MA262 **Calculus II: Notes for March 18**
Spring 2008 Kallfelz

- *TAYLOR SERIES METHODS: SOME GENERAL METHODS CONCERNING STRATEGY AND TACTICS*

  Okay, so now that we've built up a tool-chest consisting of McClaurin series representations for elementary functions like e^x, sin$x$, cos$x$, arctan$x$, arcsin$x$, (of which are summarized in the table, p. 638, text). We've also been exposed to the two very useful strategies: the 'Geometric Series Trick' (or **GST:** summarized in March 4th notes[1], pp. 10-13) as well as the 'Binomial Series Trick' (or **BST:** summarized in March 6th notes[2], pp. 3-11). In addition, substitution-methods summarized In Thm 10.22, text (p. 627) enables one in certain cases to substitute more complicated arguments within the framework of the aforementioned tricks, as well as the McClaurin Series representations.[3] Also, differentiation/integration techniques as summarized in Thm 10.21, text (p. 627) were applied to **GST** and **BST**, to build the McClaurin Series for arctan$x$, and arcsin$x$, respectively.[4] Moreover, we saw how to apply to the **GST** representations of Taylor series centered at points other than zero, as in the case of ln$x$ (centered at c = 1), as shown in p. 628, text, as well as in the more general case of finding the Taylor series centered at any point c for the function $f(x)=\frac{a}{b+kx}$ in Example 1, pp. 10-11, March 6th notes.[5]

- **So a natural question comes up:** Given the arsenal of expressions and methods, which mostly apply to *McClaurin Series* representations, i.e. power series centered at the point c = 0, how can they be employed to construct more general *Taylor Series*, i.e. power series centered at *any* point c?

  A common error is to assume willy-nilly that whatever a function's McClaurin Series representation is, then its Taylor Series is simply found by replacing the x^k terms in the series with the expression: $(x - c)^k$. One is often seduced by such

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[1] http://www.glue.umd.edu/%7Ewkallfel/MA261-2/Mar4.pdf
[2] http://www.glue.umd.edu/%7Ewkallfel/MA261-2/Mar6.pdf
[3] As some of the examples we've looked already show: For example, given $f(x)=\frac{1}{1-x}=\sum_{k=0}^{\infty}x^k$ for all x : $-1 < x < 1$, by virtue of the Thm 10.22 we can say: $\frac{1}{1+x^3}=f(-x^3)=\sum_{k=0}^{\infty}(-1)x^{3k}$ for all x: $-1 < x^3 < 1$ $\Rightarrow 1 < x < 1$. Or to name another example, given: $\cos x=\sum_{j=0}^{\infty}\frac{(-1)^k}{(2j)!}x^{2j}$ for all x: $-\infty < x < \infty$, by virtue of the substitutions implied in Thm 10.22 we can write: $\cos(2x+1)=\sum_{j=0}^{\infty}\frac{(-1)^k}{(2j)!}(2x+1)^{2j}$
[4] As summarized respectively in pp. 12-13 March 4th notes (http://www.glue.umd.edu/%7Ewkallfel/MA261-2/Mar4.pdf) and pp 6-9, March 6 notes (http://www.glue.umd.edu/%7Ewkallfel/MA261-2/Mar6.pdf).
[5] http://www.glue.umd.edu/%7Ewkallfel/MA261-2/Mar6.pdf

reasoning from the substitution-techniques as discussed above (summarized in Thm 10.21).

For instance, on the one hand it is *correct* to conclude that: $e^{(x-3)}=\sum_{k-0}^{\infty}\frac{(x-3)^k}{k!}$ based on the McClaurin Series representation: $e^x=\sum_{k-0}^{\infty}\frac{x^k}{k!}$, because of the substitution-techniques for power-series, ***on the other hand it is certainly false to conclude that*** $\sum_{k-0}^{\infty}\frac{(x-3)^k}{k!}$ ***is the Taylor Series for*** e^x***, centered at c =*** **3!** The reason why this conclusion is patently false is summarized below in two simple reasons:

