kc)\left[1+\frac{k}{(b+kc)}(x-c)\right]}$$

$$=\tfrac{a}{(b+kc)}\left(\frac{1}{1-\left(-\frac{k}{(b+kc)}(x-c)\right)}\right)=\tfrac{a}{b+kc}\sum_{j=0}^{\infty}(-1)^j\frac{k^j}{(b+kc)^j}(x-c)^j$$

This has an interval of convergence for all *x* such that:

$$\left|\tfrac{k}{b+kc}(x-c)\right|<1\Rightarrow -1<\tfrac{k}{b+kc}(x-c)<1\Rightarrow -\tfrac{b+kc}{k}<x-c<\tfrac{b+kc}{k}\Rightarrow -\left(c+\tfrac{b}{k}\right)<x-c<\left(c+\tfrac{b}{k}\right)$$

$$\Rightarrow -\tfrac{b}{k}<x<2c+\tfrac{b}{k}\Rightarrow I_c^{c+\frac{b}{k}}=\left(-\tfrac{b}{k},2c+\tfrac{b}{k}\right)$$

> **Remark 6:** Note that the how the radius of convergence changed from part i.)! To determine R rigorously in ii.), observe that the total *length* of the interval $\left(-\frac{b}{k},2c+\frac{b}{k}\right)$ is: $\left|\left(2c+\frac{b}{k}\right)-\left(-\frac{b}{k}\right)\right|=\left|2c+2\frac{b}{k}\right|=2\left|c+\frac{b}{k}\right|$. So to determine the radius, simply the half-length of the interval: $R=\left(c+\frac{b}{k}\right)$. (Absolute values dropped because all the constants *a*, *b*, *c*, *k* are positive.

For yet more complicated cases, use the method of partial fractions to simplify:

- <u>Example 2 (# 12, § 10.9)</u> $f(x)=\dfrac{4x-7}{2x^2+3x-2}=\dfrac{4x-7}{(2x-1)(x+2)}$ for $c = 0$

$$f(x)=\frac{4x-7}{(2x-1)(x+2)}=\frac{A_1}{(2x-1)}+\frac{A_2}{(x+2)}\Rightarrow 4x-7=A_1(x+2)+A_2(2x-1)$$

$$x=-2\Rightarrow -15=-5A_2\rightarrow A_2=3$$

$$x=\tfrac{1}{2}\Rightarrow 1=\tfrac{3}{2}A_1\rightarrow A_1=\tfrac{2}{3}$$

$$\therefore f(x)=\tfrac{2}{3}\left(\frac{1}{2x-1}\right)+3\left(\frac{1}{x+2}\right)=-\tfrac{2}{3}\left(\frac{1}{1-2x}\right)+\tfrac{3}{2}\left(\frac{1}{1-\left(-\frac{1}{2}x\right)}\right)=-\tfrac{2}{3}\sum_{k=0}^{\infty}(2x)^k+\tfrac{3}{2}\sum_{k=0}^{\infty}(-1)^k\left(\tfrac{x}{2}\right)^k$$

The first series has an interval of absolute convergence: $|2x|<1\Rightarrow I_0^{1/2}=\left(-\frac{1}{2},\frac{1}{2}\right)$. The second series has an interval of absolute convergence of: $\left|\frac{1}{2}x\right|<1\Rightarrow I_0^2=(-2,2)$. So both combined have an interval of absolute convergence of: $I_0^{1/2}\cap I_0^2=\left(-\frac{1}{2},\frac{1}{2}\right)\cap(2,2)=\left(-\frac{1}{2},\frac{1}{2}\right)=I_0^{1/2}$. No need to check for convergence at the endpoints, since $I_0^{1/2}$ is for the series which does not alternate;

One can express the above as:

$$\therefore f(x) = -\tfrac{2}{3}\sum_{k=0}^{\infty}(2x)^k + \tfrac{3}{2}\sum_{k=0}^{\infty}(-1)^k\left(\tfrac{x}{2}\right)^k = \sum_{k=0}^{\infty}\left[-\tfrac{2}{3}\cdot 2^k + (-1)^k\cdot\tfrac{3}{2}\cdot\tfrac{1}{2^k}\right]x^k = \sum_{k=0}^{\infty}\left[-\tfrac{1}{3}\cdot 2^k + (-1)^k\cdot\frac{3}{2^{k+1}}\right]x^k$$

- Example 3: Express ln$x$ as a power-series, determine its interval of convergence

Using Thm10.21: $\dfrac{d}{dx}\ln x = \dfrac{1}{x} = \dfrac{1}{1+x-1} = \dfrac{1}{1-(-(x-1))} = \displaystyle\sum_{k=0}^{\infty}(-1)^k(x-1)^k$

