MA262 Calculus II: Notes for March 25
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- *REVIEW PROBLEMS-EXAM II*

I. Determine the convergence/divergence of:

a.) $\sum_{k=1}^{\infty}\frac{1}{\sqrt{n^2+2n}}$

Method 1: Observe $n^2+2n\approx n^2$, when $n>>1$. The series $\sum_{k=1}^{\infty}\frac{1}{n}$ is a divergent p-series (a Harmonic series, in which $p = 1$). Hence according to **LCT:**

$$\lim_{n\to\infty}\frac{b_n}{a_n}=\lim_{n\to\infty}\frac{1/n}{1/\sqrt{n^2+2n}}=\lim_{n\to\infty}\frac{\sqrt{n^2+2n}}{n}=\lim_{n\to\infty}\sqrt{1+\tfrac{2}{n}}=1$$

Certainly[1] $0<\lim_{n\to\infty}\frac{b_n}{a_n}<\infty$, hence $\sum_{k=1}^{\infty}\frac{1}{\sqrt{n^2+2n}}$ diverges.

Method 2 (not recommended) Use the integral test:

$$\sum_{k=1}^{\infty}\frac{1}{\sqrt{n^2+2n}}\Leftrightarrow\int_1^{\infty}\frac{dx}{\sqrt{x^2+2x}}=\lim_{b\to\infty}\int_1^{b}\frac{dx}{\sqrt{x^2+2x+1-1}}=\lim_{b\to\infty}\int_1^{b}\frac{dx}{\sqrt{(x+1)^2-1}}$$

$$(x+1)=\sec\theta, dx=\sec\theta\tan\theta d\theta\Rightarrow\lim_{b\to\infty}\int_{arc\sec 2}^{arcc\sec(b+1)}\frac{\sec\theta\tan\theta}{\sqrt{\sec^2\theta-1}}d\theta$$

$$=\lim_{\beta\to\frac{\pi}{2}}\int_{arc\sec 2=\arccos(1/2)=\pi/3}^{\beta}\frac{\sec\theta\tan\theta}{\tan\theta}d\theta=\lim_{\beta\to\frac{\pi}{2}}\int_{\frac{\pi}{3}}^{\beta}\sec\theta d\theta$$

$$=\lim_{\beta\to\frac{\pi}{2}}\ln\left|\sec\theta+\tan\theta\right|\Big|_{\pi/3}^{\beta}=\lim_{\beta\to\frac{\pi}{2}}\ln\left|\sec\beta+\tan\beta\right|-\ln\left|\sec\tfrac{\pi}{3}+\tan\tfrac{\pi}{3}\right|$$

$$=\infty-\ln\left|2+\sqrt{3}\right|=\infty\Rightarrow \textit{Diverges}$$

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[1] With no loss of generality, then obviously the same holds true for $\lim_{n\to\infty}\frac{a_n}{b_n}$, since obviously the reciprocal of a finite, positive quantity is likewise positive. Observe also, as mentioned in class, that the **SCT** won't be of use here, if one adopts the same series to compare $\sum a_n$ with. Since: $\frac{1}{\sqrt{n^2+2n}}\le\frac{1}{n}\Rightarrow\sum_{n=1}^{\infty}\frac{1}{\sqrt{n^2+2n}}\le\infty$ doesn't say definitely whether or not the series $\sum a_n=\sum\frac{1}{\sqrt{n^2+2n}}$ converges or diverges. Nevertheless the exercise is not lost, insofar as whatever series you sought a simple comparison with can be used as the series to run the **LCT** on.

b.) $\sum_{k=1}^{\infty}(-1)^k \frac{\sqrt{k}}{k+1}$

This is an alternating series. Checking for absolute convergence first, note that: $\lim_{k\to\infty}\frac{\sqrt{k}}{k+1} = \lim_{k\to\infty}\frac{k^{-1/2}}{1+k^{-1}} = 0$, which means that one ought to try some other convergence test (divergence test is inconclusive…i.e. just because $a_n \to N > 0$ entails that $\sum a_n$ diverges, *it does not follow that* $a_n \to 0$ entails that $\sum a_n$ converges.)

