MA261 Calculus I: Notes for August 30
Fall 2007 Kallfelz

- ***<u>The Limit Concept</u>***

The limit concept is *essential* for Calculus. This is because Calculus deals with computing *exact* quantities like rates of change, areas, volumes, etc., for general cases which can only be handled by pre-Calculus mathematics using *approximate* techniques.

For example, if one were to ask:

1. Given an object's displacement x as a function of time is given by:

   $x = f(t) = a + bt + ct^2$ (where a, b, c are constants)

What is the object's *instantaneous* (i.e. *exact*) velocity, acceleration, at any time t ?

2. Given an area of land is bounded below by the curve: $y_1(x) = x^2 - 4$, and bounded above by the curve: $y_2(x) = 16 - x^4$

   What is the enclosed area?

…a moment's reflection might tell you that if you were to attempt to answer the above two questions using ordinary pre-Calculus methods (algebra, analytic geometry, trigonometry) there's *no way* you could give an answer that's precisely correct. If you were clever and patient enough, you may be able to give an *approximate* answer. For instance, in question 1. you might estimate the velocity at any time t by calculating the slope of a little secant line (a line that intersects $f(t)$ at two points) which connects the points:

$P_1(t, f(t))$ $P_2(t + \Delta t, f(t + \Delta t))$ (where Δt is a small increment of time)

…But then you hadn't given an exactly correct answer. You would have given an *approximate* answer, namely, the *average* velocity of the object in the time interval Δt. But an *average* velocity is *not* an *instantaneous* velocity. The *instantaneous* velocity at time t is found by allowing Δt to become infinitely small, or to become an *infinitesimal*. This requires examining the behavior of the average velocity **in the $\Delta t \to 0$ *limit*.**

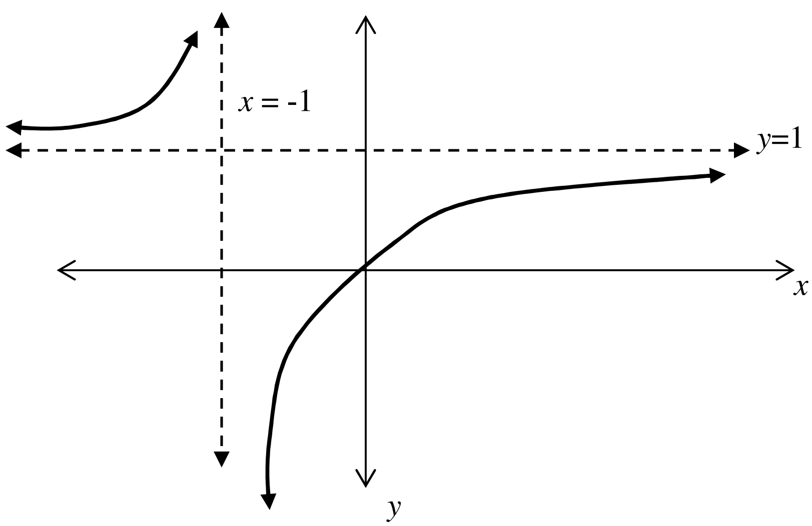
In the case of question 2., after sketching the region by intersecting the two curves and finding their intersection-points (an analytic geometry problem) you might try approximating the area of the region by fitting a bunch of skinny little rectangles (of width Δx) inside the region, and figure out their height and width and sum their areas. But, again, you wouldn't have given an answer that's exactly correct, since no matter how skinny you make the rectangles, at their tips and bases there will always be regions of the boundary they will miss, since the boundary of the land area is curved. To find the *exact* area of the region requires summing all the areas of the rectangles by allowing Δx (the width of each rectangle) to become infinitely small, or to become an *infinitesimal*. This requires examining the behavior of the "sum"[1] of the areas **in the $\Delta x \to 0$ *limit.***