This has an interval of absolute convergence for all x whenever:
$|x-1| < 1 \Rightarrow -1 < (x-1) < 1 \Rightarrow 0 < x < 2 \Rightarrow I_1^1 = (0,1) = \{x \mid 0 < x < 2\}$

Hence on $I_1^1 = (0,1) = \{x \mid 0 < x < 2\}$, according to Thm10.21

$$\ln x = \int \frac{1}{x}\,dx = \sum_{k=0}^{\infty}(-1)^k \int (x-1)^k\,dx = \sum_{k=0}^{\infty}(-1)^k\frac{(x-1)^{k+1}}{k+1}$$

At the endpoint 0: $\displaystyle\sum_{k=0}^{\infty}(-1)^k(-1)^{k+1}\frac{1}{k+1} = -\sum_{k=0}^{\infty}(-1)^{2k}\tfrac{1}{k+1} = -\sum_{k=0}^{\infty}\frac{1}{k+1}$, clearly a divergent series, as shown in previous examples above (use SCT to compare with the divergent p-series: $\tfrac{1}{2}\displaystyle\sum_{k=1}^{\infty}\frac{1}{k}$)

At the endpoint 2: $\displaystyle\sum_{k=0}^{\infty}(-1)^k(1)^{k+1}\frac{1}{k+1} = \sum_{k=0}^{\infty}(-1)^k\tfrac{1}{k+1} = \sum_{k=0}^{\infty}\frac{(-1)^k}{k+1}$, which is an alternating series absolutely diverging nevertheless conditionally converging since it passes the AST

Hence: $\ln x = \displaystyle\int \frac{1}{x}\,dx = \sum_{k=0}^{\infty}(-1)^k\int(x-1)^k\,dx = \sum_{k=0}^{\infty}(-1)^k\frac{(x-1)^{k+1}}{k+1}$ on (0 , 2]

Remark 7: From Example 1 above, part ii.), note how the above example for ln$x$ could be generalized for any nonzero c. Try it!

- Example 4 Find the power series representation and its interval of convergence for arctan$x$

Adopting the same strategy as in the above example 3:

$$\tfrac{d}{dx}\arctan x = \frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = \sum_{k=0}^{\infty}(-1)^k(x^2)^k = \sum_{k=0}^{\infty}(-1)x^{2k}$$

..whose interval of absolute convergence is, for all x:
$\left|x^2\right| < 1 \Rightarrow -1 < x < 1 \Rightarrow I_0^1 = (-1,1) = \{x \mid -1 < x < 1\}$

Hence, on $I_0^1 = (-1,1) = \{x \mid -1 < x < 1\}$, according to Thm 10.21:

$$\arctan x = \int \frac{dx}{1+x^2} = \sum_{k=0}^{\infty} (-1)^k \int x^{2k}\,dx = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{2k+1}$$

Testing convergence at the endspoints:

x = -1 : $\displaystyle\sum_{k=0}^{\infty} (-1)^k \frac{(-1)^{2k}(-1)}{2k+1} = -\sum_{k=1}^{\infty} (-1)^k \frac{1}{2k+1}$, an alternating series which diverges absolutely but conditionally converges, since it passes the AST (why?).

x = 1 : $\displaystyle\sum_{k=0}^{\infty} (-1)^k \frac{(1)^{2k+1}}{2k+1} = \sum_{k=1}^{\infty} (-1)^k \frac{1}{2k+1}$, an alternating series which diverges absolutely but conditionally converges, since it passes the AST, for the same reason as in above.

Hence: $\displaystyle\arctan x = \int \frac{dx}{1+x^2} = \sum_{k=0}^{\infty} (-1)^k \int x^{2k}\,dx = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{2k+1}$ on [-1, 1]

Thus thanks to the above methods herein, we now know automatically the following McClaurin Series for these special functions:

$$e^x = \sum_{k=0}^{\infty} \tfrac{1}{k!} x^k \ , \qquad \sin x = \sum_{k=0}^{\infty} (-1)^k \tfrac{x^{2k+1}}{(2k+1)!} \ , \qquad \cos x = \sum_{k=0}^{\infty} (-1)^k \tfrac{x^{2k}}{(2k)!}$$

..which all converge on (-∞ , ∞)

…And $\displaystyle\arctan x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{2k+1}$ converging on [-1 , 1]

…And the Taylor Series (centered at c =1) for

$\displaystyle\ln x = \sum_{k=0}^{\infty} (-1)^k \frac{(x-1)^{k+1}}{k+1}$ converging on (0 , 2]

In addition more general Series representations can be made for rational functions as well, using the Geometric Series Trick as well as Thm 10.21 as outlined in the examples above