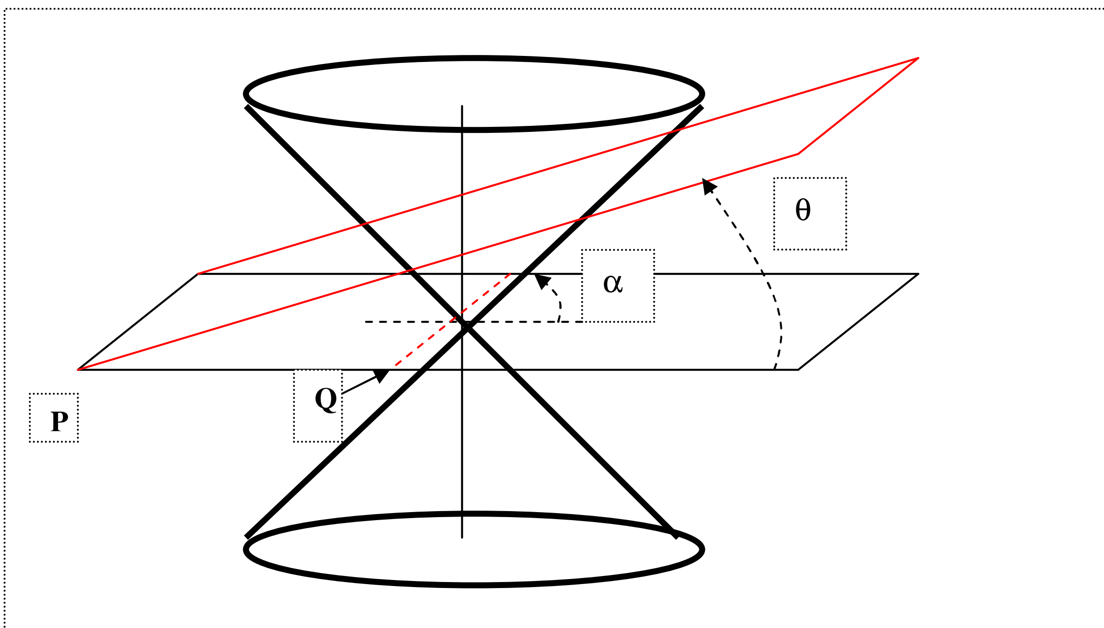
MA262 **Calculus II: Notes for March 20**
Spring 2008 Kallfelz

- *REVIEW OF THE CONIC SECTIONS: THEIR ALGEBRAIC AND GEOMETRIC PROPERTIES*

The goal in this batch of notes is to refresh what you should know already: some of the essential algebraic-geometric properties of conic sections. Familiarity of such properties is essential when we cover the material in chapters 12 and 13 (as well as in some of the material in chapters 15, 16) which inadvertently deal with such basic properties. In addition, I'll acquaint you with some aspects concerning the conics with which perhaps you are unfamiliar, nevertheless you will find of interest.

As stated in the text (p 645) 'conic sections' are classes of closed an open curves that are generated by slicing a plane at various angles through two inverted cones. Their essentially geometric properties I will summarize below were known and presented in Appolonius of Pergia's (262 -192 BC) eight treatises (of which seven survive today) as well as in Hypatia's (370-415 AD) treatises.[1] Consider the illustration below:



Tilting the plane (highlit in red) for various values of θ at the point **P** generates four distinct possibilities of kinds of classes of curves located at the intersection of the boundaries of the plane with the cone(s), along with the associated degenerate cases listed

[1] For more information, see:
http://mathdl.maa.org/convergence/1/?pa=content&sa=viewDocument&nodeId=196&bodyId=60
http://encyclopedia.farlex.com/Apollonius+of+Perga

in the illustrations in p. 645. Hence according to the above figure, depending on the value of θ relative to the angle to the cone α such possibilities are:

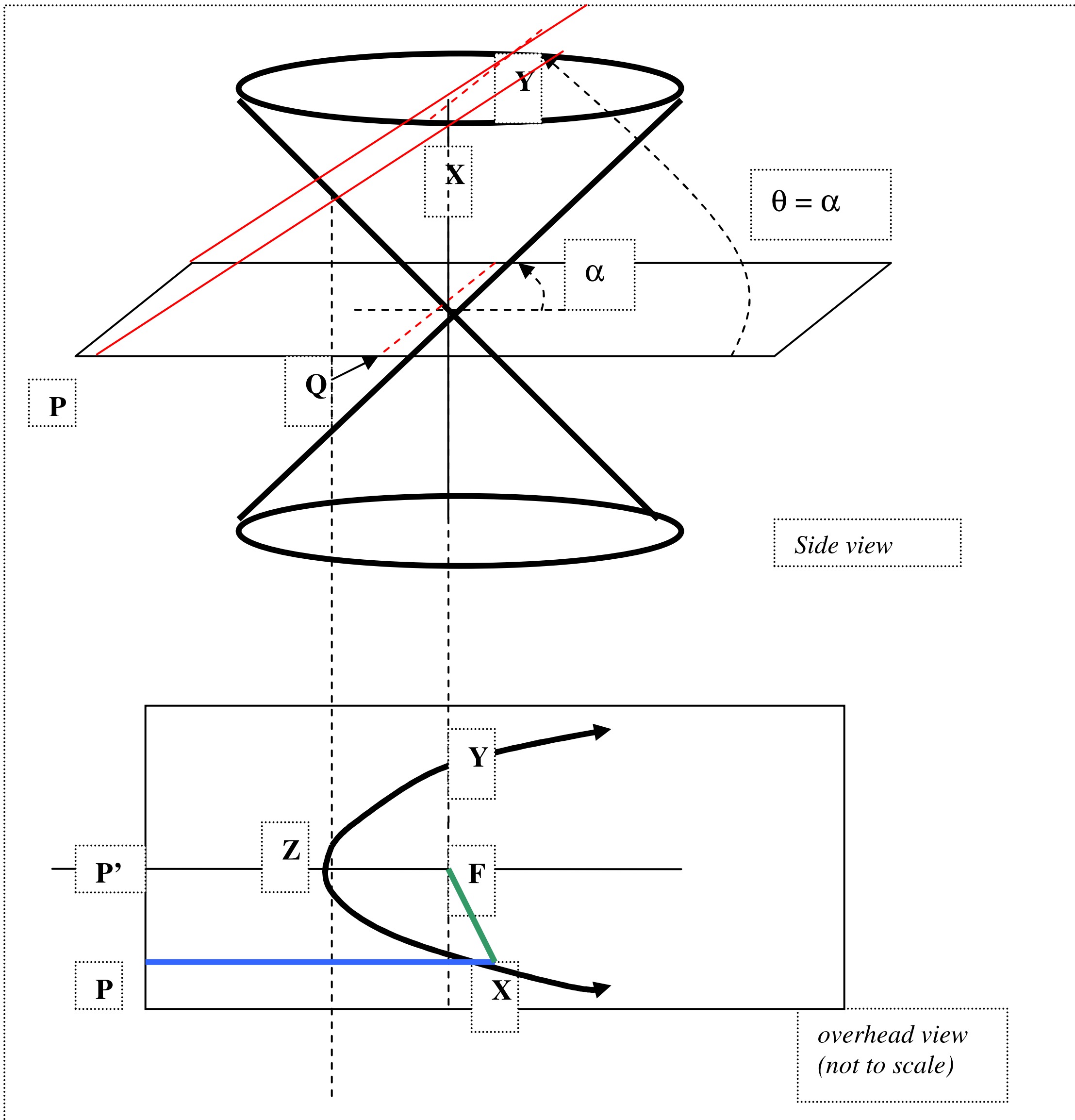
$\theta = 0$	**Circle,** whose degenerate case is a **point,** as evidenced in the figure above. Elevating the plane (with red boundaries) will produce a circle.
$0 < \theta < \alpha$	**Ellipse,** whose degenerate case is a **circle**, in the limit: $\theta \to \alpha$.
$\theta = \alpha$	**Parabola,** whose degenerate case become a **line,** in the limit: **P→ Q.**
$\alpha < \theta \leq \pi/2$	**Hyperbola,** whose degenerate case becomes a **two intersecting lines,** in the limit: **P→ Q.**

Other essential properties generating such curves were summarized in Appolonius' treatise, and were derived through various ingenious methods:[2]

1. (**Property 1)** A *parabola* is the set of all points equidistant from a focus *F* and a *directerix line.*
2. (**Property 2)** A *circle* is the set of all points equidistant from a point *C* (the center).
3. (**Property 3)** An *ellipse* is the set of all points whose sum of distances from two foci is constant.
4. (**Property 4)** An *hyperbola* is the set of all points whose difference of distances from two foci is constant.

For example, in the case of the parabola:

[2] It bears emphasizing that Greek mathematicians in Antiquity did *not* have the powerful language of algebra at their disposal, much less the method of analytic geometry (i.e. the synthesis of algebra and geometry as developed by Renee Descartes in the 17th century). Hence all their derivations proceeded according to Euclid's *Analects*: that is, by strict deduction from essential postulates and theorems, using arguments involving ratios.



Property 1 is evidenced by the following reason[3]: since the angle θ of the plane is identical to the angle of the cone α, then[4] **d(P'Z) = d(ZF)**, i.e. point **Z** is halfway between points **F** and **P'**. What is true about **Z** is true about any point **X** located on the cone, i.e. **d(PX) = d(FX)** (note that the drawing is not to scale). Hence the parabola is generated by the green line having the same distance as the blue line, for *any* point on the parabola. Note the points **X** and **Y** were chosen to be collinear to the line intersecting **F** (the focus). This line is known as the *latus rectum* ("straight side" trnsl. From Latin) of the cone. It turns out (for reasons we'll see later herein) that 4**d(ZF) = d(XY)**.

Consider the case of the ellipse:

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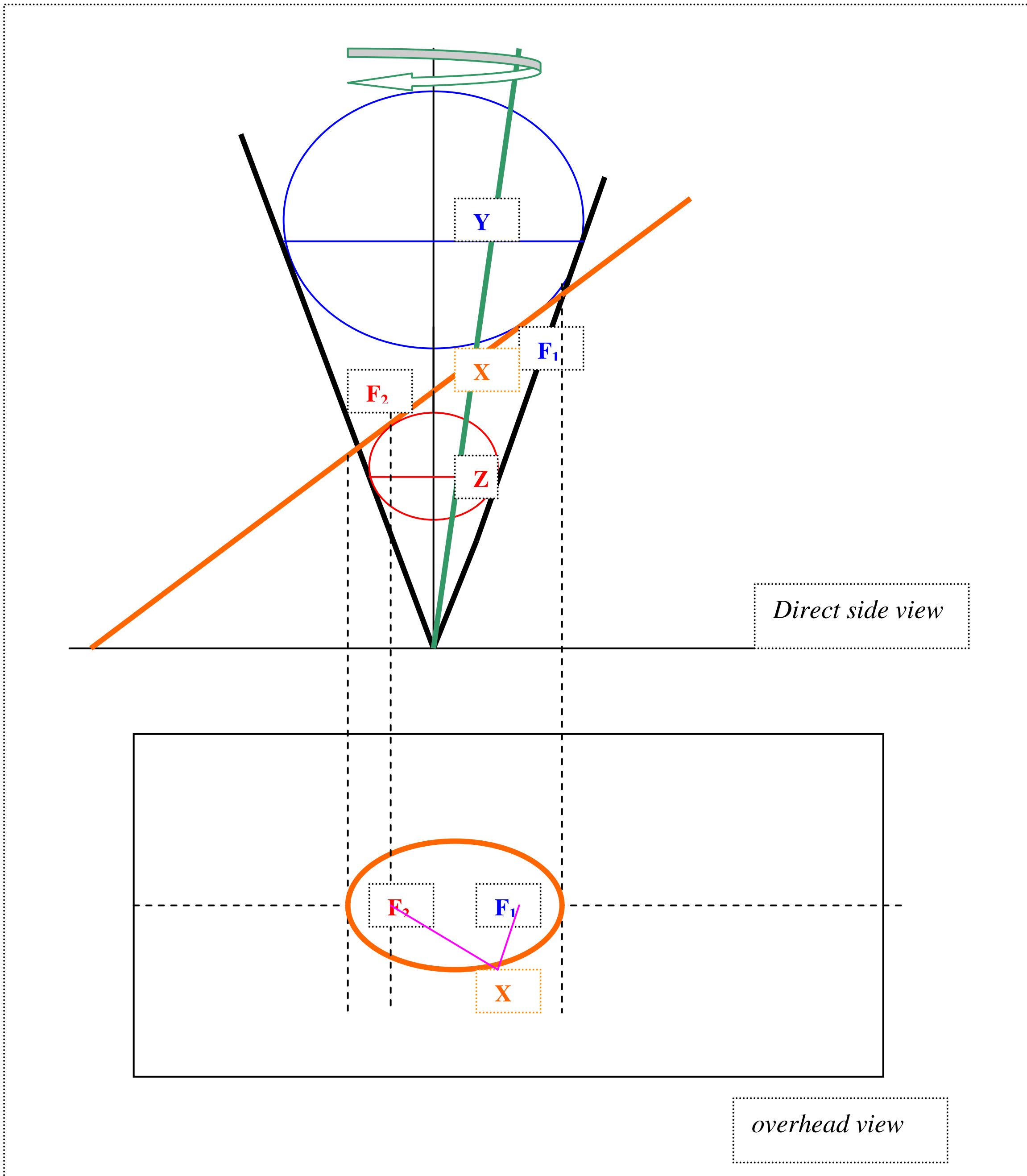
[3] The reason I am offering is of course not as rigorous and explicitly logical as Appolonius', who resorted to Euclid's unprecendented achievement in logic and deduction to derive everything though explicit argument with the utmost precision. For the sake of efficiency, my reasoning appeals to your intuition.
[4] Where d(…) stands for the distance of the line segment.



Imagine two spheres inside the cone. The large blue sphere is tangent to the cone marked by the blue line (which is a latitude on the sphere). The small red sphere is tangent to the cone marked by the red line (which is a latitude on the sphere). The orange plane will touch the two spheres at two points: **F_2** and **F_1**. Appolonius showed that these two points (the *foci*) were equidistant from the center of the ellipse, and lay on the major axis of the ellipse. Now, imagine anchoring the **green** line segment in such a manner that it sits at the base of the cone and is attached to the wall of the cone. The green line will touch the latitude of the small sphere at point **Z,** and will touch the latitude of the small sphere at point **Y**, and will touch the **ellipse at point X**. Now imagine swiveling this green line all around the axis of symmetry of the cone (so that it of course stays attached to the face of the cone) as denoted by the arrow in the overhead drawing. Certainly throughout the rotations:

d(ZY) = constant.

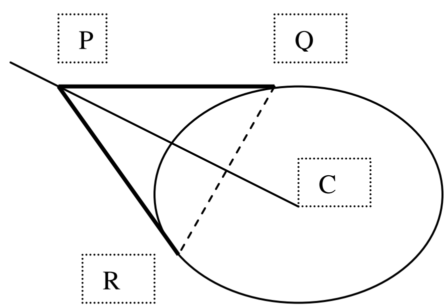
However, further note from the drawing that:

d(ZY) = d(ZX) + d(XY)

Therefore: **d(ZX) + d(XY) = constant.**

- Now, the line segment: $\overline{ZX}$ is tangent to the red circle at point **Z.** The line segment $\overline{XF_2}$ is tangent to the red circle at point **F_2.** Therefore: **d(ZX) = d(XF_2).**
- Also the line segment: $\overline{ZY}$ is tangent to the blue circle at point **Y.** The line segment $\overline{XF_1}$ is tangent to the blue circle at point **F_1.** Therefore: **d(ZY) = d(XF_1).**

The reason why the equalities hold is because (as shown rigorously by Euclid) two tangent lines to a sphere/circle intersecting at a common point are equidistant. As suggested in the figure below:



…The distances **d(PQ) = d(PR)**. Hence the triangle PQR is isosceles, whose base is the secant line QR.