1. The obvious reason why the above is false is that $e^{(x-3)}=\sum_{k-0}^{\infty}\frac{(x-3)^k}{k!}$, which is power series representation for the *function*: $g(x)=e^{(x-3)}$. To go ahead and call that also a power series (which according to Thms 10.23, 10.24, *is* the Taylor series) centered at c = 3 for the *function* $f(x)=e^x$ is to confuse functions. It would be like saying: $e^{(x-3)}=e^x$, which is obviously incorrect.
2. The second (perhaps less obvious but more precise) reason why such reasoning proves false is because one is presupposing that the McCluarin coefficients $\frac{1}{k!}f^{(k)}(0)$ have the same value for *any* value of c, i.e. to assume that: $\frac{1}{k!}f^{(k)}(0)=\frac{1}{k!}f^{(k)}(c)$ for all k and any c. But already in Calculus I you realized that there are only very trivial cases of functions which behave like that (i.e. functions whose values of derivatives for any order k remain the same regardless of what point c they're evaluated on)—namely the *constant* functions whose derivatives (of any order k) are of course zero everywhere. But for anything besides that trivial case, it's certainly wrong to assume that $\frac{1}{k!}f^{(k)}(0)=\frac{1}{k!}f^{(k)}(c)$ for all k and any c for *any* function $f(x)$.[6]

In addition, Corey Heiges pointed out in class today yet another major strategic error inherent in the false reasoning in the example above of assuming from:

$$f(x)=e^x=\sum_{k=0}^{\infty}\frac{1}{k!}f^{(k)}(0)x^k=\sum_{k=0}^{\infty}\frac{1}{k!}x^k ,$$

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[6] Counterexamples are easy to come up with. Consider $f(x)=x^2$. Then $f'(x)=2x$ which means that for the case k = 1 and c = 1, $f^{(1)}(0)=0\neq f^{(1)}(1)=2$

that: $$f(x)=e^x=\sum_{k=0}^{\infty}\tfrac{1}{k!}f^{(k)}(3)(x-3)^k=\sum_{k=0}^{\infty}\tfrac{1}{k!}(x-3)^k$$

One is in essence substituting *selectively* (i.e. not *consistently*). One can see immediately the lack of follow-through in the above error: by keeping the expansion coefficients the same (i.e. the coefficients $\tfrac{1}{k!}f^{(k)}(0)$) in both cases, one merely performed a substitution on only one part of the series: $x^k \mapsto (x-3)^k$, but forgot to follow through with the other part of the series, i.e. forgot to follow though on the substitution: $\tfrac{1}{k!}f^{(k)}(0)\mapsto\tfrac{1}{k!}f^{(k)}(3)$. The analogy Corey adopted was from the case of u-substitutions in integration: substituting the integrand $f(x)$ with a u-composition $f(u)$ but forgetting to compensate dx with the appropriate du term.

So now we see from the above instance the *wrong* way to construct general Taylor Series building upon the toolchest we have so far, the question naturally comes up: what is the *right* way? Answer: the "trick of 0": work with the original function $f(x)$ by applying the trick of 0 to it: I.e., it's certainly obviously correct to say: $f(x)=f(x+c-c)=f(c+(x-c))$ for *any* c. Now the term in the parenthesis $(x-c)$ is the term on can effectively substitute into a McClaurin Series represention. **However, please note: this is the McClaurin Series for $f(c+u)$ (where $u=x-c$), and <u>not</u> the McCluarin Series for just $f(x)$!** As some of the examples indicate below, sometimes the connection between $f(x)$ and $f(c+u)$ can get rather messy.

- <u>Example 1:</u> Find the Taylor Series representation for $f(x)=e^x$ (i.e. its power-series representation centered at any point c) from its McCluarin Series $f(x)=e^x=\sum_{k=0}^{\infty}\tfrac{1}{k!}x^k$.

  Answer: Using the 'trick of 0': $f(x)=e^x=e^{x+c-c}=e^c e^{(x-c)}$. Hence we can use the McClaurin Series representation for the term: $e^u=\sum_{k=0}^{\infty}\tfrac{1}{k!}u^k$, where: $u=(x-c)$

  $$f(x)=e^x=e^{x+c-c}=e^c e^{(x-c)}=e^c\sum_{k=0}^{\infty}\tfrac{1}{k!}(x-c)^k=\sum_{k=0}^{\infty}\tfrac{1}{k!}e^c(x-c)^k$$

