TUESDAY - JAN. 22ND

MA262 SELECTED ANSWERS, FROM EXERCISES ASSIGNED FOR WEEKS OF JAN. 7TH, JAN. 14TH

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§9.1, TEXT (pp ~~497~~ 497-498)

7.) $\int \frac{t^2-3}{-t^3+9t+1}\,dt = \int (-t^3+9t+1)^{-1}(t^2-3)\,dt$

- STRATEGY: Note that the $(\dots)^{-1}$ term is a deg. 3 polynomial, and that the remaining terms are deg. 2 polynomials. <u>So a u-substitution will work</u> (where $u(x)$ is what's inside the $(\dots)^{-1}$ term)

$u(t) = -t^3+9t+1$

$\frac{du}{dt} = -3t^2+9$

... Compare to $(t^2-3)dt$: $\frac{du}{dt} = -3t^2+9 = -3(t^2-3)$

$\therefore\ du = -3(t^2-3)dt$

$\boxed{\therefore\ -\frac{1}{3}du = (t^2-3)dt}$

$\therefore\ \int(-t^3+9t+1)^{-1}(t^2-3)dt = \int u^{-1}(-\frac{1}{3}du) = -\frac{1}{3}\int u^{-1}du = -\frac{1}{3}\{\ln|u|+c\}$

SINCE $u(t) = -t^3+9t+1 \Rightarrow -\frac{1}{3}\{\ln|u|+c\} = \boxed{-\frac{1}{3}\ln|-t^3+9t+1| + C}$

- <u>Note 1</u>: Since C is undetermined, then distributing across by $-\frac{1}{3}$ makes no difference, i.e. we can still say (logically) that "$-\frac{1}{3}C = C$," even if (algebraically) it looks incorrect. This is only because C is <u>undetermined</u>, (i.e. we don't know, and don't care, what its actual value may be, given some initial condition(s). (Such initial conditions of course aren't supplied in this problem, hence it's an <u>indefinite</u> integral.

- <u>Note 2</u>: The above answer of course can be written in other forms as well, for example: $-\frac{1}{3}\ln|-(t^3+9t+1)|$

$= -\frac{1}{3}\ln|-(t^3-9t-1)| = -\frac{1}{3}\ln|t^3-9t^2-1|$

(using property of absolute value)

Also: $\Rightarrow -\frac{1}{3}\ln|t^3-9t-1| = \ln(|t^3-9t-1|^{-1/3}) = \ln\left(\left|\frac{1}{\sqrt[3]{t^3-9t-1}}\right|\right)$

(USING EXPONENTIAL PROPERTY OF ln, and definition of rational exponents.)

12) $\int \frac{x^2}{x-1}\,dx$

STRATEGY: This is a rational expression, with (degree of numerator) > (degree of denominator). So simplify first by polynomial division:

$$\begin{array}{r} x+1 \\ x-1 \overline{)\,x^2} \\ -(x^2-x) \\ \hline x \\ -(x-1) \\ \hline 1 \end{array} \leftarrow \text{Remainder}$$

So: $\frac{x^2}{x-1} = x+1+\frac{1}{x-1}$

$\therefore \int \frac{x^2}{x-1}\,dx = \int (x+1+\frac{1}{x-1})\,dx$

$$\boxed{= \frac{1}{2}x^2 + x + \ln|x-1| + C}$$

• Note 1: $\int [x+1+\frac{1}{x-1}]\,dx = \int x\,dx + \int 1\cdot dx + \int (x-1)\,dx$

do a u-substitution to obtain $\int \frac{du}{u}$

(13) $\int t\sin t^2\,dt$ STRATEGY: The argument of sin is a degree 2 polynomial. The remaining terms are degree 1 so a u-substitution will work:

$\int \sin(t^2)\,t\,dt = \int \sin(u)\,t\,dt$ where $u = t^2$

$\therefore \frac{du}{dt} = 2t\,dt$

$\therefore t\,dt = \frac{1}{2}du$

$\Rightarrow \int \sin(u)\,t\,dt = \int \sin(u)\cdot\frac{1}{2}du = \frac{1}{2}\int \sin u\,du$

