MA262 **Calculus II: Notes for Jan31**
Spring 2008 Kallfelz

- ***HINTS ON SOME EXERCISES (ASSIGNMENT I-SELECTED FROM Q & A)***

II.a) $\int \frac{\sqrt{4x^2+9}}{x^4}dx$

Note that $\int \frac{\sqrt{4x^2+9}}{x^4}dx = \int \frac{\sqrt{(2x)^2+3^2}}{x^4}dx$, so let $u = 2x$. Hence:

$$\int \frac{\sqrt{(2x)^2+3^2}}{x^4}dx = \int \frac{\sqrt{u^2+3^3}}{\left(\frac{1}{2}u\right)^4}\tfrac{1}{2}du = 8\int \frac{\sqrt{u^2+3^2}}{u^4}du$$

Let $u = 3\tan\theta \Rightarrow du = 3\sec^2\theta d\theta$

$$\therefore 8\int \frac{\sqrt{u^2+3^2}}{u^4}du = 8\int \frac{3\sec\theta}{3^4\tan^4\theta}\cdot 3\sec^2\theta d\theta = \tfrac{8}{9}\int \frac{\sec^3\theta}{\tan^4\theta}d\theta$$

One could re-express everything in terms of $\sec\theta$ or $\tan\theta$, but that can get cumbersome. A more direct approach involves re-expressing the integral in terms of elementary trigonometric functions sine and cosine:

$$\tfrac{8}{9}\int \frac{\sec^3\theta}{\tan^4\theta}d\theta = \tfrac{8}{9}\int \frac{\cos^4\theta}{\sin^4\theta}\cdot\frac{1}{\cos^3\theta}d\theta = \tfrac{8}{9}\int \sin^{-4}\theta\cos\theta d\theta$$

…which is an integral that can be resolved via a simple *u*-substitution. **However, note that your final answer must be expressed in terms of *x*!** The above integral will give an answer in terms of $\sin^{-3}\theta = \csc^3\theta$. This can be defined in terms of *u* via (recall Method 2 in Jan. 29th and Jan. 24th notes):

$$\tan\theta = \frac{u}{3} = \frac{OPP}{ADJ} \Rightarrow HYP = \sqrt{OPP^2 + ADJ^2} = \sqrt{u^2+9} \Rightarrow \csc\theta = \frac{HYP}{OPP} = \frac{\sqrt{u^2+9}}{u}$$

III.a) $\int \frac{x^3-x+3}{x^2+x-2}dx$

Recall problem #9, discussed as an example in page 13, **<u>Jan. 24th Notes</u>**. Here we have a similar situation: the degree of the numerator > degree of the denominator. So you should reduce this fraction first, either by polynomial long division or via the following trick:

$$\frac{x^3-x+3}{x^2+x-2}=\frac{x^3+x^2-2x-x^2+x+3}{x^2+x-2}=\frac{x(x^2+x-2)}{x^2+x-2}-\frac{x^2-x-3}{x^2+x-2}$$
$$x-\frac{x^2+x-2-2x-1}{x^2+x-2}=x-\frac{x^2+x-2}{x^2+x-2}-\frac{2x+1}{x^2+x-2}=x-1-\frac{2x+1}{x^2+x-2}$$

(Where multiples of the denominator term are teased out in the numerator via the "trick of 0.")

The last remainder term $\frac{2x+1}{x^2+x-2}$ can be resolved via partial fractions using strategy A (the denominator can be factored into a product of two distinct linear terms: $x^2+x-2=(x+2)(x-1)$

III.c) $\int\frac{6x^2+1}{x^2(x-1)^3}dx$

Recall the discussion of problem 24 in pp. 9-11 **<u>Jan 29th Notes</u>**. One can adopt two different methods here as well: either consider the quadratic term as the product of two linear repeating terms, thus treating the whole expression as a Strategy C case, or consider the quadratic term in terms of Strategy D, and hence consider the entire expression as a mixture of C and D strategies. Either way, you'll obtain five constants you'll have to solve for.