- **Note1:** Observe how the Taylor Coefficients changed their value: $\tfrac{1}{k!}f^{(k)}(0)=\tfrac{1}{k!}\mapsto\tfrac{1}{k!}f^{(k)}(c)=\tfrac{1}{k!}e^c$. This should confirm with your intuitions concerning the exponential function, since: $\frac{d^k}{dx^k}e^x=e^x$, for all k. So certainly $f^{(k)}(c)=e^c$.
- **Note2:** We can always check to see that the Taylor Series representation should reduce to the McClaurin Series representation in the special case $c = 0$:

$$\sum_{k-0}^{\infty}\tfrac{1}{k!}e^c(x-c)^k\xrightarrow{c=0}\sum_{k=0}^{\infty}\tfrac{1}{k!}e^0(x-0)^k=\sum_{k=0}^{\infty}\tfrac{1}{k!}x^k$$

- Example 2: Find the Taylor Series representation for $f(x) = \sin x$ (i.e. its power-series representation centered at any point c) from its McCluarin Series $f(x) = \sin x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1}$.

  Answer: Using the ‘trick of 0’:

$$\sin x = \sin(x + c - c) = \sin(c + (x - c)) = \sin c \cos(x - c) + \cos c \sin(x - c)$$
$$= \sin c \cos u + \cos c \sin u$$
(where $u = (x - c)$)

  Hence inserting the McClaurin Series:

$$\sin x = \sin c \cos u + \cos c \sin u = \sin c \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} u^{2k} + \cos c \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} u^{2k+1}$$
$$= \sin c \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} (x - c)^{2k} + \cos c \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} (x - c)^{2k+1}$$
$$= \sum_{k=0}^{\infty} (-1)^k \left[\frac{\sin c}{(2k)!} (x - c)^{2k} + \frac{\cos c}{(2k+1)!} (x - c)^{2k+1} \right]$$
$$= \sum_{k=0}^{\infty} (-1)^k \left[\frac{\sin c}{(2k)!} + \frac{\cos c}{(2k+1)!} (x - c) \right] (x - c)^{2k}$$

  …whose Taylor coefficients are now certainly a far cry from the McCluarin coefficients!

- **Note 3:** Nevertheless, one can check the above result and see that it collapses back down to the usual McClaurin series representation for $c = 0$:

$$\sum_{k=0}^{\infty} (-1)^k \left[\frac{\sin 0}{(2k)!} + \frac{\cos 0}{(2k+1)!} (x - 0) \right] (x - 0)^{2k} = \sum_{k=0}^{\infty} (-1)^k \left[0 + \frac{x}{(2k+1)!} \right] x^{2k} = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!}$$

- **Note 4:** Moreover, one can check the above result and see that it collapses back down to the usual McClaurin series representation for cos$x$ in the case $c = {}^{\pi}/_{2}$, since: $\sin\left(x + \frac{\pi}{2}\right) = \cos x$

$$\sum_{k=0}^{\infty} (-1)^k \left[\frac{\sin\frac{\pi}{2}}{(2k)!} + \frac{\cos\frac{\pi}{2}}{(2k+1)!} \left(x - \frac{\pi}{2}\right) \right] \left(x - \frac{\pi}{2}\right)^{2k} = \sum_{k=0}^{\infty} (-1)^k \left[\frac{1}{(2k)!} + \frac{0}{(2k+1)!} \left(x - \frac{\pi}{2}\right) \right] \left(x - \frac{\pi}{2}\right)^{2k}$$
$$= \sum_{k=0}^{\infty} (-1)^k \frac{1}{(2k)!} \left(x - \frac{\pi}{2}\right)^{2k} = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} u^{2k} = \cos u$$

  The necessity of the u-term appears since the series is *centered* at $c = {}^{\pi}/_{2}$, which is equivalent to the McCluarin representation for cosine at $u = (x - {}^{\pi}/_{2})$.

- Example3: Find the Taylor series representation for $f(x)=\dfrac{a}{b+kx}$ at any point *c*, using the **GST** (whose geometric series representation is the McCluarin Series)

  Answer: The details (as mentioned obliquely in the first page of these notes) are summarized in pp. 10-11, March 4th notes[7]. To check the answer (i.e. show that the Taylor series in page 11 reduces to the McCluarin series) observe:

$$f(x)=\frac{a}{b+kx}=\tfrac{a}{b+kc}\sum_{j=0}^{\infty}(-1)^j\frac{k^j}{(b+kc)^j}(x-c)^j \xrightarrow{c=0} \tfrac{a}{b+0}\sum_{j=0}^{\infty}(-1)^j\frac{k^j}{(b+0)^j}(x-0)^j$$
$$=\tfrac{a}{b}\sum_{j=0}^{\infty}\frac{k^j}{b^j}x^j$$

  ...which of course reduces to the result on page 10 in the March 4th notes.