$= \frac{1}{2}\{-\cos u\} + C = \boxed{-\frac{1}{2}\cos(t^2) + C}$

17) $\int \frac{(1+e^t)^2}{e^t}\,dt = \int (1+e^t)^2 e^{-t}\,dt$

STRATEGY: Note that here, the $(\ldots)^2$ involves e^t. But the remaining terms involve e^{-t}. So a u-substitution cannot work (since $\frac{d}{dt}e^t = e^t$, not e^{-t}). So your remaining choice is to foil the $(\ldots)^2$ term and simplify:

$$(1+e^t)^2 e^{-t} = (1+2e^t+e^{2t})e^{-t} = (e^{-t}+2+e^t)$$

$$\therefore \int (1+e^t)^2 e^{-t}\,dt = \int (e^{-t}+2+e^t)\,dt = \int e^{-t}\,dt + 2\int dt + \int e^t\,dt$$

$$= \boxed{-e^{-t}+2t+e^t+C}$$

Note: The first integral, of course, was performed using a u-substitution:

$$\int e^{-t}\,dt \qquad u=-t \Rightarrow du=-dt \Rightarrow dt=-du \Rightarrow \int e^{-t}\,dt = \int e^u(-du) = -\int e^u\,du = -e^u = -e^{-t}$$

21) $\int \frac{2}{(e^{-x}+1)}\,dx = 2\int (e^{-x}+1)^{-1}\,dx$

STRATEGY: Note that the $(\ldots)^{-1}$ term involves an e^{-x} term, but the remaining terms involve no such term at all! Also, "foiling" (as in #17) above) won't do any good either, since the $(e^{-x}+1)^{-1}$ term is already expressed in simplest possible form.

Note, however: $\frac{1}{e^{-x}+1} = \frac{1}{e^{-x}(1+e^x)} = \frac{e^x}{(1+e^x)}$

(which can also be achieved using the "trick of $1 = \frac{e^x}{e^x}$"
i.e., $\frac{1}{e^{-x}+1}\cdot\frac{e^x}{e^x} = \frac{e^x}{(1+e^x)}$

So: $2\int (e^{-x}+1)^{-1}\,dx = 2\int \frac{e^x}{e^x+1}\,dx$, which can be solved using the following u-substitution: $u(x) = e^x+1$

$$\frac{du}{dx} = e^x \Rightarrow e^x\,dx = du$$

Hence: $2\int \frac{e^x\,dx}{e^x+1} = 2\int \frac{du}{u} = 2\ln|u| + C$

$= 2\ln|e^x+1| + C$

• Note, however, that $e^x+1 > 0$ for all x (real). $\therefore$ we can simplify by writing: $2\ln|e^x+1| + C = \boxed{2\ln(e^x+1) + C}$

(which, of course, can also be written as: $\ln(e^x+1)^2 + C$)

23) $\int \frac{1}{1-\cos x}\,dx = \int (1-\cos x)^{-1}\,dx$

STRATEGY: The $(\ldots)^{-1}$ involves a $\cos x$ term, but there are no sinusoidal terms remaining. So a simple u-substitution won't work! Instead, multiply by the complement $(1+\cos x)$ using "trick of 1" and simplify via a Pythagorean Identity:

$$\int (1-\cos x)^{-1}\,dx = \int \frac{1}{1-\cos x}\cdot\frac{1+\cos x}{1+\cos x}\,dx = \int \frac{1+\cos x}{1-\cos^2 x}\,dx$$

$$= \int \frac{1+\cos x}{\sin^2 x}\,dx = \int (\csc^2 x + \cot x \csc x)\,dx$$

$$= \int \csc^2 x\,dx + \int \csc x \cot x\,dx$$

$$\boxed{= -\cot x - \csc x + C}$$

(Recall: $\frac{d}{dx}\cot x = -\csc^2 x$ and $\frac{d}{dx}\csc x = -\csc x \cot x$)