Observe that for $k >> 1$: $\frac{\sqrt{k}}{k+1} \approx \frac{\sqrt{k}}{k} = \frac{1}{k^{1/2}}$. The term on the right is of a divergent *p*-series ($p = {}^1/_2 \le 1 \Rightarrow$ diverges). However adopting the **SCT** won't be of use, since $k+1 > k \Rightarrow \frac{\sqrt{k}}{k+1} < \frac{\sqrt{k}}{k} = \frac{1}{k^{1/2}}$. Note, however, when using the **LCT:**

$$\lim_{k\to\infty}\frac{b_n}{a_n} = \lim_{k\to\infty}\frac{1}{\sqrt{k}}\cdot\frac{k+1}{\sqrt{k}} = \lim_{k\to\infty}\frac{k+1}{k} = \lim_{k\to\infty}\left(1+\tfrac{1}{k}\right) = 1$$

$$\therefore 0 < \lim_{k\to\infty}\frac{b_n}{a_n} < \infty \Rightarrow \sum a_n Diverges$$

Hence the series diverges in absolute value, though it may still conditionally converge. Adopting the **AST:**

i.) Monotonicty:

$$\frac{a_{k+1}}{a_k} = \frac{\sqrt{k+1}}{k+2}\cdot\frac{k+1}{\sqrt{k}} = \frac{k+1}{k+2}\sqrt{\frac{k+1}{k}} = \frac{k+1}{k+2}\sqrt{1+\tfrac{1}{k}} = \frac{\sqrt{1+\frac{1}{k}}}{\frac{k+1+1}{k+1}}$$

$$= \frac{\sqrt{1+\frac{1}{k}}}{1+\frac{1}{k+1}} \le 1, for\_all\_k$$

Note that the monotone decreasing property can be more rigorously established via:

$$a_k = f(k) = \frac{\sqrt{k}}{k+1} \Leftrightarrow f(x) = \frac{\sqrt{x}}{x+1} = \frac{1}{x^{1/2}+x^{-1/2}}$$

$$\Rightarrow f'(x) = \frac{d}{dx}\left(x^{1/2}+x^{-1/2}\right)^{-1} = -\frac{\left(\frac{1}{2}x^{-1/2}-\frac{1}{2}x^{-3/2}\right)}{\left(x^{1/2}+x^{-1/2}\right)^2} = -\frac{(x-1)}{2x^{3/2}\left(x^{1/2}+x^{-1/2}\right)^2} < 0$$

$$for\_all\_x \ge 1$$

ii.) $a_n \xrightarrow{?} 0$

$$\lim_{k\to\infty} \frac{\sqrt{k}}{k+1} = \lim_{k\to\infty} \frac{k^{-1/2}}{1+k^{-1}} = \frac{0}{1} = 0$$

Hence the series conditionally converges.

II. Determine the interval of convergence for:

a.) $\sum_{k=1}^{\infty} \frac{\ln k}{k}(x+1)^k$

<u>Method 1:</u> Using **RaT:**

$$\rho(x) = \lim_{k\to\infty} \frac{|a_{k+1}(x)|}{|a_k(x)|} < 1 \Rightarrow \lim_{k\to\infty} \frac{\ln(k+1)}{k+1} \cdot \frac{k}{\ln k} \cdot \frac{|x+1|^{k+1}}{|x+1|^k} = |x+1| \lim_{k\to\infty} \frac{k\ln(k+1)}{(k+1)\ln k}$$

$$= |x+1| \lim_{k\to\infty} \frac{1}{1+\frac{1}{k}} \cdot \frac{\ln(k+1)}{\ln k} = |x+1| \lim_{k\to\infty} \frac{\ln(k+1)}{(1+k^{-1})\ln k}$$

$$\xrightarrow{LHR} |x+1| \lim_{k\to\infty} \frac{\frac{1}{k+1}}{(1+k^{-1})\frac{1}{k} - k^{-2}\ln k} = |x+1| \lim_{k\to\infty} \frac{1}{(k+1)\left[\frac{k+1}{k^2} - \frac{\ln k}{k^2}\right]}$$

$$= |x+1| \lim_{k\to\infty} \frac{k^2}{(k+1)^2 - (k+1)\ln k} \xrightarrow{LHR} |x+1| \lim_{k\to\infty} \frac{2k}{2(k+1) - (k+1)\frac{1}{k} - \ln k}$$

$$= |x+1| \lim_{k\to\infty} \frac{2k}{2k - 1 - \frac{1}{k} - \ln k} \xrightarrow{LHR} |x+1| \lim_{k\to\infty} \frac{2}{2 + \frac{1}{k^2} - \frac{1}{k^2}} = |x+1| \cdot 1 < 1$$

$$\Rightarrow |x+1| < 1 \to -1 < x+1 < 1 \to -2 < x < 0, \therefore I_{-1}^{1} = (-2,0) = \{x \mid -2 < x < 0\}$$