As you'll see in this course, being able to answer questions 1. and 2. require two central concepts you'll learn in Calculus I: calculating a ***derivative*** (in the case of question 1) and calculating an ***integral* (or *anti-derivative*)** (in the case of question 2). Both concepts are based on the concept of ***limit.***

- Actually you've already seen this concept of limit before, when you examined the behavior of function's *asymptotes* (whether vertical, oblique, or horizontal). For example, given the function:

$$y = f(x) = \frac{x}{x+1} = 1 - \frac{1}{(x+1)}$$

  The function has a horizontal asymptote $y = 1$ and a vertical asymptote $x = -1$, as indicated by the graph of the curve below:



So based on the above, we can make the following statements:

"As x ***approaches*** $\pm\infty$, $f(x)$ ***approaches*** the line $y = 1$."

[1] As you'll see in the week of **October 1**, there's a for the quotation marks here. For this is no ordinary sum! Making a bunch of rectangles infinitely skinny in a finite region means I can fit infinitely many of them in that region. So we're 'summing' the areas of an infinite number of objects that are infinitely skinny.

"As x ***approaches*** 1 on the right-hand side (or 1^+ for shorthand) $y = f(x)$ ***approaches*** $-\infty$."

"As x ***approaches*** 1 on the left-hand side (or 1^- for shorthand) $y = f(x)$ ***approaches*** $+\infty$."

It's obviously tedious to write these three statements out each and every time, so in mathematical shorthand we write:

$$\lim_{x \to \pm\infty} \frac{x}{x+1} = 1$$

$$\lim_{x \to -1^+} \frac{x}{x+1} = -\infty$$

$$\lim_{x \to -1^-} \frac{x}{x+1} = +\infty$$

So in general, when we say:

"As x ***approaches*** c on the right-hand side, for any $c \in (-\infty, \infty)$ (i.e., $-\infty < c < \infty$) $y = f(x)$ ***approaches*** R"

"As x ***approaches*** c on the left-hand side, for any $c \in (-\infty, \infty)$ (i.e., $-\infty < c < \infty$) $y = f(x)$ ***approaches*** L"

"As x ***approaches*** c on *either side*, for any $c \in (-\infty, \infty)$ (i.e., $-\infty < c < \infty$) $y = f(x)$ ***approaches*** M"

we write:

$$\lim_{x \to c^+} f(x) = R$$

$$\lim_{x \to c^-} f(x) = L$$

$$\lim_{x \to c} f(x) = M$$

- Note that a ***necessary condition*** for a limit to exist at any point is for its right and left hand limits to agree at the same value. That is to say, *if* $\lim_{x \to c} f(x)$ exists, *then* $\lim_{x \to c^+} f(x) = \lim_{x \to c^-} f(x)$. (Or $L = R = M$)

As a counteraxmple, note that in the function mentioned above:

$$\lim_{x\to 1}\frac{x}{x+1} DNE, because:$$
$$\lim_{x\to 1^+}\frac{x}{x+1}=-\infty$$
$$\lim_{x\to 1^-}\frac{x}{x+1}=+\infty$$

Or, recall the other example discussed in class: (the "floor" function[2]: $y = f(x) = [|x|] =$ glib(x) (where "glib" = "greatest lower integer bound". For instance: $[|e|] = 2$, $[|\pi|] = 3$, etc.) Not that when x approaches *any* integer n:

$$\lim_{x\to n}[|x|] DNE, because:$$
$$\lim_{x\to n^+}[|x|]=n$$
$$\lim_{x\to n^-}[|x|]=n-1$$

So you can see that the issue involving the *existence* of a limit is separate from the issue of whether or not that limit turns out to be *infinite*. This is discussed in detail in section 2.4 of your text. For instance:

$$\lim_{x\to 0}\frac{1}{x^2}=\infty, because$$
$$\lim_{x\to 0^+}\frac{1}{x^2}=\infty=\lim_{x\to 0^-}\frac{1}{x^2}$$

- ***Evaluating Limits***

  It's one thing to *grasp* the concept of a limit, but another to *apply* it. It can be tedious, if you start each and every time with first principles. Technically speaking, the "first principles" involve the so-called δ,ε definition of limit which is the ***technically precise*** definition, and covered in detail in section 2.5 of the text. In section 2.1, numerous short-cut theorems and properties are stated without proof, in the forms of **Theorems 2.1 – 2.7**.