31) $\int \frac{-1}{\sqrt{1-(2t-1)^2}}\,dt$

STEP 1: Let $u = 2t-1$, then $du/dt = 2 \Rightarrow du = 2dt \Rightarrow dt = \frac{1}{2}du$

$$= \int \frac{-1}{\sqrt{1-u^2}}\cdot\frac{1}{2}\,du = -\frac{1}{2}\int \frac{du}{\sqrt{1-u^2}} = -\frac{1}{2}\arcsin(u) + C$$

$$\boxed{= -\frac{1}{2}\arcsin(2t-1) + C}$$

<u>Note.</u>: For a review of derivatives and integrals of inverse trigonometric functions, and ~~der~~ integrals involving terms like $(a^2-u^2)^{-1/2}$, see §8.5, §8.6 text, as well as handouts posted on course website (mid-November 2007, Calc. I materials)

37) $\displaystyle\int \frac{\sec^2 x}{4+\tan^2 x}\,dx$

<u>STRATEGY</u>: Note that denominator term can be rewritten: $(4+\tan^2 x) = 2^2 + (\tan x)^2$. Note that $\frac{d}{dx}\tan x = \sec^2 x$
So using a u-substitution: $(u = \tan x)$

$$\int \frac{\sec^2 x}{4+\tan^2 x}\,dx = \int \frac{\sec^2 x}{2^2+(\tan x)^2}\,dx = \int \frac{du}{2^2+u^2}$$

$$= \frac{1}{2}\arctan\left(\frac{u}{2}\right) + C$$

$$\boxed{= \frac{1}{2}\arctan\left(\frac{\tan x}{2}\right) + C}$$

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§9.2 (p 506)

9) $\displaystyle\int x^3 \ln x\,dx$

<u>STRATEGY</u>: An integral of the form $\int u^n (\ln u)^m\,du$. Use integration by parts, where:

$$u = \ln x \qquad dv = x^3\,dx$$
$$du = \frac{dx}{x} \qquad v = \frac{1}{4}x^4$$

$$\therefore \int x^3 \ln x\,dx = \int u\,dv = uv - \int v\,du$$

$$= \frac{1}{4}x^4\ln x - \frac{1}{4}\int x^4\frac{dx}{x}$$

$$= \frac{1}{4}x^4\ln x - \frac{1}{4}\int x^3\,dx$$

$$= \frac{1}{4}x^4\ln x - \frac{1}{16}x^4 + C$$

$$\boxed{= \frac{x^4}{16}(4\ln x - 1) + C}$$

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11) $\int t \ln(t+1)\,dt$

STEP 1: $u(t) = t+1 \Rightarrow t = u-1 \Rightarrow dt = du$

$$\therefore \int t \ln(t+1)\,dt = \int (u-1)\ln u\,du = \int u \ln u - \int \ln u$$

(A) (B)

STEP 2: These two integrals can be computed using Integration by Parts:

(A) $\int u\ln u\,du \qquad U = \ln u \quad dV = u\,du$

$dU = du/u \quad V = \frac{1}{2}u^2$

$$\therefore \int u\ln u\,du = \int U\,dV = UV - \int V\,dU = \frac{1}{2}u^2\ln u - \int \frac{1}{2}u^2\,du/u$$

$$= \frac{1}{2}u^2\ln u - \frac{1}{2}\int u\,du = \frac{1}{2}u^2\ln(u) - \frac{1}{4}u^2 + C$$

(B) $\int \ln u\,du \qquad U = \ln u \quad dV = du$

$dU = du/u \quad V = u$

$$= \int U\,dV = UV - \int V\,dU = u\ln u - \int du = u\ln u - u$$

$$\therefore \text{(A)} - \text{(B)} = \frac{1}{2}u^2\ln u - \frac{1}{4}u^2 - (u\ln u - u) + C$$

$$= \ln u\left(\frac{1}{2}u^2 - u\right) + u - \frac{1}{4}u^2$$

$$= \ln|t+1|\left[\frac{1}{2}(t+1)^2 - (t+1)\right] + (t+1) - \frac{1}{4}(t+1)^2$$

$$= \ln|t+1|\left[\frac{1}{2}t^2 + \cancel{t} + \frac{1}{2} - \cancel{t} - 1\right] + (t+1) - \frac{1}{4}t^2 - \frac{1}{2}t - \frac{1}{4} + C$$

$$= \ln|t+1|\left[\frac{1}{2}t^2 - \frac{1}{2}\right] - \frac{1}{4}t^2 + \frac{1}{2}t + C - \frac{1}{4}$$

$$= \frac{1}{4}\ln|t+1|\left[2t^2 - 2\right] - \frac{1}{4}(t^2 - 2t) + C - \frac{1}{4}$$

$$= \frac{1}{4}\ln|t+1|\left[2(t^2-1)\right] + \frac{1}{4}(-t^2 + 2t) + C - 1/4$$

$$\boxed{= \frac{1}{4}\left\{\ln|t+1|\left[2(t^2-1)\right] + 2t - t^2\right\} + C}$$