- Example 4: (#7, §10.9) Find the Taylor Series (and its interval of convergence) for the function $f(x)=\dfrac{1}{2x-5}$, centered at $c = -3$.

  Answer: First apply the **GST** to *f*(*x*) to find its McClaurin Series:

$$f(x)=\frac{1}{2x-5}=-\tfrac{1}{5}\left[\frac{1}{1-\frac{2}{5}x}\right]=-\tfrac{1}{5}\sum_{k=0}^{\infty}\left(\tfrac{2}{5}x\right)^k=-\tfrac{1}{5}\sum_{k=0}^{\infty}\tfrac{2^k}{5^k}x^k$$

  Now apply the "Trick of 0":

$$f(x+3-3)=\frac{1}{2(x+3-3)-5}=-\tfrac{1}{5}\left[\frac{1}{1-\frac{2}{5}(x+3-3)}\right]=-\tfrac{1}{5}\left[\frac{1}{1-\frac{2}{5}(x+3)+\frac{6}{5}}\right]$$
$$=-\tfrac{1}{5}\left[\frac{1}{\frac{11}{5}-\frac{2}{5}(x+3)}\right]=-\left[\frac{1}{11-2(x+3)}\right]=-\tfrac{1}{11}\left[\frac{1}{1-\frac{2}{11}(x+3)}\right]=-\tfrac{1}{11}\left[\frac{1}{1-u}\right]$$
$$=-\tfrac{1}{11}\sum_{k=0}^{\infty}u^k=-\tfrac{1}{11}\sum_{k=0}^{\infty}\left[\tfrac{2}{11}(x+3)\right]^k=-\tfrac{1}{11}\sum_{k=0}^{\infty}\tfrac{2^k}{11^k}(x+3)^k$$

  To find its interval of convergence, the series converges provided $|u| < 1$, i.e.:

$$-1<u<1\Rightarrow -1<\tfrac{2}{11}(x+3)<1\Rightarrow -\tfrac{11}{2}<x+3<\tfrac{11}{2}\Rightarrow -\tfrac{11}{2}-3<x<\tfrac{11}{2}-3$$
$$\Rightarrow -\tfrac{17}{2}<x<\tfrac{5}{2}$$
$$\text{Hence: } I_{-3}^{11/2}=\left(-3-\tfrac{11}{2},-3+\tfrac{11}{2}\right)=\left(-\tfrac{17}{2},\tfrac{5}{2}\right)=\left\{x\middle|-\tfrac{17}{2}<x<\tfrac{5}{2}\right\}$$

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[7] http://www.glue.umd.edu/%7Ewkallfel/MA261-2/Mar4.pdf

- **Note 5:** As depicted in the above notation $I_{-3}^{11/2}$, it's helpful to have the information concerning the center (c = -3) and radius (R = $^{11}/_{2}$) of the interval of convergence as well as its endpoints.
- **Note 6:** Since the problem asked specifically for the Taylor Series centered at c = -3, one can't apply a checking procedure to see if we get a reduction to McClaurin series (i.e., one can't simply replace -3 with 0, since the whole point is that the Taylor coefficients likewise depend on c = -3! To think otherwise would be to commit the error discussed in page 2 above). However one can apply the general procedure (for *any c*) and show that it reduces to the above two cases (c = -3, c = 0):

$$f(x+c-c)=\frac{1}{2(x+c-c)-5}=-\tfrac{1}{5}\left[\frac{1}{1-\frac{2}{5}(x+c-c)}\right]=-\tfrac{1}{5}\left[\frac{1}{1-\frac{2}{5}(x-c)-\frac{2}{5}c}\right]$$

$$=-\tfrac{1}{5}\left[\frac{1}{1-\frac{2}{5}c-\frac{2}{5}(x-c)}\right]=-\tfrac{1}{5}\left[\frac{1}{\frac{5-2c}{5}-\frac{2}{5}(x-c)}\right]=-\left[\frac{1}{(5-2c)-2(x-c)}\right]=$$