  Thm 2.1 basically tells you what to do in the case of a removable discontinuity (or removable singularity), when the graph has a "hole" in it. Basically, as shown in the examples we discussed in class, removable singularaites occur in the case

[2] I am using the textbook's notation here (see page 66). Note that other texts specify the notation as: $\lfloor x \rfloor$.

of rational functions, in which one ends up with a 0/0 indeterminacy.[3] There, the function has a value that ***doesn't exist*** (**DNE**). Still we can factor and cancel into simpler form and simply **substitute** the point *c* in which the removable singularity occurs, to find the function's *limit* at that point. We're not contradicting ourselves here, since we're not interested in the behavior of $f(x)$ at the *arrival* of the point $x = c$, simply the behavior of $f(x)$ at the *approach* of the point $x \to c$.

For example (p. 68, #9)

$$\lim_{x\to -2}\frac{x^3+8}{x+2} = \lim_{x\to -2}\frac{(x+2)(x^2-2x+4)}{x+2} = \lim_{x\to -2}\left(x^2-2x+4\right) = 12$$

- **Note (1):** In the subsequent evaluation, note how **Thms2.2,2.3** are invoked:

$\lim_{x\to -2}(x^2 - 2x + 4) = \lim_{x\to -2} x^2 - \lim_{x\to -2} 2x + \lim_{x\to -2} 4$ (using **Thm2.3 property 2)**
$= (-2)^2 - 2\lim_{x\to -2} x + 4$ (using **Thm2.2 properties 2 & 3, and Thm 2.3 property 1)**
$= (-2)^2 - 2(-2) + 4 = 12$ (Using **Thm2.2 property 2)**

- **Note (2):** The numerator was factored using the difference of cubes formula:

$$(x^3 \pm a^3) = (x \pm a)\,(x^2 \mp ax + a^2)$$

For example (p. 68, #14)

$$\lim_{\Delta x\to 0}\frac{(1+\Delta x)^3 - 1}{\Delta x} = \lim_{\Delta x\to 0}\frac{1+3\Delta x+3\Delta^2 x+\Delta^3 x-1}{\Delta x}$$
$$= \lim_{\Delta x\to 0}\frac{\Delta x\left(3+3\Delta x+\Delta^2 x\right)}{\Delta x} = \lim_{\Delta x\to 0}\left(3+3\Delta x+\Delta^2 x\right) = 3$$

- **Note (3):** The numerator term was expanded using the **Binomial theorem** which states:

$$(x+y)^n = \sum_{k=0}^{n}\binom{n}{k}x^{n-k}y^k = \binom{n}{0}x^{n-0}y^0 + \binom{n}{1}x^{n-1}y^1 + \ldots + \binom{n}{n-1}x^{n-(n-1)}y^{n-1} + \binom{n}{n}x^{n-n}y^n$$

The Binomial coefficients: $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ are tedious to evaluate from the formula alone. A useful shortcut is ***Pascal's Triangle***:

[3] You'll learn to analyze such indeterminacies more precisely later in the course, when you study L'Hopital's Rule.