Recall Note 1, page 1

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17) $\int \frac{x e^{2x}}{(2x+1)^2} dx$ Like in #11) A U-subst. must be performed first, to simplify the fractional term, prior to integrating by parts.

STEP 1 : Let $u = 2x+1 \Rightarrow du = 2dx \Rightarrow dx = \frac{1}{2} du$

Also : $x = \frac{1}{2}(u-1)$

$$\therefore \int \frac{x e^{2x}}{(2x+1)^2} dx = \int \frac{\frac{1}{2}(u-1)\, e^{(u-1)}}{u^2} \cdot \frac{1}{2} du$$

$$= \frac{1}{4} e^{-1} \int (u^{-1} - u^{-2}) e^u du = \frac{1}{4} e^{-1} \Big\{ \overset{Ⓐ}{\int u^{-1} e^u du} - \overset{Ⓑ}{\int u^{-2} e^u du} \Big\}$$

Consider Ⓐ First: $\int u^{-1} e^u du$. Let $U = u^{-1}$ $\quad dV = e^u du$
$dU = -u^{-2} du \quad V = e^u$

So: $\int u^{-1} e^u du = UV - \int V dU = u^{-1} e^u - \int (-u^{-2} du) e^u$

$$= \frac{1}{u} e^u + \int u^{-2} e^u du \quad + C$$

↑ Note that this is −Ⓑ, so they cancel!

$$\therefore \frac{1}{4} e^{-1} \Big\{ \frac{1}{u} e^u + \cancel{\int u^{-2} e^u du} - \cancel{\int u^{-2} e^u du} \Big\} + C$$

$$= \frac{1}{4} e^{-1} \Big\{ \frac{1}{(2x+1)} e^{(2x+1)} \Big\} = \boxed{\frac{e^{2x}}{4(2x+1)} + C}$$

21) $\int x \sqrt{x-1}\, dx$

STRATEGY: Like in the previous two cases, start off with a u-subst. to simplify the $\sqrt{\dots}$ term, i.e. let $u = x-1$

Then: $u = x-1 \Rightarrow x = u+1 \Rightarrow dx = du$

$$\int x\sqrt{x-1}\, dx = \int (u+1) u^{1/2} du = \int (u^{3/2} + u^{1/2}) du = \frac{2}{5} u^{5/2} + \frac{2}{3} u^{3/2} + C$$

$$= \frac{2}{5}(x-1)^{5/2} + \frac{2}{3}(x-1)^{3/2} + c = \frac{2}{15}(x-1)^{3/2}[3(x-1)+5] + C$$

$$= \frac{2}{15}(x-1)^{3/2}[3x-3+5] + C = \boxed{\frac{2}{15}(x-1)^{3/2}[3x+2] + C}$$

23) $\displaystyle\int \frac{x^2}{\sqrt{2+3x}}\,dx = \int x^2(2+3x)^{-1/2}\,dx$

<u>STRATEGY</u>: As in #21) : $u = 2+3x \Rightarrow du = 3dx \Rightarrow dx = \frac{1}{3}du$
and $x = \frac{1}{3}(u-2)$

$$\Rightarrow \int x^2(2+3x)^{-1/2}\,dx = \int \frac{1}{9}(u-2)^2 \cdot u^{-1/2} \cdot \frac{1}{3}\,du$$

$$= \frac{1}{27}\int (u-2)^2 u^{-1/2}\,du = \frac{1}{27}\int (u^2-4u+4)u^{-1/2}\,du$$