<u>Method 2:</u> Using **RoT:**

$$r(x) = \lim_{k\to\infty} \sqrt[k]{|a_k|} < 1 \Rightarrow \lim_{k\to\infty} \left(\frac{\ln k}{k}|x+1|^k\right)^{1/k} = |x+1| \lim_{k\to\infty} \left(\frac{\ln k}{k}\right)^{1/k}$$

Taking the natural logarithm of the last limit:

$$\lim_{k\to\infty} \ln\left(\left(\frac{\ln k}{k}\right)^{1/k}\right) = \lim_{k\to\infty} \frac{\ln\left(\frac{\ln k}{k}\right)}{k} = \lim_{k\to\infty} \frac{\ln(\ln k) - \ln k}{k}$$

$$\xrightarrow{LHR} \lim_{k\to\infty} \frac{\frac{1}{\ln k} \cdot \frac{1}{k} - \frac{1}{k}}{1} = \lim_{k\to\infty} \frac{\frac{1}{\ln k} - 1}{k} \xrightarrow{LHR} \lim_{k\to\infty} \frac{-\frac{1}{(\ln k)^2} \cdot \frac{1}{k}}{1} = 0$$

$$\Rightarrow \lim_{k\to\infty} \left(\frac{\ln k}{k}\right)^{1/k} = e^0 = 1 \Rightarrow |x+1| < 1 \Rightarrow -2 < x < 0 \Rightarrow I_{-1}^{1} = (-2,0)$$

b.) $\sum_{k=1}^{\infty}(-1)^k \frac{(x-2)^k}{k^k}$

Using **RoT:**

$$r(x) = \lim_{k\to\infty} \sqrt[k]{|a_k(x)|} < 1 \Rightarrow \lim_{k\to\infty} \frac{\sqrt[k]{|x-2|^k}}{\sqrt[k]{k^k}} = |x-2| \lim_{k\to\infty} \frac{1}{k}$$

$$= |x-2| \cdot 0 \Rightarrow I = (-\infty, \infty)$$

III.) Given $\sum_{k=1}^{\infty} \frac{k}{1-k^2}$

a.) Use the integral test to examine convergence/divergence:

Method 1 (using a simple *u*-substitution)

$$\sum_{k=1}^{\infty} \frac{k}{1-k^2} \Leftrightarrow \int_1^{\infty} \frac{xdx}{1-x^2} = \lim_{b\to\infty} \int_1^b \frac{xdx}{1-x^2} \Rightarrow u = 1-x^2, du = -2xdx \Rightarrow xdx = -\tfrac{1}{2}du$$

$$\Rightarrow -\tfrac{1}{2}\lim_{b\to\infty} \int_{u(1)}^{u(b)} \frac{du}{u} = -\tfrac{1}{2}\lim_{b\to\infty} \int_0^{1-b^2} \frac{du}{u} = -\tfrac{1}{2}\lim_{b\to\infty} \ln u \Big|_0^{1-b^2} = -\tfrac{1}{2}\lim_{b\to\infty} \ln\left|1-b^2\right| + \ln 0$$

$\Rightarrow$ *Diverges*

Method 2 (using a trig substitution-not recommended)

$$\sum_{k=1}^{\infty} \frac{k}{1-k^2} \Leftrightarrow \int_1^{\infty} \frac{xdx}{1-x^2} = \lim_{b\to\infty} \int_1^b \frac{xdx}{1-x^2} \Rightarrow x = \sin\theta, dx = \cos\theta d\theta$$

$$= \lim_{b\to\infty} \int_{\arcsin(1)=\pi/2}^{\arcsin b} \frac{\sin\theta\cos\theta}{1-\sin^2\theta} d\theta = \lim_{b\to\infty} \int_{\pi/2}^{\beta=\arcsin b} \tan\theta d\theta = \lim_{b\to\infty} \ln|\cos\theta| \Big|_{\pi/2}^{\beta}$$

$$= \lim_{b\to\infty} \ln|\cos(\arcsin b)| - \ln\left|\cos\tfrac{\pi}{2}\right| \Rightarrow \textit{diverges}$$

b.) Use the **SCT** or **LCT** to confirm your answer

Method 1 (**LCT)**

As $k >> 1$ $a_k = \frac{k}{1-k^2} \approx -\frac{1}{k} = b_k$, and $\sum_{k=1}^{\infty} \frac{1}{k} = \sum b_k$ is a diverging *p*-series ($p = 1$)

Using the **LCT:**

$$\lim_{k\to\infty}\frac{b_n}{a_n}=\lim_{k\to\infty}\frac{-\frac{1}{k}}{\frac{k}{1-k^2}}=\lim_{k\to\infty}\frac{k^2-1}{k^2}=\lim_{k\to\infty}\left(1-\tfrac{1}{k^2}\right)=1$$