1 — $n = 0$

1 1 — $n=1$

1 2 1 — $n=2$

1 3 3 1 — $n=3$

1 4 6 4 1 — $n=4$

1 5 10 10 5 1 — $n=5$

Using the simple addition procedure (indicated by the diagonal arrows)automatically generates the binomial coefficients. For example:

$$(x + y)^5 = \mathbf{1}\cdot x^{(5-0)}y^0 + \mathbf{5}\cdot x^{(5-1)}y^1 + \mathbf{10}\cdot x^{(5-2)}y^2 + \mathbf{10}\cdot x^{(5-3)}y^3 + \mathbf{5}\cdot x^{(5-4)}y^4 + \mathbf{1}\cdot x^{(5-5)}y^5$$
$$= x^5 + 5\cdot x^4 y + 10x^3y^2 + 10x^2y^3 + 5\cdot xy^4 + y^5$$

<u>For example (p. 68, #36)</u>

$$\lim_{x\to 4^-} \frac{\sqrt{x}-2}{x-4} = \lim_{x\to 4^-} \frac{\sqrt{x}-2}{(\sqrt{x}+2)(\sqrt{x}-2)} = \lim_{x\to 4^-} \frac{1}{\sqrt{x}+2} = \tfrac{1}{4}$$

<u>For example (p. 68, #18)</u>

$$\lim_{x\to 0} \frac{\sqrt{2+x}-\sqrt{2}}{x} = \lim_{x\to 0} \frac{(\sqrt{2+x}-\sqrt{2})(\sqrt{2+x}+\sqrt{2})}{x(\sqrt{2+x}+\sqrt{2})} = \lim_{x\to 4^-} \frac{2+x-2}{x(\sqrt{2+x}+\sqrt{2})}$$
$$= \lim_{x\to 0} \frac{x}{x(\sqrt{2+x}+\sqrt{2})} = \lim_{x\to 0} \frac{1}{\sqrt{2+x}+\sqrt{2}} = \frac{1}{2\sqrt{2}} = \frac{\sqrt{2}}{4}$$

The above examples all dealt with *finite* results. Here are some instances of *infinite* limits:

For example (p. 82, #32)

$$\lim_{x\to 0^-}\frac{x^2-2x}{x^3}=\lim_{x\to 0^-}\frac{x(x-2)}{x^3}=\lim_{x\to 0^-}\frac{x-2}{x^2}=\lim_{x\to 0^-}\left(\frac{1}{x}-\frac{2}{x^2}\right)=-\infty$$

- **Note (4):** There are a few important subtleties in this problem. Note that we're being asked to evaluate a left-sided limit only (what happens to the function as x approaches 0 on the left hand side?) That's easy to answer, since $^1/_x \to -\infty$ when $x \to 0^-$ and $^2/_{x^2} \to \infty$ when $x \to 0^-$, so subtracting aterm that goes to $+\infty$ from a term that goes to $-\infty$ obviously produces a result that goes to $-\infty$.

  However, ask yourself: what if we were asked to evaluate the *right-handed* limit? You might be tempted to say "0" since $^1/_x \to +\infty$ when $x \to 0^+$ and $^2/_{x^2} \to \infty$ when $x \to 0^+$, so subtracting term that goes to $+\infty$ from a term that goes to $+\infty$ produces a result that goes to 0. **This reasoning, however, is wrong!** **(It's an example of subtracting infinities, which turns out to be ill-defined.)** Notice that $^2/_{x^2} \to \infty$ "faster than" $^1/_x \to +\infty$ when $x \to 0^+$. (For example, when $x = 0.1$, $^1/_x = 10$, but $^2/_{x^2} = 200$.) **So as x becomes an arbitrarily small positive real number, the $^2/_{x^2}$ term dominates. Hence:**

  $$\lim_{x\to 0^+}\left(\frac{1}{x}-\frac{2}{x^2}\right)=\lim_{x\to 0^+}-\frac{2}{x^2}=-\infty$$

  So we have an example when a limit *exists* at x =0 (since both left and right hand limits agree) and is equal to $-\infty$

For example (p. 82, #17)

$$f(x)=\frac{x}{x^2+x-2}$$

To find its behavior in the $x \to \pm\infty$ limits (i.e. to find if there are horizontal asymptotes) one divides the numerator and denominator by the leading term:

$$\lim_{x\to\pm\infty}\frac{x}{x^2+x-2}=\lim_{x\to\pm\infty}\frac{x/x^2}{\left(1+1/x-2/x^2\right)}=\lim_{x\to\pm\infty}\frac{1/x}{2+1/x-2/x^2}=\frac{0}{2+0-0}=0$$

So the function has a horizontal asymptote : y =0 (i.e., the x – axis)

Factoring the denominator $x^2 + x - 2 = (x + 2)(x - 1)$ indicates that vertical asymptotes occur at $x = -2$, and $x = 1$ (since a 0 is produced in the denominator but the numerator is non-zero, so we have an essential, not a removable, singularity).