$$= \frac{1}{27}\int [u^{3/2} - 4u^{1/2} + 4u^{-1/2}]\,du$$

$$= \frac{1}{27}\left\{\frac{2}{5}u^{5/2} - \frac{8}{3}u^{3/2} + 8u^{1/2}\right\} + c$$

$$= \frac{u^{1/2}}{27}\left\{\frac{2}{5}u^2 - \frac{8}{3}u + 8\right\} + c$$

$$= \frac{u^{1/2}}{405}\left\{6u^2 - 40u + 120\right\} + c$$

$$= \frac{2u^{1/2}}{405}\left\{3u^2 - 20u + 60\right\} + c$$

$$= \frac{2}{405}\sqrt{2+3x}\left\{3(2+3x)^2 - 20(2+3x) + 60\right\} + c$$

$$= \frac{2}{405}\sqrt{2+3x}\left\{3(9x^2+12x+4) - 40 - 60x + 60\right\} + c$$

$$= \frac{2}{405}\sqrt{2+3x}\left\{27x^2 + 36x + 12 - 40 - 60x + 60\right\} + c$$

$$\boxed{= \frac{2}{405}\sqrt{2+3x}\left\{27x^2 - 24x + 32\right\} + c}$$

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27) $\int x\sec^2 x\,dx$

Let $u = x$ $\quad dv = \sec^2 x$

$du = dx$ $\quad v = \tan x$

$$\Rightarrow \int u\,dv = uv - \int v\,du = x\tan x - \int \tan x\,dx = x\tan x + \ln|\cos x| + C$$

Recall: $\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx$

So $u = \cos x$
$du = -\sin x$ $\Big\}$ $\Rightarrow \int -\frac{du}{u}$
$= -\ln|u|$

31) $\int \arctan x\,dx$

Let $u = \arctan x$ $\quad dv = dx$

$du = \frac{dx}{1+x^2}$ $\quad v = x$

$$= uv - \int v\,du = x\arctan x - \int \frac{x}{1+x^2}\,dx$$

$\Updownarrow$

Let $u = 1 + x^2$

$\therefore du = 2x\,dx \Rightarrow x\,dx = \frac{1}{2}du$

$\therefore \int \frac{x}{1+x^2}\,dx = \frac{1}{2}\int \frac{du}{u} = \frac{1}{2}\ln|u|$

$$= x\arctan x - \frac{1}{2}\ln(1+x^2) + C$$

since $1+x^2 > 0$, so simply by dropping the absolute value

SECTION 9.3 (P 516)

5) $\int \sin^5 x \cos^2 x\,dx = \int \sin x \sin^4 x \cos^2 x\,dx = \int \sin x\,(\sin^2 x)^2 \cos^2 x\,dx$

$= \int \sin x\,(1-\cos^2 x)^2 \cos^2 x\,dx = \int \sin x\,(1 - 2\cos^2 x + \cos^4 x)\cos^2 x\,dx$

$= \int \sin x\,[\cos^2 x - 2\cos^4 x + \cos^6 x]\,dx$

Let $u = \cos x \quad du = -\sin x\,dx$

$= \int -(du)\,[u^2 - 2u^4 + u^6] = \int [-u^2 + 2u^4 - u^6]\,du = -\frac{1}{3}u^3 + \frac{2}{5}u^5 - \frac{1}{7}u^7 + c$

$= \boxed{-\frac{1}{3}\cos^3 x + \frac{2}{5}\cos^5 x - \frac{1}{7}\cos^7 x + C}$

9) $\int \sin^4 \pi x\,dx \qquad$ Let $u = \pi x \quad du = \pi\,dx \Rightarrow dx = \frac{1}{\pi}\,du$

$= \frac{1}{\pi}\int \sin^4 u\,du = \frac{1}{\pi}\int (\sin^2 u)^2\,du = \frac{1}{\pi}\int \left(\frac{1}{2}(1-\cos 2u)\right)^2 du$

$= \frac{1}{4\pi}\int (1-\cos 2u)^2\,du = \frac{1}{4\pi}\int (1 - 2\cos 2u + \cos^2 2u)\,du$