$$-\tfrac{1}{(5-2c)}\left[\frac{1}{1-\frac{2}{(5-2c)}(x-c)}\right]=-\tfrac{1}{(5-2c)}\left[\frac{1}{1-u}\right]=-\tfrac{1}{(5-2c)}\sum_{k=0}^{\infty}u^{k}=-\tfrac{1}{(5-2c)}\sum_{k=0}^{\infty}\left[\frac{2(x-c)}{5-2c}\right]^{k}$$

$$=-\tfrac{1}{(5-2c)}\sum_{k=0}^{\infty}\tfrac{2^{k}}{(5-2c)^{k}}(x-c)^{k}$$

Which in the case c = 0:

$$-\tfrac{1}{(5-2c)}\sum_{k=0}^{\infty}\tfrac{2^{k}}{(5-2c)^{k}}(x-c)^{k}\xrightarrow{c=0}-\tfrac{1}{(5-0)}\sum_{k=0}^{\infty}\tfrac{2^{k}}{(5-0)^{k}}(x-0)^{k}=-\tfrac{1}{5}\sum_{k=0}^{\infty}\tfrac{2^{k}}{5^{k}}x^{k}$$

…reduces to our friend the McCluarin Series representation depicted in page 5 above.

And in the case c = -3:

$$-\tfrac{1}{(5-2c)}\sum_{k=0}^{\infty}\tfrac{2^{k}}{(5-2c)^{k}}(x-c)^{k}\xrightarrow{c=-3}-\tfrac{1}{(5+6)}\sum_{k=0}^{\infty}\tfrac{2^{k}}{(5+6)^{k}}(x--3)^{k}=-\tfrac{1}{11}\sum_{k=0}^{\infty}\tfrac{2^{k}}{11^{k}}(x+3)^{k}$$

…which reduces the previous answer as well

- Example 5: Find the Taylor Series for arctan$x$ given its McClaurin Series

$$\arctan x=\sum_{k=0}^{\infty}(-1)^{k}\frac{x^{2k+1}}{2k+1}\qquad\text{converging on } [-1\ ,\ 1]$$

Answer: Here it wouldn't prove helpful to apply the trick of 0 to arctan$x$ itself, since there exists no simple 'addition formula' (contrary to the case discussed in Example 2, page 4 aobve) to isolate terms like arctan(x – c). Hence it's best to proceed using the derivative of arctan$x$:

$$\tfrac{d}{dc}\arctan x = \frac{1}{1+x^2} \xrightarrow{GST} \sum_{k=0}^{\infty}(-1)^k x^{2k}$$

Using the trick of 0:

$$f(x+c-c) = f(c+(x-c)) = \frac{1}{1+[c+(x-c)]^2} = \frac{1}{1+c^2+2c(x-c)+(x-c)^2}$$

$$= \frac{1}{\left[1+2cx-c^2\right]+(x-c)^2} = \tfrac{1}{(1+2cx-c^2)}\left[\frac{1}{1+\frac{(x-c)^2}{1+2cx-c^2}}\right] = \tfrac{1}{(1+2cx-c^2)}\left[\frac{1}{1+\left(\frac{x-c}{\sqrt{1+2cx-c^2}}\right)^2}\right]$$

$$= \tfrac{1}{(1+2cx-c^2)}\left[\frac{1}{1+u^2}\right] = \tfrac{1}{(1+2cx-c^2)}\sum_{k=0}^{\infty}(-1)^k u^{2k} = \tfrac{1}{(1+2cx-c^2)}\sum_{k=0}^{\infty}(-1)^k\left[\frac{x-c}{\sqrt{1+2cx-c^2}}\right]^{2k}$$

So:

$$\arctan x = \int \tfrac{1}{(1-2cx-c^2)}\sum_{k=0}^{\infty}(-1)^k\left[\frac{x-c}{\sqrt{1+2cx-c^2}}\right]^{2k} dx = \sum_{k=0}^{\infty}(-1)^k\int\left[\frac{x-c}{\left(1+2cx-c^2\right)^{3/2}}\right]^{2k} dx$$