However, the behavior of the function indicates that:

$$\lim_{x\to -2^-} f(x) = -\infty, \lim_{x\to -2^+} f(x) = \infty$$
$$\therefore \lim_{x\to 2} f(x) DNE$$
$$\lim_{x\to 1^-} f(x) = -\infty, \lim_{x\to 1^+} f(x) = \infty$$
$$\therefore \lim_{x\to 1} f(x) DNE$$

In other words, the limits don't exist at the respective vertical asymptotes because the function diverges in opposite directions on either side of the vertical asymptote

- ***Continuity***

As discussed in section 2.3, a function is continuous on a closed interval $[a , b]$ provided:

a.) For any $c \in (a , b)$ $f(c)$ is finite and $\lim_{x\to c} f(x) = f(c)$
b.) $\lim_{x\to a^+} f(x) = f(a)$ and $\lim_{x\to b^-} f(x) = f(b)$

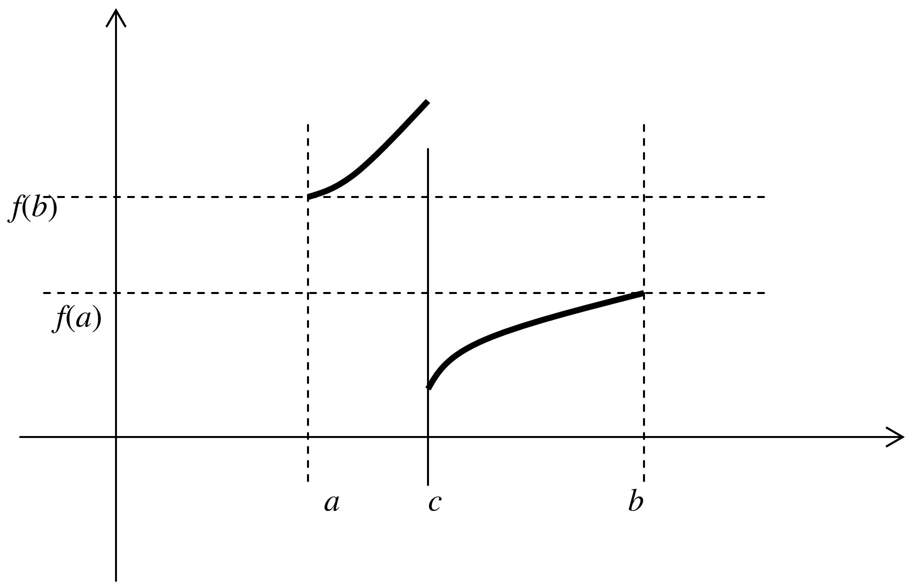
Statements a.) and b.) express the conditions discussed in pp71-73 in a more condensed fashion.

The **Intermediate Value Theorem (IVT) (Thm 2.11, page 74)** is of central importance. As the text mentions, the proof of the theorem requires advanced techniques beyond the scope of beginning Calculus. However, despite its subtle proof, note that the IVT expresses a very attractive feature about continuous functions, namely that for any function f continuous on $[a , b]$ we are *guaranteed* there exists *at least one point in the y-axis* $f(c)$ such that $f(a) < f(c) < f(b)$ for *any* $c \in (a , b)$. In other words, any point between the endpoints on the domain is going to produce *at least one* y-value between the points $f(a)$ and $f(b)$ in the range.