$= \frac{1}{4\pi}\left\{u - \sin 2u + \int \cos^2 2u\,du\right\}$

Note: $\int -2\cos 2u\,du = \int -\cos w\,dw \quad$ where $w = 2u$
$\therefore dw = 2du$

$= \frac{1}{4\pi}\left\{u - \sin 2u + \int \frac{1}{2}(1+\cos 4u)\,du\right\}$

$= \frac{1}{4\pi}\left\{u - \sin 2u + \frac{1}{2}u + \frac{1}{8}\sin 4u + c\right\}$

$= \frac{1}{4\pi}\left\{\frac{3}{2}u - \sin 2u + \frac{1}{8}\sin 4u\right\} + C$

$= \frac{1}{4\pi}\left\{\frac{3}{2}\pi x - \sin 2\pi x + \frac{1}{8}\sin 4\pi x\right\} + C$

$= \boxed{\frac{1}{32\pi}\left\{12\pi x - 8\sin 2\pi x + \sin 4\pi x\right\} + C}$

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(13) $\int x\sin^2 x\,dx = \int x\cdot\frac{1}{2}(1-\cos 2x)\,dx = \frac{1}{2}\int x\,dx - \frac{1}{2}\int x\cos 2x\,dx$

$= \frac{1}{4}x^2 - \frac{1}{2}\int x\cos 2x\,dx$

$\rightarrow u = x \quad dv = \cos 2x\,dx$

$du = dx \quad v = \frac{1}{2}\sin 2x$

$\therefore \int x\cos 2x = \frac{1}{2}x\sin 2x - \frac{1}{2}\int \sin 2x\,dx$

$= \frac{1}{2}x\sin 2x + \frac{1}{4}\cos 2x + c$

$= \frac{1}{4}x^2 - \frac{1}{2}\left\{\frac{1}{2}x\sin 2x + \frac{1}{4}\cos 2x + c\right\}$

$= \frac{1}{4}x^2 - \frac{1}{4}x\sin 2x - \frac{1}{8}\cos 2x + c$

$\boxed{= \frac{1}{8}(2x^2 - 2x\sin 2x - \cos 2x) + C}$

(17) $\int \sec^4 5x\,dx \qquad u = 5x \quad du = 5dx \Rightarrow dx = \frac{1}{5}du$

$\frac{1}{5}\int \sec^4 u\,du = \frac{1}{5}\int \sec^2 x\sec^2 x\,dx = \frac{1}{5}\int (1+\tan^2 x)\sec^2 x\,dx$

$= \frac{1}{5}\left\{\int \sec^2 u\,dx + \int \tan^2 u\sec^2 u\,du\right\}$

$= \frac{1}{5}\left\{\tan u + \int U^2\,dU\right\} = \frac{1}{5}\left\{\tan u + \frac{1}{3}\tan^3 u\right\} + C$

$= \frac{1}{15}\left\{3\tan u + 5\tan^3 u\right\} + C = \frac{1}{15}\left\{3\tan 5x + 5\tan^3 5x\right\} + c$

$\boxed{= \frac{\tan 5x}{15}\left[3 + 5\tan^2 5x\right] + C}$

19) $\int \sec^3 x\,dx = \int \sec^2 x\sec x\,dx = \int (1+\tan^2 x)\sec x\,dx$

~~$dv = \sec^2 x\,dx$~~ ~~$v = \tan x$~~ / ~~$u = \sec x$~~ ~~$du = \sec x\tan x\,dx$~~

$$= \int \sec x\,dx + \int \tan^2 x \sec x\,dx$$

$$= \ln|\sec x + \tan x| + \int \tan x \cdot \underbrace{\sec x \tan x\,dx}$$

$$dV = \sec x \tan x\,dx$$

$$V = \sec x$$

$$u = \tan x$$

$$du = \sec^2 x\,dx$$

$$\therefore \int \tan x \sec x \tan x = \sec x \tan x - \int \sec^3 x\,dx$$

$$\therefore \int \sec^3 x\,dx = \ln|\sec x + \tan x| + \sec x \tan x - \int \sec^3 x\,dx$$

$$\therefore 2\int \sec^3 x\,dx = \ln|\sec x + \tan x| + \sec x \tan x \quad + C$$

$$\int \sec^3 x\,dx = \frac{1}{2}\ln|\sec x + \tan x| + \frac{1}{2}\sec x \tan x + C$$