Hence: $0<\lim_{k\to\infty}\frac{b_n}{a_n}<\infty \Rightarrow \sum a_n$ diverges

<u>Method 2</u> (**SCT**)

A little trickier, but notice: $1-k<k$, for all $k\geq 1$

Hence: $\frac{1-k}{1-k^2}<\frac{k}{1-k^2}$ for all $k\geq 1$

The term on the left can be simplified:

$\frac{1-k}{1-k^2}=\frac{1-k}{(1-k)(1+k)}=\frac{1}{k+1}$ which via an index shifting trick can be shown to be a diverging p –series: $\sum_{k=1}^{\infty}\frac{1}{k+1}=\sum_{j=2}^{\infty}\frac{1}{j}$

Hence, for all $k\geq 1$:

$$\sum_{k=1}^{\infty}\frac{1}{k+1}<\sum_{k=1}^{\infty}\frac{k}{1-k^2}\Rightarrow \sum a_k=\sum_{k=1}^{\infty}\frac{k}{1-k^2} \text{ diverges}$$

c.) Confirm your answer using partial fractions:

$$\frac{k}{1-k^2}=\frac{k}{(1-k)(1+k)}=\frac{A_1}{(1-k)}+\frac{A_2}{(1+k)}\Rightarrow k=(k+1)A_1+(1-k)A_2$$

$$\Rightarrow k=1\to A_1=\tfrac{1}{2}, k=-1\to A_2=-\tfrac{1}{2}$$

$$\therefore \sum_{k=1}^{\infty}\frac{k}{1-k^2}=\tfrac{1}{2}\sum_{k=1}^{\infty}\left(\frac{1}{1-k}-\frac{1}{1+k}\right)$$

The second series diverges (as was shown in b.) and the first diverges as well (easily demonstrated via index-shifting, or **SCT**, or integral test)

IV.) Find a power series representation, and the interval of convergence for:

a.) $f(x)=\int_0^x \frac{\sin t}{t}\,dt$

The McClaurin Series: $\sin t=\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)!}t^{2k+1}$

Hence:

$$f(x) = \int_0^x \frac{\sin t}{t}\,dt = \int_0^x \frac{1}{t}\sum_{k=0}^{\infty} \tfrac{(-1)^k}{(2k+1)!} t^{2k+1}\,dt = \sum_{k=0}^{\infty} \tfrac{(-1)^k}{(2k+1)!}\int_0^x t^{2k}\,dt = \sum_{k=0}^{\infty} \tfrac{(-1)^k}{(2k+1)!}\frac{t^{2k+1}}{k+1}\Bigg|_0^x$$

$$= \sum_{k=0}^{\infty} \tfrac{(-1)^k}{(2k+1)!(k+1)} x^{2k+1}$$

To find the interval of convergence[2]: (Using **RaT**)

$$\rho(x) = \lim_{k\to\infty} \frac{|a_{k+1}(x)|}{|a_k(x)|} < 1 \Rightarrow \lim_{k\to\infty} \frac{(2k+1)!(k+1)}{(2k+3)!(k+2)} \cdot \frac{|x|^{2k+3}}{|x|^{2k+1}} = |x|^2 \lim_{k\to\infty} \frac{(k+1)}{(2k+3)(2k+2)(k+2)}$$

$$= |x|^2 \cdot 0 < 1 \Rightarrow I = (-\infty, \infty)$$

b.) $g(x) = \dfrac{2x}{3-x^2}$

<u>Method 1:</u> (Using **GST**)

$$g(x) = \frac{2x}{3-x^2} = \tfrac{2}{3}x\left(\frac{1}{1-\frac{x^2}{3}}\right) = \tfrac{2}{3}x\left(\frac{1}{1-\left(\frac{x}{\sqrt{3}}\right)^2}\right) = \tfrac{2}{3}x\sum_{k=0}^{\infty}\left(\tfrac{x}{\sqrt{3}}\right)^{2k} = \tfrac{2}{3}\sum_{k=0}^{\infty} 3^{-k} x^{2k+1}$$

To establish interval of convergence, note that: $g(x) = \tfrac{2}{3}x\left(\dfrac{1}{1-\left(\frac{x}{\sqrt{3}}\right)^2}\right) = h(x)l(x)$

- The interval of convergence for $h(x) = \tfrac{2}{3}x$ is $(-\infty, \infty)$
- The interval of convergence for $l(x) = \dfrac{1}{1-\left(\frac{x}{\sqrt{3}}\right)^2}$ is:

$$-1 < \left(\frac{x}{\sqrt{3}}\right)^2 < 1 \Rightarrow I_0^{\sqrt{3}} = \left(-\sqrt{3}, \sqrt{3}\right)$$