Therefore, from the above argument, we have derived **Property 3:**

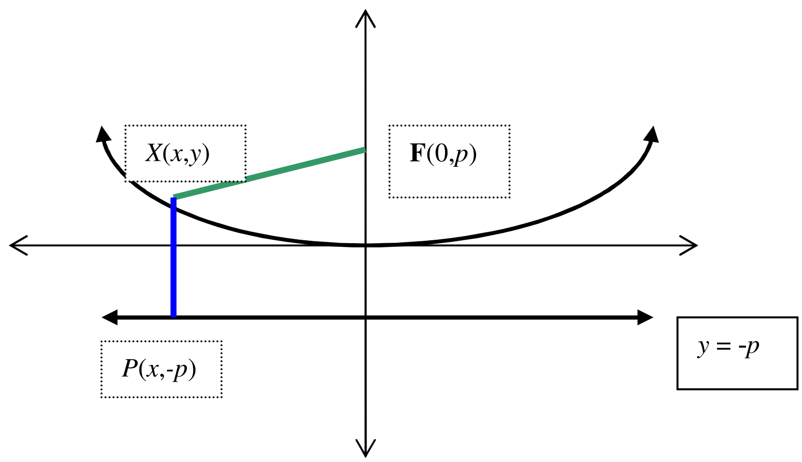
d(ZX) + d(XY) = **d(XF_2) + d(XF_1) = constant.**

Property 4 for the hyperbola is derived in a similar manner. **Property 2** for the circle is of course trivial, but note how it exists of course as a degenerate case of **Property 3**: if **F_1 = F_2 = C** (where **C** is the center), then certainly **d(XC) + d(XC) = 2d(XC) = constant = diameter.**

Truth be told, Appolonius and his surrogates in Antiquity derived *all* the important essential geometric properties of the conic sections without the aid of anything resembling present-day analytic geometry. A truly astounding achievement, when you think of it!

Of course by the 17th and early 18th centuries **Properties 1-4** were given precise algebraic characterizations, known respectively as the *characteristic equations* of the conic sections.

To derive the characteristic equation of the parabola, assume (with no loss of generality) that it's centered at the origin, and define: p = **d(P'Z) = d(ZF)**



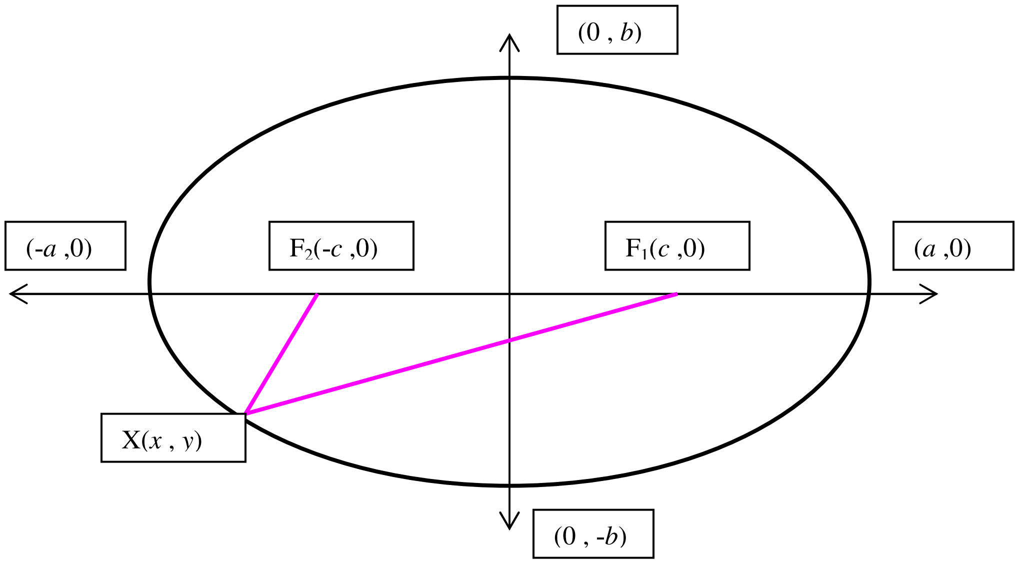
Using the distance formula: $d(P_1P_2) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

$$d(PX) = d(XP) \Rightarrow \sqrt{(x-x)^2 + (y-(-p))^2} = \sqrt{(x-0)^2 + (y-p)^2}$$
$$\Rightarrow \sqrt{(y+p)^2} = \sqrt{x^2 + (y-p)^2} \Rightarrow (y+p)^2 = x^2 + (y-p)^2$$
$$\Rightarrow y^2 + 2py + p^2 = x^2 + y^2 - 2py + p^2 \Rightarrow x^2 = 4py$$

Similarly if the bisecting axis of symmetry of the parabola were chosen to be the x-axis, the equation (in standard form) would yield: $y^2 = 4px$.

- For any parabola located at any center (h,k) other than the origin, then its standard form equations are: $(x-h)^2 = 4p(y-k)$ (for bisecting y axis)
  $(y-k)^2 = 4p(x-h)$ (for bisecting x axis)

To derive the generating equation for an ellipse again assume (without loss of generality) that its center **C** is at the origin, and define c = **d(CF$_1$) = d(CF$_2$),** and define a as the length of its semimajor axis, and b as the length of its seminor axis. Then, as suggested by the figure below:



According to **Property 3: d(XF$_2$) + d(XF$_1$) = constant.** Based on the choice of the above parameters, one can define what that constant term is. Consider the special case when **X** is coincident with the right endpoint of the ellipse. Then **d(XF$_2$) =** $a + c$ and **d(XF$_1$) =** $a - c$. Hence: **d(XF$_2$) + d(XF$_1$)** $= a + c + (a - c) = 2a$.