~~Note: "x" = πx~~

$$\therefore \int \sec^3 u\,du = \frac{1}{2}\ln|\sec u + \tan u| + \frac{1}{2}\sec u \tan u + C$$

$$u = \pi x$$

$$\int \sec^3 \pi x\,dx = \frac{1}{\pi}\int \sec^3 u\,du = \boxed{\frac{1}{2\pi}\left\{\ln|\sec \pi x + \tan \pi x| + \sec \pi x \tan \pi x\right\} + C}$$

(23) $\int \tan^5 \frac{x}{4}\,dx \qquad u = \frac{x}{4} \quad du = \frac{1}{4}dx \Rightarrow dx = 4du$

$$4\int \tan^5 u\,du = 4\int \tan^4 u \tan u\,du = 4\int (\overset{\sec^2}{\cancel{\tan^2}}u - 1)^2 \tan u\,du$$

$$= 4\int (\sec^2 u - 1)^2 \tan u\,du = 4\int (\sec^4 u - 2\sec^2 u + 1)\tan u\,du$$

$$= 4\int \sec^4 u \tan u - 8\int \sec^2 u \tan u + 4\int \tan u\,du$$

$$= 4\int \sec^3 u (\sec u \tan u)\,du - 8\int \sec u (\sec u \tan u)\,du + 4\int \tan u\,du$$

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$$= 4\int U^3 dU - 8\int U\,dU + -4\ln|\cos u| + C$$

$$= U^4 - 4U^2 - 4\ln|\cos u| + C$$

$$= (\sec^4 u) - 4(\sec^2 u) - 4\ln|\cos u| + C$$

$$= \sec^4(x/4) - 4\sec^2(x/4) - 4\ln|\cos(x/4)| + C$$

Typo in text. (Note error in book)

Check: $\frac{d}{dx}\sec^4(x/4) = 4\sec^3(x/4)\cdot\sec(x/4)\tan(x/4)\cdot 1/4$

$= \sec^4(x/4)\tan(x/4)$

$\frac{d}{dx}\sec^2(x/4) = 2\sec(x/4)\sec(x/4)\tan x(x/4)\cdot 1/4$

$= \frac{1}{2}\sec^2(x/4)\tan(x/4)$

$-\frac{d}{dx}\ln|\cos(x/4)| = \frac{1}{4}\tan(x/4)$

$\therefore \frac{d}{dx}\left(\sec^4(x/4) - 4\sec^2(x/4) - 4\ln|\cos(x/4)| + c\right)$

$= \sec^4(x/4)\tan(x/4) - 2\sec^2(x/4)\tan(x/4) + \tan(x/4)$

$= (\tan(x/4))\left[\sec^4(x/4) - 2\sec^2(x/4) + 1\right]$

$= \tan(x/4)\left[\sec^2(x/4) - 1\right]^2$

$= \tan(x/4)\left[\tan^2(x/4)\right]^2 = \tan^5(x/4)$ ✓

29) $\int \sec^5 \pi x \tan \pi x\, dx$ $\quad u = \pi x \quad du = \pi dx \Rightarrow dx = \frac{1}{\pi} du$

$= \frac{1}{\pi}\int \sec^5 u \tan u\, du = \frac{1}{\pi}\int \sec^4 u \sec u \tan u\, du = \frac{1}{\pi}\int v^4\, dv$

$= \frac{1}{\pi}\left\{\frac{1}{5}v^5 + C\right\} = \frac{1}{5\pi}v^5 + C = \boxed{\frac{1}{5\pi}\sec^5 \pi x + C}$

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31) $\int \sec^6 4x \tan 4x \, dx$

$u = 4x \quad du = 4dx \Rightarrow dx = \frac{1}{4} du$

$= \frac{1}{4}\int \sec^6 u \tan u \, du = \frac{1}{4}\int \sec^5 u \sec u \tan u \, du$