We can use this intuitive appeal of this property to present an informal proof of the IVT using a counter-argument. (Assuming the opposite and showing we contradict ourselves when doing so.)

PROOF:

Suppose f is continuous on $[a , b]$ and that for *any* $c \in (a , b)$
$f(a) < f(c) < f(b)$ does **not** hold. In other words, there exists **no** $y = f(c)$ such that for *any* $c \in (a , b)$ we have the property: $f(a) < f(c) < f(b)$. But that would imply (for instance) the following graph:



This graph is clearly discontinuous!

Example (problem 30, pg 75)

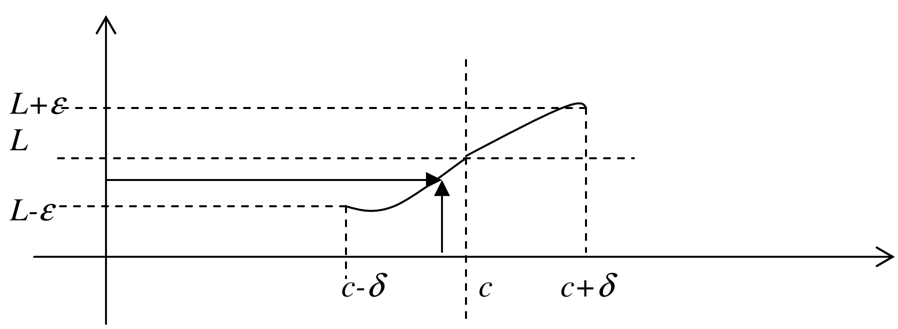
$f(x) = x^{-1/2}$ is continuous on $[0, \infty)$ and $g(x) = x^{-1}$ is continuous on $(-\infty,0) \cup (0,\infty)$ (i.e. at all real numbers except 0. Hence $h = f(g(x)) = (x^{-1})^{-1/2} = x^{½}$ is continuous on $[0, \infty)$

- ***The ε-δ Definition of Limits***

Stated precisely $\lim_{x\to c} f(x) = L$ means:

For any arbitrarily small ε such that $0 < |f(x) - L| < \varepsilon$ there exists an arbitrarily small δ such that $0 < |x - c| < \delta$ such that $\delta \to 0$ as $\varepsilon \to 0$

Again, a picture is worth a thousand words here. See below:



So what the definition essentially says is that $\lim_{x \to c} f(x) = L$ means that for any point $y = f(x)$ bound in an ε-interval around L (i.e., for any $y \in (L - \varepsilon, L + \varepsilon)$) for *any* arbitrarily small ε , that there exists an arbitrarily small δ guaranteed that the domain point x will be bound in a δ-interval around c (i.e., $x \in (c - \delta, c - \delta)$).

We can use this exact defnition, for example, to prove **Thm 2.3 property 1:**

$$\lim_{x \to c} bf(x) = b\lim_{x \to c} f(x)$$

Proof:

Suppose $\lim_{x \to c} f(x) = L$. Then for any arbitrarily small ε such that $0 < |f(x) - L| < \varepsilon$ there exists an arbitrarily small δ such that $0 < |x - c| < \delta$ such that $\delta \to 0$ as $\varepsilon \to 0$. Since we're guaranteed such a δ, consider a simple case in which $\delta = \varepsilon$

Now Consider the function $g(x) = bf(x)$. We aim to show that $\lim_{x \to c} g(x) = bL$. Consider: $|bf(x) - bL|$. By the Triangle Inequality:
$|bf(x) - bL| \leq |b||f(x) - L| < |b|\varepsilon$

So in the case of this new function $g(x)$, we can define a ε_2 related to the previous ε for $f(x)$ via the expression: $\varepsilon_2 = |b|\varepsilon$. Note that as ε gets arbitrarily small, so does ε_2 , since the quantity b is finite. Then the associated δ_2 for g is related to the previous δ for f by the formula: $\delta_2 = \varepsilon / |b|$. As $\varepsilon \to 0$ certainly $\delta_2 \to 0$.