- Hence the interval is: $(-\infty, \infty) \cap \left(-\sqrt{3}, \sqrt{3}\right) = \left(-\sqrt{3}, \sqrt{3}\right)$

<u>Method 2</u>: Using partial fractions and **GST**

$$g(x) = \frac{2x}{3-x^2} = \frac{2x}{\left(\sqrt{3}-x\right)\left(\sqrt{3}+x\right)} = \frac{A_1}{\left(\sqrt{3}-x\right)} + \frac{A_2}{\left(\sqrt{3}+x\right)} \Rightarrow 2x = A_1\left(\sqrt{3}+x\right) + A_2\left(\sqrt{3}-x\right)$$

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[2] One can also establish the interval of convergence via **Thm 10.21** and **Thm 10.22**

$$\therefore x=\sqrt{3}\rightarrow A_1=1, x=-\sqrt{3}\rightarrow A_2=-1$$

$$g(x)=\frac{1}{\sqrt{3}-x}-\frac{1}{\sqrt{3}+x}=\tfrac{\sqrt{3}}{3}\left(\frac{1}{1-\frac{x}{\sqrt{3}}}-\frac{1}{1+\frac{x}{\sqrt{3}}}\right)=\tfrac{\sqrt{3}}{3}\left\{\sum_{k=0}^{\infty}\left(\tfrac{x}{\sqrt{3}}\right)^k-\sum_{k=0}^{\infty}\left(-\tfrac{x}{\sqrt{3}}\right)^k\right\}$$

$$=\tfrac{\sqrt{3}}{3}\sum_{k=0}^{\infty}\left[\left(\tfrac{x}{\sqrt{3}}\right)^k\left(1-(-1)^k\right)\right]=\tfrac{\sqrt{3}}{3}\sum_{k=0}^{\infty}2\left(\frac{x}{\sqrt{3}}\right)^{2k+1}$$

(I.e., only the odd terms will survive in the above series, and summing the two terms which are the same in the odd number of cases of course produces the coefficient 2.)

Simplifying:

$$\tfrac{\sqrt{3}}{3}\sum_{k=0}^{\infty}2\left(\frac{x}{\sqrt{3}}\right)^{2k+1}=2\,\tfrac{\sqrt{3}}{3}\sum_{k=0}^{\infty}\tfrac{1}{\sqrt{3}}\cdot\left(3^{-1/2}\right)^{2k}x^{2k+1}=\tfrac{2}{3}\sum_{k=0}^{\infty}3^{-k}x^{2k+1}$$

V.) Given $f(x)=e^{(2x+1)}$

a.) Find its McClaurin Series, given that $e^x=\sum_{k=0}^{\infty}\frac{x^k}{k!}$

<u>Method 1:</u> $f(x)=e^{(2x+1)}=e\cdot e^{2x}=e\sum_{k=0}^{\infty}\frac{(2x)^k}{k!}=\sum_{k=0}^{\infty}\frac{2^k e}{k!}x^k$

<u>Method 2:</u>[3] $f(x)=e^{(2x+1)}=\sum_{k=0}^{\infty}\frac{(2x+1)^k}{k!}$

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[3] To show the equivalence, note: $\sum_{k=0}^{\infty}\frac{2^k e}{k!}x^k=\sum_{k=0}^{\infty}e\frac{(2x)^k}{k!}=\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\frac{1}{m!}\frac{(2x)^k}{k!}=\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\frac{k!}{m!k!}\cdot\frac{(2x)^k}{k!}$

(by expanding e the McClaurin Series of e^x, with $x=1$. I.e. $e^1=\sum_{m=0}^{\infty}\frac{1^m}{m!}=\sum_{m=0}^{\infty}\frac{1}{m!}$

Note also the 'trick of 1' in the last step: $\frac{1}{m!}=\frac{k!}{m!k!}$

Now, the term $\frac{k!}{m!k!}$ is the Binomial coefficient: $\binom{k}{k-m}=\binom{k}{m}=\frac{k!}{m!(k-m)!}$

Hence $\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\frac{k!}{m!k!}\cdot\frac{(2x)^k}{k!}=\sum_{k-0}^{\infty}\sum_{m=0}^{\infty}\binom{k}{m}\cdot\frac{(2x)^k}{k!}=\sum_{k=0}^{\infty}\frac{(2x+1)^k}{k!}$

b.) Find its Taylor Series, at any point c and show you answer agreeing with a.) in the $c \to 0$ limit.