…which (for obvious reasons) the terms of series will remain in the above integral form. However, in the special case $c = 0$ note how the answer reduces to the McClaurin series:

$$\sum_{k=0}^{\infty}(-1)^k\int\left[\frac{x-c}{\left(1+2cx-c^2\right)^{3/2}}\right]^{2k} dx \xrightarrow{c=0} \sum_{k=0}^{\infty}(-1)^k\int\left[\frac{x-0}{(1+0+0)^{3/2}}\right]^{2k} dx = \sum_{k=0}^{\infty}(-1)^k\int x^{2k}dx$$

$$= \sum_{k=0}^{\infty}(-1)^k\frac{x^{2k+1}}{2k+1}$$

…The above five examples should give you a good enough sense of how useful the McClaurin series lists are, along with the associated tricks like the **GST** and **BST** and substitution methods, to construct Taylor Series for a large garden variety of functions without having to painstakingly derive the Taylor coefficients $\tfrac{1}{k!}f^{(k)}(c)$ from scratch (or from first principles). Nevertheless, as some of the examples show, sometimes the algebra can be get messy. Nevertheless, it's a comparatively *far* more efficient procedure than deriving the Taylor coefficients $\tfrac{1}{k!}f^{(k)}(c)$ from scratch (which should only be used as a last resort).

Of course, everything mentioned here so far is just one facet of series methods—specifically the strategy of deriving a Taylor Series without having to calculates its coefficients. Other applications of Taylor Series Methods include estimating integrals that are otherwise difficult if not impossible to evaluate directly, based on all the techniques of integration you've learned thus far. For instance:

- Example (#43, §10.10) Estimate with an error < 10^{-4} the integral $\int_{0.1}^{0.3} \sqrt{1+x^3}\,dx$

1. Using **BST** find the McClaurin series for $f(x) = \left(1+x^3\right)^{1/2}$:

Answer

$$(1+u)^{1/2} = \sum_{k=0}^{\infty} \binom{1/2}{k} u^k = \sum_{k=0}^{\infty} \tfrac{1}{k!}\left[\tfrac{1}{2}\cdot\left(\tfrac{1}{2}-1\right)\cdot\ldots\cdot\left(\tfrac{1}{2}-k+1\right)\right]u^k$$

$$= \sum_{k=0}^{\infty} \tfrac{1}{k!}\left[\tfrac{1}{2}\cdot\tfrac{(1-2)}{2}\cdot\ldots\cdot\tfrac{(1-2k+2)}{2}\right]u^k = \sum_{k=0}^{\infty} \tfrac{1}{k!}(-1)^{k-1}\left[\frac{1\cdot 1\cdot 3\cdot 5\cdot\ldots\cdot(2k-3)}{2^k}\right]u^k$$

$$= \sum_{k=0}^{\infty} \tfrac{1}{k!}(-1)^{k-1}\,\tfrac{1}{2}\left[\frac{1\cdot 3\cdot 5\cdot\ldots\cdot(2(k-1)-1)}{2^{k-1}}\right]u^k = \sum_{k=0}^{\infty} \tfrac{1}{2k}(-1)^{k-1}\left[\frac{1\cdot 3\cdot 5\cdot\ldots\cdot(2(k-1)-1)}{2^{k-1}(k-1)!}\right]u^k$$

$$= \sum_{k=0}^{\infty} \tfrac{(-1)^{k-1}}{2k}\left[\frac{1\cdot 3\cdot\ldots\cdot(2(k-1)-1)}{[2(k-1)]\cdot[2(k-2)]\cdot\ldots\cdot[4]\cdot[2]}\right]u^k = \sum_{k=0}^{\infty} \tfrac{(-1)^{k-1}}{2k}\left[\frac{(2k-2)(2k-3)(2k-4)\cdot\ldots\cdot 2\cdot 1}{[2(k-1)]^2[2(k-2)]^2\cdot\ldots\cdot 4^2\cdot 2^2}\right]u^k$$

$$= \sum_{k=0}^{\infty} \tfrac{(-1)^{k-1}}{2k}\cdot\frac{2(k-1)!}{\left(2^{k-1}\cdot(k-1)!\right)^2}u^2 = \sum_{k=0}^{\infty} \tfrac{(-1)^{k-1}}{2k}\cdot\frac{2(k-1)!}{\left(2^{k-1}(k-1)!\right)^2}x^{3k}$$