$= \frac{1}{4}\int v^5 dv = \frac{1}{24} v^6 + C = \frac{1}{24}\sec^6 u + C$

$= \boxed{\frac{1}{24}\sec^6 4x + C}$

37) $\int \cot^3 2x \, dx \qquad u = 2x \qquad du = 2dx \qquad dx = \frac{1}{2} du$

$\frac{1}{2}\int \cot^3 u \, du = \frac{1}{2}\int \cot^2 u \cot u \, du = \frac{1}{2}\int (\csc^2 u - 1)\cot u \, du$

$= \frac{1}{2}\int \csc^2 u \cot u \, du - \frac{1}{2}\int \cot u \, du$

$= \frac{1}{2}\int \csc u \, (\csc u \cot u) du - \frac{1}{2}\ln|\sin u| + C$

$= -\frac{1}{2}\int v \, dv - \frac{1}{2}\ln|\sin u| + C$

$= -\frac{1}{4}\cdot v^2 - \frac{1}{2}\ln|\sin u| = \frac{1}{4}\csc^2 u - \frac{1}{2}\ln|\sin u| + C$

$= \frac{1}{4}\left[-\csc^2 2x + 2\ln|\csc 2x|\right] + C$

$= \frac{1}{4}\left[-\cot^2 2x + \underbrace{1] 2\ln|\csc 2x|] + C}_{\text{absorb in term}}\right.$

$= \frac{1}{4}\left[2\ln|\csc 2x| - \cot^2 2x\right] + C$

41) $\int \frac{\cot^2 t}{\csc t} dt = \int \frac{\cos^2 t}{\sin^2 t}\sin t = \int \frac{\cos^2 t}{\sin t} dt$ ~~$\int \cot t \csc t \, dt$~~

~~$= -\csc t + C$~~

$= \int \underset{v}{\cot t} \, \underset{dV}{\cos t \, dt} \qquad v = \cot$

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41) $\int \frac{\cot^2 t}{\csc t}\,dt = \int \frac{\csc^2 t - 1}{\csc t} = \int \csc t - \sin t$

$= \boxed{\ln|\csc t - \cot t| + \cos t + C}$

45) $\int \sin\theta \sin 3\theta \, d\theta = \int \frac{1}{2}[\cos(-2\theta) - \cos 4\theta]\,d\theta$

$= \frac{1}{2}\int \cos 2\theta - \frac{1}{2}\int \cos 4\theta \, d\theta$

$= \frac{1}{4}\sin 2\theta - \frac{1}{8}\sin 4\theta + C$

$= \boxed{\frac{1}{8}(2\sin 2\theta - \sin 4\theta) + C}$

55) $\int_0^{\pi/2} \frac{\cos t}{1+\sin t}\,dt$ $\qquad u = 1+\sin t$

$du = \cos t\,dt$

$$\int_{u(0)}^{u(\pi/2)} \frac{du}{u} = \int_1^2 \frac{du}{u} = \ln|u|\Big|_1^2 = \ln 2 - \ln 1 = \boxed{\ln 2}$$

WALLIS' FORMULAE: (Alternate derivation)

$$\int_0^{\pi/2} \cos^n x\,dx = \int_0^{\pi/2} \cos^{n-1} x \cos x\,dx$$

Suppose $n = 2k+1$ (n odd)

$$\int_0^{\pi/2} \cos^n x\,dx = \int_0^{\pi/2} \cos^{2k} x \cos x\,dx = \int_0^{\pi/2} (\cos^k x)^2 \cos x\,dx$$

$$= \int_0^{\pi/2} (\cos^2 x)^k \cos x\,dx = \int_0^{\pi/2} (1-\sin^2 x)^k \cos x\,dx$$

$u = \sin x \qquad du = \cos x$

$$= \int_{u(0)}^{u(\pi/2)} (1-u^2)^k du = \int_0^1 (1-u^2)^k du$$

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$$= \int_0^1 \sum_{j=0}^{k} \underbrace{\binom{k}{j}(-1)^j u^{2j}}_{\text{Binomial Expansion}} \, du = \sum_{j=0}^{k} \binom{k}{j} (-1)^j \frac{u^{2j+1}}{2j+1} \Big|_0^1$$

$$= \sum_{j=0}^{k} \binom{k}{j} (-1)^j \frac{1}{2j+1}$$