$$f(x) = e^{(2x+1)} = e^{(2(x-c)+2c+1)} = e^{2c+1}e^{2(x-c)} = e^{2c+1}\sum_{k=0}^{\infty}\frac{(2(x-c))^k}{k!} = e^{2c+1}\sum_{k=0}^{\infty}\tfrac{2^k}{k!}(x-c)^k$$

Note when $c \to 0$: $e^{2c+1}\sum_{k=0}^{\infty}\tfrac{2^k}{k!}(x-c)^k \xrightarrow{c\to 0} e\sum_{k=0}^{\infty}\tfrac{2^k}{k!}x^k$ (in agreement with the previous answer)

Bonus) Represent $f(x) = \int_0^x \sqrt{a+bt}\,dt$ as a power-series(where a, b are non-zero constants)

Using **BST:**

$$f(x) = \int_0^x \sqrt{a+bt}\,dt = \int_0^x \sqrt{a}\left(1+\tfrac{b}{a}t\right)^{1/2}dt = \sqrt{a}\int_0^x \sum_{k=0}^{\infty}\binom{\frac{1}{2}}{k}\left(\tfrac{b}{a}t\right)^k dt = \sum_{k=0}^{\infty}\binom{\frac{1}{2}}{k}a^{\left(\frac{1}{2}-k\right)}b^k\int_0^x t^k\,dt$$

$$= \sum_{k=0}^{\infty}\binom{\frac{1}{2}}{k}a^{\left(\frac{1}{2}-k\right)}b^k \cdot \left.\frac{t^{k+1}}{(k+1)}\right|_0^x = \sum_{k=0}^{\infty}\binom{\frac{1}{2}}{k}\frac{a^{\left(\frac{1}{2}-k\right)}b^k}{(k+1)}x^{k+1}$$

- *PARAMETRIC REPRESENTATIONS OF CURVES*

  By 'paramater' it is meant an independent variable that nevertheless plays a crucial role in fixing the values of other quantities that functionally depend on it.

  Consider, for example, projectile motion. Neglecting air resistance, the position ($x(t)$, $y(t)$) has the following parameterization:

  $$x(t) = x_0 + v_{0,x}t = x_0 + v_0\cos\alpha t$$
  $$y(t) = y_0 + v_{0,y}t - \tfrac{1}{2}gt^2 = y_0 + v_0\sin\alpha t - \tfrac{1}{2}gt^2$$

Where (x_0, y_0) is the initial position of the projectile, v_0 is its initial launch speed (at angle α) and g is the magnitude of the surface gravity field ($= 9.8\frac{m}{\sec^2} = 32.0\frac{ft}{\sec^2}$)

Given a parameterization (x(t), y(t)) = (g(t), h(t)) , it is always possible to 'de-parameterize' by eliminating t such that: $(x(t), y(t)) \xrightarrow{DP} (x, y), y = f(x)$

For instance, in the above:

$$x(t) = x_0 + v_{0,x}t = x_0 + v_0 \cos\alpha t \Rightarrow t = \frac{(x - x_0)}{v_0 \cos\alpha}$$

Inserting into $y(t)$:

$$y(t) = y_0 + v_0 \sin\alpha t - \tfrac{1}{2} g t^2 = y_0 + v_0 \sin\alpha \cdot \frac{(x - x_0)}{v_0 \cos\alpha} - \tfrac{g}{2} \cdot \frac{(x - x_0)^2}{v_0^2 \cos^2\alpha}$$

$$\Rightarrow (y - y_0) = \tan\alpha (x - x_0) - \tfrac{g}{2v_0^2} \sec^2\alpha (x - x_0)^2$$

Denote: $\Delta x = x - x_0, \Delta y = y - y_0$. Then:

$$\begin{aligned}
\Delta y &= -\tfrac{g}{2v_0^2} \sec^2\alpha \left[(\Delta x)^2 - \tfrac{2v_0^2}{g} \sin\alpha \cos\alpha (\Delta x)\right] \\
&= -\tfrac{g}{2v_0^2} \sec^2\alpha \left[(\Delta x)^2 - \tfrac{v_0^2}{g} \sin 2\alpha (\Delta x) + \tfrac{v_0^4}{g^2} \sin^2\alpha \cos^2\alpha\right] + \tfrac{v_0^2}{2g} \sin^2\alpha \\
&= -\tfrac{g}{2v_0^2} \sec^2\alpha \left[(\Delta x) - \tfrac{v_0^2}{g} \sin\alpha \cos\alpha\right]^2 + \tfrac{v_0^2}{2g} \sin^2\alpha \\
&\Rightarrow \left((\Delta y) - \tfrac{v_0^2}{2g} \sin^2\alpha\right) = -\tfrac{g}{2v_0^2} \sec^2\alpha \left((\Delta x) - \tfrac{v_0^2}{2g} \sin 2\alpha\right)^2 \\
&\Rightarrow \left(x - \left(x_0 + \tfrac{v_0^2}{2g} \sin 2\alpha\right)\right)^2 = -\tfrac{2v_0^2}{g} \cos^2\alpha \left(y - \left(y_0 + \tfrac{v_0^2}{2g} \sin 2\alpha\right)\right)
\end{aligned}$$