2. Integrating:

$$\int_{0.1}^{0.3} \sqrt{1+x^3}\,dx = \sum_{k=0}^{\infty} \tfrac{(-1)^{-1k}}{2k}\cdot\frac{2(k-1)!}{\left(2^{k-1}(k-1)!\right)}\int_{0.1}^{0.3} x^{3k}\,dx = \sum_{k=0}^{\infty} \tfrac{(-1)^{k-1}}{2k}\cdot\tfrac{2(k-1)!}{\left(2^{k-1}\cdot(k-1)!\right)^2}\cdot\frac{\left(\tfrac{3}{10}\right)^{3k+1}-\left(\tfrac{1}{10}\right)^{3k+1}}{(3k+1)}$$

$$= \sum_{k=0}^{\infty} \tfrac{(-1)^{k-1}}{2k}\cdot\tfrac{2(k-1)!}{\left(2^{k-1}(k-1)!\right)^2}\cdot\tfrac{3^{3k+1}-1}{(3k+1)\cdot 10^{3k+1}}$$

(Note that the limits of the integral a = 0.1, b = 0.3, fall well within the interval of absolute convergence: -1 < x < 1)

Now, left in purely binomial coefficient form, the McClaurin Series above can also be depicted more simply schematically[8] as:

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[8] Although then, in terms of practical computation, the binomial coefficients would require a 'macro' or subroutine to be evaluated.

$$\int \sqrt{1+x^3}\,dx = \sum_{k=0}^{\infty}\binom{1/2}{k}\int x^{3k}\,dx = \sum_{k=0}^{\infty}\binom{1/2}{k}\cdot\frac{x^{3k+1}}{3k+1}$$

And according to Taylor's Thm, the above can be re-written as:

$$\int \sqrt{1+x^3}\,dx = \sum_{k=0}^{\infty}\binom{1/2}{k}\int x^{3k}\,dx = \sum_{k=0}^{\infty}\binom{1/2}{k}\cdot\frac{x^{3k+1}}{3k+1} = \sum_{k=0}^{n}\binom{1/2}{k}\cdot\frac{x^{3k+1}}{3k+1} + R_n(x)$$

Where: $R_n(x) = \dfrac{f^{(n+1)}(z)}{(n+1)!}x^{n+1}$ for any z in the interval (-1,1)

Now in order to find the minimum n in the above integral estimate, we must ensure that the maximum of $R_n(x) = \dfrac{f^{(n+1)}(z)}{(n+1)!}x^{n+1}$ for any z and x in the interval (0.1, 0.3) not exceed 10^{-4}. For the part of the error formula involving x^{n+1}, this is straightforward enough, for x^{n+1} is monotone increasing the in the interval (0.1, 0.3). Hence the maximum value = $(0.3)^{n+1} = \frac{3^{n+1}}{10^{n+1}}$. To maximize the $(n + 1)^{\text{th}}$ derivative of $f(x) = \int\sqrt{1+x^3}\,dx$ is equivalent to maximizing the n^{th} derivative of $g(x) = \sqrt{1+x^3}$. No matter how complicated the numerator term gets for such derivatives (of order n) the denominator term will have the form: $(1+x^3)^{n/2}$. Now it's certainly true that: $1 = \dfrac{1}{(1+0^3)^{n/2}} > \dfrac{1}{(1+x^3)^{n/2}}$ for all x in the interval (0.1, 0.3). Hence, without further ado, we can look at the n^{th} derivative of $g(x) = \sqrt{1+x^3}$ *at the origin. But this is just the nth McClaurin Coefficient in the, which we already found in step 1 above!* Hence we need to solve the inequality:

$$\frac{(2n)!}{(2^n n!)^2}\cdot\frac{3^{n+1}}{10^{n+1}} < 10^{-4},$$

which can be done via trial and error. Using EXCEL:

n	***Rn***
1	0.045
2	0.010125
3	0.002531
4	0.000664
5	0.000179
6	**4.93E-05**

Hence $\displaystyle\int_{0.1}^{0.3}\sqrt{1+x^3}\,dx \approx \sum_{k=0}^{6}\frac{(-1)^{k-1}}{2k}\cdot\frac{2(k-1)!}{\left(2^{k-1}(k-1)!\right)^2}\cdot\frac{3^{3k+1}-1}{(3k+1)\cdot 10^{3k+1}}$ within an error < 10^{-4}