Moreover, note that when **X** is coincident with top endpoint of the ellipse, according to the Pythagorean Theorem and **Property 3:**

$$d(F_1X) + d(F_2X) = 2\sqrt{c^2 + b^2} = 2a \Rightarrow c^2 + b^2 = a^2 \Rightarrow b^2 = a^2 - c^2$$

As suggested by the above, the 'departure from circularity' (or *eccentricity*) is marke by the discrepancy between b and a. (Certainly if $a = b$ we'd have a circle or degenerate ellipse, and $c = 0$). The eccentricity e is defined by the following parameter:

$$e = \frac{c}{a} = \frac{\sqrt{a^2 - b^2}}{a}$$

…*if* one *defines* b to be the length of the *semiminor axis.* If one on the other hand relaxes this stipulation, calling b to be merely the distance away from the center on the *vertical* symmetry axis of the ellipse, then:

$$e = \frac{c}{a} = \frac{\sqrt{|a^2 - b^2|}}{a(or\_b)}$$

(since there's no especial reason in this more general case why b must be less than a)

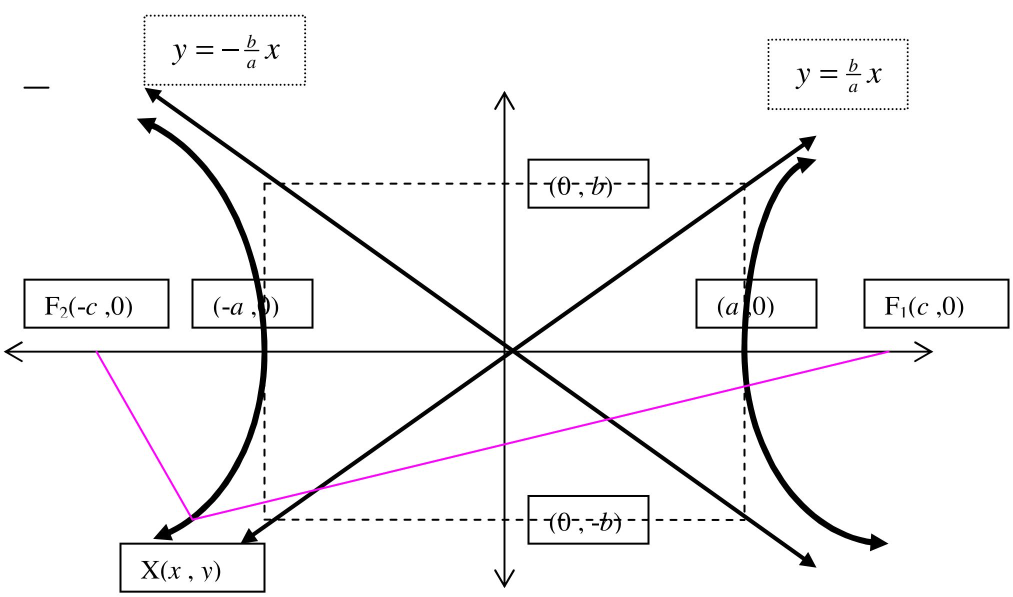
Hence using the distance formula:

$$
\begin{aligned}
& d(XF_2)+d(XF_1)=2a \Rightarrow \sqrt{(x+c)^2+y^2}+\sqrt{(x-c)^2+y^2}=2a \\
& \Rightarrow \sqrt{(x+c)^2+y^2}=2a-\sqrt{(x-c)^2+y^2} \\
& \Rightarrow (x+c)^2+y^2=4a^2-4a\sqrt{(x-c)^2+y^2}+(x-c)^2+y^2 \\
& \Rightarrow x^2+2cx+c^2=4a^2-4a\sqrt{(x-c)^2+y^2}+x^2-2cx+c^2 \\
& \Rightarrow 4cx-4a^2=-4a\sqrt{(x-c)^2+y^2} \Rightarrow a-\left(\tfrac{c}{a}\right)x=\sqrt{(x-c)^2+y^2} \\
& \Rightarrow a^2-2cx+\left(\tfrac{c^2}{a^2}\right)x^2=(x-c)^2+y^2 \Rightarrow a^2-2cx+\left(\tfrac{c^2}{a^2}\right)x^2=x^2-2cx+c^2+y^2 \\
& \Rightarrow a^2+\left[\left(\tfrac{c}{a}\right)^2-1\right]x^2-y^2=c^2 \Rightarrow \left(\tfrac{c^2-a^2}{a^2}\right)x^2-y^2=c^2-a^2 \\
& \Rightarrow -\tfrac{b^2}{a^2}x^2-y^2=-b^2 \Rightarrow \frac{x^2}{a^2}+\frac{y^2}{b^2}=1
\end{aligned}
$$

And when the center is located at some other point (h , k) besides the origin, in standard form the equation of the ellipse becomes:

$$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$$

And the case of the hyperbola is virtually identical to that of the ellipse:



According to **Property 4: d(XF$_1$) - d(XF$_2$) = constant.** Based on the choice of the above parameters, one can define what that constant term is. Consider the special case when **X** is coincident with the left endpoint of the ellipse. Then **d(XF$_2$) =** c - a and **d(XF$_1$) =** c +a. Hence: **d(XF$_1$) - d(XF$_2$)** = c + a - (c – a) = 2a. Note however that $a^2 + b^2 = c^2$. Hence:

$$d(XF_1) - d(XF_2) = 2a \Rightarrow \sqrt{(x-c)^2 + y^2} - \sqrt{(x+c)^2 + y^2} = 2a$$
$$\Rightarrow \sqrt{(x-c)^2 + y^2} = 2a + \sqrt{(x+c)^2 + y^2}$$
$$\Rightarrow (x-c)^2 + y^2 = 4a^2 + 4a\sqrt{(x+c)^2 + y^2} + (x+c)^2 + y^2$$
$$\Rightarrow x^2 - 2cx + c^2 = 4a^2 + 4a\sqrt{(x+c)^2 + y^2} + x^2 + 2cx + c^2$$
$$\Rightarrow -4cx - 4a^2 = 4a\sqrt{(x+c)^2 + y^2} \Rightarrow -a - \left(\tfrac{c}{a}\right)x = \sqrt{(x+c)^2 + y^2}$$
$$\Rightarrow a^2 + 2cx + \left(\tfrac{c^2}{a^2}\right)x^2 = (x+c)^2 + y^2 \Rightarrow a^2 + 2cx + \left(\tfrac{c^2}{a^2}\right)x^2 = x^2 + 2cx + c^2 + y^2$$
$$\Rightarrow a^2 + \left[\left(\tfrac{c}{a}\right)^2 - 1\right]x^2 - y^2 = c^2 \Rightarrow \left(\tfrac{c^2-a^2}{a^2}\right)x^2 - y^2 = c^2 - a^2$$
$$\Rightarrow \tfrac{b^2}{a^2}x^2 - y^2 = b^2 \Rightarrow \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