So after a few trigonometric identities, one recovers the de-paramaterized version of the trajectory of the particle, which is immediately recognized as a downward-opening parabola. From the standard equation of the parabola:

$$(x - h)^2 = 4p(y - k)$$

One can obtain the information of the vertex, focal distance p directly from the initial dynamical parameters:

$$h = x_0 + \tfrac{v_0^2}{2g} \sin 2\alpha, k = y_0 + \tfrac{v_0^2}{2g} \sin 2\alpha$$

$$p = \tfrac{v_0^2}{2g} \cos^2\alpha$$

In physical terms, a parametric representation depicts the *dynamics* of a particular system (i.e., it actual evolution in time given initial conditions), whereas the de-parameterized quantities depict the *kinematics*: i.e. the constraints on all its possible motion. For example, assuming no air resistance, regardless of the initial values, the particle in the problem of projectile motion is constrained to move along a parabolic curve.

Based on the Pythagorean Identities, conic sections like the ellipse and hyperbola for instance admit the following 'natural' parameterizations:

$$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1 \Rightarrow x=h+a\cos t,\, y=k+a\sin t$$
$$\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1 \Rightarrow x=h+a\cosh t,\, y=k+a\sinh t$$

Certainly there is no *unique* correspondence between a parameterization of a curve and the curve itself: In principle, there are a countless variety[4] of different parameterizations which (when de-parameterized) all describe the same curve or class of curves. For the above could have just as well been parameterized via:

$$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1 \Rightarrow x=h+a\sin t,\, y=k+a\cos t$$

$$\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1 \Rightarrow x=h+a\tan t,\, y=k+a\sec t$$

(though the second set describe different 'dynamics,' or another kind of behavior of a particle in motion constrained nevertheless on the same curve)

- Example (§12.1, #20)

$$x(\theta)=\cos\theta,\, y(\theta)=2\sin^2\theta$$
$$\sin^2\theta=\frac{y}{2},\cos^2\theta=x^2 \Rightarrow x^2+\tfrac{1}{2}y=1 \Rightarrow \tfrac{1}{2}(y-2)=-x^2 \quad \text{Parabola}$$

- Example (§12.1, #50)

$$x=h+a\sqrt{t+1},\, y=k+b\sqrt{t}$$
$$\sqrt{t}=\tfrac{1}{b}(y-k) \Rightarrow t=\frac{(y-k)^2}{b^2}$$
$$\sqrt{t+1}=\tfrac{1}{a}(x-h) \Rightarrow t+1=\frac{(x-h)^2}{a^2}=\frac{(y-k)^2}{b^2}+1 \Rightarrow \frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$$

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[4] Certainly adding physical laws into the picture greatly constrains what paramaterizations are physically possible.

- *CALCULUS ON PARAMETRIC REPRESENTATIONS OF CURVES*

Via the Chain Rule, it is easily shown:

$$\frac{dy}{dx} = \frac{dy}{dt}\frac{dx}{dt} = \frac{dy/dt}{dx/dt}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dt}\left[\frac{dy}{dx}\right]\frac{dt}{dx} = \frac{\frac{d}{dt}\left[\frac{dy}{dx}\right]}{dx/dt} = \frac{\frac{d}{dt}\left[\frac{dy/dt}{dx/dt}\right]}{dx/dt}$$

And by recursion[5]: $$\frac{d^n y}{dx^n} = \frac{\frac{d}{dt}\left[\frac{d^{(n-1)}y}{dx^{(n-1)}}\right]}{dx/dt}$$

In addition, in terms of the parametric representation, the infinitesimal arc length element takes on the following elegant form:

$ds = \sqrt{(\dot{x})^2 + (\dot{y})^2}\,dt$ (where the dot superscript is shorthand for $\frac{d}{dt}$)

Hence arclength become: $$L = \int_a^b \sqrt{(\dot{x})^2 + (\dot{y})^2}\,dt$$

- **Note1:** By deparameterizing, note how the original arclength formula can be recovered:

$$L = \int_a^b \sqrt{(\dot{x})^2 + (\dot{y})^2}\,dt = \int_{x(a)}^{x(b)} \sqrt{\left(\tfrac{dx}{dt}\right)^2 + \left(\tfrac{dy}{dt}\right)^2}\,dt = \int_{x_a}^{x_b} \sqrt{1 + \left(\frac{dy/dt}{dx/dt}\right)^2}\,\frac{dx}{dt}\,dt = \int_{x_a}^{x_b} \sqrt{1 + \left(\tfrac{dy}{dx}\right)^2}\,dx$$

Surface area: (revolved around x-axis and y-axis, respectively)

$$S_x = 2\pi \int_a^b y(t)\sqrt{(\dot{x})^2 + (\dot{y})^2}\,dt \qquad S_y = 2\pi \int_a^b x(t)\sqrt{(\dot{x})^2 + (\dot{y})^2}\,dt$$

- Example (§12.2, #15)

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[5] Where the expression $\frac{d^{(n-1)}y}{d^{(n-1)}x}$ has already been expressed explicitly in terms of *t*-derivatives.

Find the equation of the line tangent to curve $x(\theta) = 2\cot\theta, y(\theta) = 2\sin^2\theta$ at the point $\theta = \frac{\pi}{4}$

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{2\sin\theta\cos\theta}{-2\csc^2\theta} = -\sin^3\theta\cos\theta \Rightarrow \left.\frac{dy}{dx}\right|_{\theta=\pi/4} = -\left(\tfrac{\sqrt{2}}{2}\right)^3\left(\tfrac{\sqrt{2}}{2}\right) = -\tfrac{1}{2}$$

The point at the tangent line is: $x_0 = x\left(\frac{\pi}{4}\right) = 2\cot\frac{\pi}{4} = 2, y_0 = 2\sin^2\frac{\pi}{4} = 2\cdot\frac{1}{2} = 1$
Using the point-slope formula:

$$y - y_0 = m(x - x_0)$$
$$\Rightarrow y - 1 = -\tfrac{1}{2}(x-2) \Rightarrow y = -\tfrac{1}{2}x + 2 \Rightarrow 2y + x - 4 = 0$$

- Example (§12.2, #44)

Find the surface area of revolution around the *y*-axis for
$x(t) = t^3, y = t + 2, 1 \le t \le 2$

$$S_y = 2\pi\int_a^b x(t)\sqrt{(\dot{x})^2 + (\dot{y})^2}\,dt = 2\pi\int_1^2 t^3\sqrt{\left(3t^2\right)^2 + 1}dt = 2\pi\int_1^2 t^3\sqrt{1+9t^4}\,dt$$

$$\tan\theta = 3t^2 \Rightarrow \theta = \arctan\left(3t^2\right), \sec^2\theta = 6tdt$$

$$\Rightarrow 2\pi\int_{\alpha=\arctan(3)}^{\beta=\arctan(12)}\left(\tfrac{1}{3}\cdot 3t^2\right)\left(\tfrac{1}{6}\cdot 6tdt\right)\sqrt{1+9t^4} = \tfrac{1}{9}\pi\int_\alpha^\beta \tan\theta\sec^2\theta d\theta\sqrt{1+\tan^2\theta}$$
$$= \tfrac{1}{9}\pi\int_\alpha^\beta \sec^3\theta\tan\theta d\theta = \tfrac{1}{9}\pi\int_\varepsilon^\beta \sec^2\theta(\sec\theta\tan\theta d\theta) = \tfrac{1}{9}\pi\int_{\sec\alpha}^{\sec\beta} u^2 du$$
$$= \tfrac{\pi}{27}u^3\Big|_{\sec\alpha}^{\sec\beta} = \tfrac{\pi}{27}\left(\sec^3(\arctan 12) - \sec^3(\arctan 3)\right)$$

$$\beta = \arctan 12 \Rightarrow \tan\beta = \frac{12}{1} = \frac{OPP}{ADJ} \Rightarrow HYP = \sqrt{144+1} = \sqrt{145} \Rightarrow \sec\beta = \frac{HYP}{ADJ} = \sqrt{145}$$

$$\alpha = \arctan 3 \Rightarrow \tan\beta = \frac{3}{1} = \frac{OPP}{ADJ} \Rightarrow HYP = \sqrt{9+1} = \sqrt{10} \Rightarrow \sec\beta = \frac{HYP}{ADJ} = \sqrt{10}$$

Hence: $S_y = \frac{\pi}{27}\left(145^{3/2} - 10^{3/2}\right) \approx 199.5$