And when the hyperbola is located at some other point (h , k) besides the origin, in standard form the equation becomes (for the x bisecting axis):

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$

And: $$\frac{(y-k)^2}{b^2} - \frac{(x-h)^2}{a^2} = 1$$ (for the y bisecting axis).

…which can be combined into the result: $$\pm\frac{(x-h)^2}{a^2} \mp \frac{(y-k)^2}{b^2} = 1$$

As evidenced in the case of the hyperbola, since $c^2 = a^2 + b^2 \Rightarrow e = \tfrac{c}{a} > 1$

In terms of the eccentricity parameter:

$e = \tfrac{c}{a} = 0$	Circle
$0 < e < 1$	Ellipse
e= 1	Parabola
$e > 1$	Hyperbola

- *GENERAL CHARACTERIZATION OF CONIC SECTIONS*

As evidenced by the three characteristic equations of the conic sections (parabola, ellipse, hyperbola, respectively[5]) *any* conic section can be expressed as a degree-2 polynomial equation with respect to two variables:

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

- In the case in which $B = 0$, then the axes of symmetry of the conic coincide in orientation with respect to the x, y axes (i.e. the conic's angle θ of orientation = 0 or $^{\pi}/_{2}$, depending on whether the horizontal symmetry axis is parallel to the x or to the y axis, respectively).

- In the unrotated ($B = 0$) case, based on the above derivations, it is immediately apparent whenever the following properties and constrainrs hold:

Property/constraint[6]	**Conic**
$A = C,\ D^2 + E^2 - 4AF \geq 0$	Circle
$\mathrm{sgn}(A) = \mathrm{sgn}(C) = \mathrm{sgn}\left(\frac{D^2}{A} + \frac{E^2}{C} - 4F\right)$ (sgn stands for sign, either + or -)	Ellipse
$A = 0$ or $C = 0$ (but not both)	Parabola
$\mathrm{sgn}(A) = -\mathrm{sgn}(C)$ $\mathrm{sgn}(A) = \mathrm{sgn}\left(\frac{D^2}{A} + \frac{E^2}{C} - 4F\right)$ or $\mathrm{sgn}(C) = \mathrm{sgn}\left(\frac{D^2}{A} + \frac{E^2}{C} - 4F\right)$	Hyperbola

It's a matter of simple algebra (completing the square) to convert to standard form. In the case of the circle:

$$Ax^2 + Ay^2 + Dx + Ey + F = 0 \Rightarrow A\left(x^2 + \tfrac{D}{A}x\right) + A\left(y^2 + \tfrac{E}{A}y\right) = -F$$
$$\Rightarrow A\left(x + \tfrac{D}{2A}\right)^2 + A\left(y + \tfrac{E}{2A}\right)^2 = -F + \tfrac{1}{4A}\left(D^2 + E^2\right)$$
$$\Rightarrow \left(x + \tfrac{D}{2A}\right)^2 + \left(y + \tfrac{E}{2A}\right)^2 = \tfrac{D^2 + E^2 - 4AF}{4A^2}$$

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[5] The characteristic equation of the second conic section, i.e. the circle, can be obviously derived as a special case of the ellipse: If $e = 0$, then $c = 0$, and hence $a = b$. Therefore:

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{a^2} = 1 \Rightarrow (x-h)^2 + (y-k)^2 = a^2$$

[6] The second inequality/equality constraints involving F, D, E arise to ensure that one doesn't encounter an impossible case: i.e. a circle with a negative radius squared, or an (impossible) "ellipse" with characteristic equation: $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{a^2} = -1$. Furthermore note that no restriction apply in terms of the relative sign of terms necessarily involving F, D, E in the case of the parabola. (Why?)

Hence comparing to standard form, the coefficients of the above are related to the circle's geometric parameters in the following fashion:

$h = -\frac{D}{2A}, k = -\frac{E}{2A}, R = \frac{1}{2A}\sqrt{D^2 + E^2 - 4AF}$

- In the case of the ellipse:

$$Ax^2 + Cy^2 + Dx + Ey + F = 0 \Rightarrow A\left(x^2 + \tfrac{D}{A}x\right) + C\left(y^2 + \tfrac{E}{C}y\right) = -F$$
$$\Rightarrow A\left(x + \tfrac{D}{2A}\right)^2 + C\left(y + \tfrac{E}{2C}\right)^2 = -F + \tfrac{1}{4}\left(\tfrac{D^2}{A} + \tfrac{E^2}{C}\right)$$
$$\Rightarrow A\left(x + \tfrac{D}{2A}\right)^2 + C\left(y + \tfrac{E}{2C}\right)^2 = \tfrac{CD^2 + AE^2 - 4ACF}{4AC}$$
$$\Rightarrow \frac{\left(A + \frac{D}{2A}\right)^2}{\left[\frac{CD^2 + AE^2 - 4ACF}{4A^2C}\right]} + \frac{\left(y + \frac{E}{2C}\right)^2}{\left[\frac{CD^2 + AE^2 - 4ACF}{4AC^2}\right]} = 1$$

Comparing to standard form, the coefficients of the above are related to the ellipse's geometric parameters in the following fashion:

$$h = -\tfrac{D}{2A}, k = -\tfrac{E}{2C}, a = \tfrac{1}{2A}\sqrt{\tfrac{CD^2 + AE^2 - 4ACF}{C}}, b = \tfrac{1}{2C}\sqrt{\tfrac{CD^2 + AE^2 - 4ACF}{A}}$$

(Similar results apply in the case of the hyperbola)

- In the case of the parabola:

$$Ax^2 + Dx + Ey + F = 0 \Rightarrow A\left(x^2 + \tfrac{D}{A}x\right) = -F - Ey$$
$$\Rightarrow A\left(x + \tfrac{D}{2A}\right)^2 = -\left(Ey + F - \tfrac{D^2}{4A}\right)$$
$$\Rightarrow \left(x + \tfrac{D}{2A}\right)^2 = -\tfrac{E}{A}\left(y + \tfrac{F}{E} - \tfrac{D^2}{4EA}\right)$$

Comparing to standard form:

$$h = -\tfrac{D}{2A}, p = -\tfrac{E}{4A}, k = -\tfrac{D^2}{4EA}$$

(Similar results occur in the opposite case ($A = 0$, C nonzero)

In the more general case ($B \neq 0$) it always possible to rotate by to a new coordinate system (x',y') parallel to the axes of symmetry of the conic section:

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0 \xrightarrow{\theta} A'x'^2 + C'y'^2 + D'x' + E'y' + F' = 0$$

As derived in the proof of Thm 11. 7 (p. 671, text) the angle of orientation is:

$$\cot 2\theta = \frac{A - C}{B} \Rightarrow \theta = \arctan\left(\frac{B}{2(A - C)}\right)$$

(Note the special case of the circle: $A = C$. Of course no rotation of the axes is necessary since circles exhibit rotational symmetry anyway! Note how this result bears itself out in the above formula: $\cot 2\theta = \frac{0}{B} \Rightarrow \theta = \arctan\left(\frac{B}{0}\right) = \arctan(\infty) = \frac{\pi}{2}$, which states that if you're going to insist on rotating, the symmetry is such that any rotation is equivalent to a trivial 90 degree one)

…and the equations of coordinate transformations are:

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix} \Rightarrow \begin{pmatrix} x'\cos\theta - y'\sin\theta \\ x'\sin\theta + y'\cos\theta \end{pmatrix}$$

Last of all, the classification of the conics in the general expression can be classified by its *discriminant* $Disc = B^2 - 4AC$, which happens also to be a *rotation invariant*, i.e.: $B^2 - 4AC = B'^2 - 4A'C'$. If $Disc$ < 0 => ellipse, $Disc$ = 0 => parabola, $Disc$ > 0 => hyperbola.

- <u>Example1</u> Given the equation of the general conic

  $$x^2 + xy + y^2 + x + 2y - 1 = 0$$

  Determine its orientation, and specify it according to standard form

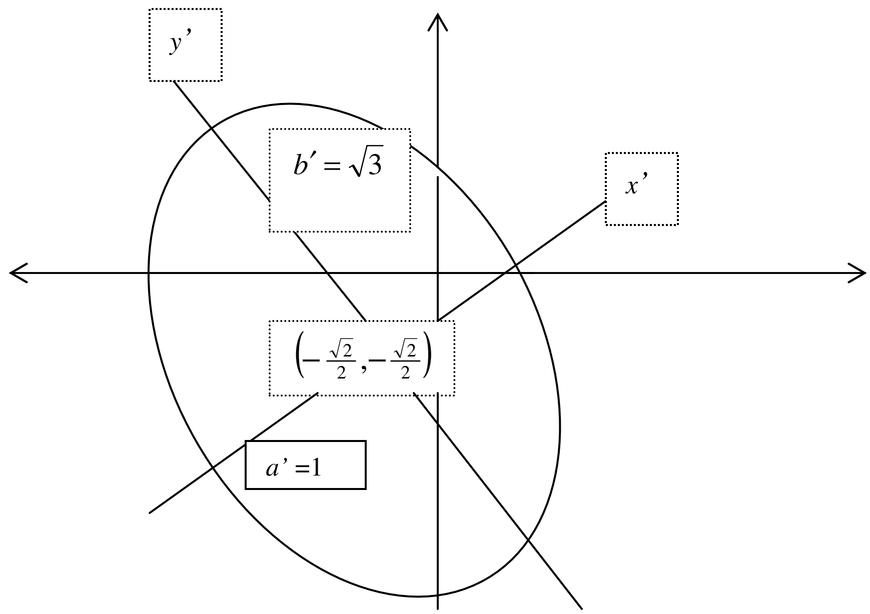
  $$\cot 2\theta = \frac{A - C}{B} = 0 \Rightarrow 2\theta = \frac{\pi}{2} \Rightarrow \theta = \frac{\pi}{4}$$

  Hence: $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x'\cos\frac{\pi}{4} - y'\sin\frac{\pi}{4} \\ x'\sin\frac{\pi}{4} + y'\cos\frac{\pi}{4} \end{pmatrix} = \frac{\sqrt{2}}{2}\begin{pmatrix} x' - y' \\ x' + y' \end{pmatrix}$

So:

$$x^2 + xy + y^2 + x + 2y - 1 = 0$$
$$\Rightarrow \tfrac{1}{2}\left(x'^2 - 2x'y' + y'^2\right) + \tfrac{1}{2}\left(x'^2 - y'^2\right) + \tfrac{1}{2}\left(x'^2 + 2x'y' + y'^2\right) + \tfrac{\sqrt{2}}{2}(x' - y') + \sqrt{2}(x' + y') = 1$$
$$\Rightarrow \tfrac{3}{2}x'^2 + \tfrac{1}{2}y'^2 + \tfrac{\sqrt{2}}{2}(3x' + y') = 1 \Rightarrow 3x'^2 + y'^2 + \sqrt{2}(3x' + y') = 2$$
$$\Rightarrow 3\left(x'^2 + \sqrt{2}x'\right) + \left(y' + \sqrt{2}y'\right) = 2 \Rightarrow 3\left(x' + \tfrac{\sqrt{2}}{2}\right)^2 + \left(y' + \tfrac{\sqrt{2}}{2}\right)^2 = 2 + \tfrac{3}{4} + \tfrac{1}{4}$$
$$\Rightarrow \frac{\left(x' + \frac{\sqrt{2}}{2}\right)^2}{1} + \frac{\left(y' + \frac{\sqrt{2}}{2}\right)^2}{\left(\sqrt{3}\right)^2} = 1$$

..which is an ellipse, tilted by $\frac{\pi}{4}$ with respect to the (*x*, *y*) system, centered at $\left(-\frac{\sqrt{2}}{2},-\frac{\sqrt{2}}{2}\right)$ whose half-length in its "vertical" symmetry axis (*y*') is $b'=\sqrt{3}$, and whose half-length in its 'horizontal' symmetry axis (*x*') is $a'=1$. See figure below:



Note that its discriminant confirms that it's an ellipse, and that it's a rotation invariant:

$$D=B^2-4AC=1-4=-3=B'^2-4A'C'=0-4\cdot\tfrac{3}{2}\cdot\tfrac{1}{2}$$

- <u>Example 2:</u> Given the above ellipse, construct an expression for the rate of change (with respect to *x* and with respect to *x*') of the slope of a line tangent to it:

  In the unrotated frame, differentiating implicitly:

$$\frac{d}{dx}(x^2+xy+y^2+x+2y-1)=0$$

$$2x+x\tfrac{dy}{dx}+y+2y\tfrac{dy}{dx}+1+2\tfrac{dy}{dx}=0$$

$$\Rightarrow \tfrac{dy}{dx}\left(x+2y+2\right)=-\left(2x+y+1\right)\Rightarrow \tfrac{dy}{dx}=-\frac{(y+2x+1)}{\left(x+2y+2\right)}$$

  In the rotated frame, differentiating implicitly:

$$\frac{d}{dx'}\left(\tfrac{\sqrt{3}}{2}x'^2+\tfrac{1}{2}y'^2+\tfrac{3\sqrt{2}}{2}x'+\tfrac{\sqrt{2}}{2}y'\right)=0$$

$$\sqrt{3}x'+y'\frac{dy'}{dx'}+\tfrac{3}{2}\sqrt{2}+\tfrac{\sqrt{2}}{2}\frac{dy'}{dx'}=0\Rightarrow\frac{dy'}{dx'}\left(y'+\tfrac{\sqrt{2}}{2}\right)=-\left(\sqrt{3}x'+\tfrac{3}{2}\sqrt{2}\right)$$

$$\Rightarrow\frac{dy'}{dx'}=-\frac{\left(\sqrt{3}x'+\tfrac{3}{2}\sqrt{2}\right)}{\left(y'+\tfrac{\sqrt{2}}{2}\right)}$$

…as expected, the rates of change (slopes) of tangent lines wouldn't have the same expression in the two coordinate systems (i.e. a derivative is *not* a rotation invariant since it's clearly coordinate-dependent!)

- Example 3: Find the area of the ellipse, and construct an expression for its circumference.

  In the *x'*,*y'* system, without loss of generality one can center the ellipse at the origin (call this the *u,v* coordinate system). Then:

$$\frac{\left(x'+\tfrac{\sqrt{2}}{2}\right)^2}{1}+\frac{\left(y'+\tfrac{\sqrt{2}}{2}\right)^2}{3}=1\mapsto\frac{u^2}{1}+\frac{v^2}{3}=1\Rightarrow v=\pm\sqrt{3}\sqrt{1-u^2}$$

$$A=4\sqrt{3}\int_0^1\sqrt{1-u^2}\,du=4\sqrt{3}\int_0^{\pi/2}\sqrt{1-\sin^2\theta}\cos\theta d\theta$$

$$=4\sqrt{3}\int_0^{\pi/2}\cos^2\theta d\theta=2\sqrt{3}\int_0^{\pi/2}(1+\cos 2\theta)d\theta$$

$$=\left(2\sqrt{3}+\tfrac{1}{2}\sin 2\theta\right)\Big|_0^{\pi/2}=\sqrt{3}\pi$$

(confirming the area formula for an ellipse, derived previously, that: $A=\pi ab$)

On the other hand, the circumference represents an unsolvable (by exact means) example of an *elliptic integral* (as shown in class, and also in the text, p. 658)

In this particular case in the *u*, *v* system: $e=\frac{\sqrt{a^2-b^2}}{a}=\frac{\sqrt{2}}{\sqrt{3}}=\sqrt{\frac{2}{3}}$

So: $C=4a\int_0^{\pi/2}\sqrt{1-e^2\sin^2\theta}d\theta=4\sqrt{3}\int_0^{\pi/2}\sqrt{1-\tfrac{2}{3}\sin^2\theta}d\theta$

(note that *a* is the length of the *semimajor* axis)