<u>Method 1:</u>

$$\frac{6x^2+1}{x\cdot x\cdot(x-1)^3}=\frac{A_1}{x}+\frac{A_2}{x^2}+\frac{A_3}{(x-1)}+\frac{A_4}{(x-1)^2}+\frac{A_5}{(x-1)^3}$$
$$\Rightarrow 6x^2+1=A_1x(x-1)^3+A_2(x-1)^3+A_3x^2(x-1)^2+A_4x^2(x-1)+A_5x^2$$

Note by inserting $x = 0$ into the above equation, you can solve for A_2
Note by inserting $x = 1$ into the above equation, you can solve for A_4

Insert these results for A_2 and A_4 back into the above equation:

$$6x^2+1=A_1x(x-1)^3+A_2(x-1)^3+A_3x^2(x-1)^2+A_4x^2(x-1)+A_5x^2$$

Which will reduce it to an equation involving three unknowns: A_1, A_3, A_5.
They can be easily solved by method of equating coefficients of powers of x (for further details, see **<u>Jan 29th Notes</u>** , pp. 6-11.

<u>Method 2:</u>

$$\frac{6x^2+1}{x^2(x-1)^3} = \frac{A_1x+B_1}{x^2} + \frac{A_2}{(x-1)} + \frac{A_3}{(x-1)^2} + \frac{A_4}{(x-1)^3}$$
$$\Rightarrow 6x^2+1 = (A_1x+B_1)(x-1)^3 + A_2x^2(x-1)^2 + A_3x^2(x-1)^1 + A_4x^2$$

Inserting $x = 0$ into top equation enables one to solve for B_1. Inserting $x = 1$ into top equation enables you to solve for A_4. As in Method 1, insert the values of these two constants into the equation to solve for the remaining three constants by equating coefficients of powers of x.

- ***PARTIAL FRACTIONS…MORE EXAMPLES***

- Example (26, § 9.5)

$$\int \frac{x^2+x+3}{x^4+6x^2+9}dx$$

Factoring the denominator reveals it's a Case D strategy (repeated quadratic irreducible):

$$\frac{x^2+x+3}{(x^2+3)^2} = \frac{A_1x+B_1}{(x^2+3)} + \frac{A_2x+B_2}{(x^2+3)^2} \Rightarrow x^2+x+3 = (A_1x+B_1)(x^2+3)+(A_2x+B_2)$$

Inserting $x = 0$ into top equation:

$$0^2+0+3 = 3 = (A_1\cdot 0+B_1)(0^2+3)+(A_2\cdot 0+B_2) = 3B_1+B_2$$
$$\Rightarrow 3 = 3B_1+B_2 \Rightarrow B_2 = 3(1-B_1)$$

Inserting this result for B_2 in the equation:

$$x^2+x+3 = (A_1x+B_1)(x^2+3)+(A_2x+3(1-B_1))$$
$$x^2+x+3 = A_1x^3+B_1x^2+3A_1x+3B_1+A_2x-3B_1+3$$
$$x^2+x = A_1x^3+B_1x^2+(3A_1+A_2)x$$

Equating in terms of coefficients of powers of x:

x^3 : $0 = A_1$
x^2 : $1 = B_1 \Rightarrow B_2 = 3(1-1) = 0$
x^1 : $1 = 3A_1+A_2 \Rightarrow A_2 = 1$

Hence:

$$\frac{x^2+x+3}{\left(x^2+3\right)^2}=\frac{A_1x+B_1}{\left(x^2+3\right)}+\frac{A_2x+B_2}{\left(x^2+3\right)^2}=\frac{1}{\left(x^2+3\right)}+\frac{x}{\left(x^2+3\right)^2}$$

Check: $$\frac{1}{\left(x^2+3\right)}+\frac{x}{\left(x^2+3\right)^2}=\frac{\left(x^2+3\right)+x}{\left(x^2+3\right)^2}=\frac{x^2+x+3}{\left(x^2+3\right)^2}$$

So:

$$\int\frac{x^2+x+3}{x^4+6x^2+9}dx=\int\left[\frac{1}{\left(x^2+3\right)}+\frac{x}{\left(x^2+3\right)^2}\right]dx$$

$$=\int\frac{dx}{\left(x^2+\left(\sqrt{3}\right)^2\right)}dx+\int\frac{x}{\left(x^2+3\right)^2}dx$$

The first integral is solvable via a trigonometric substitution[1] :

$$x=\sqrt{3}\tan\theta\Rightarrow dx=\sqrt{3}\sec^2\theta d\theta$$
$$\therefore\tan\theta=\frac{x}{3}\Rightarrow\theta=\arctan\left(\frac{x}{\sqrt{3}}\right)$$

$$\therefore\int\frac{dx}{\left(x^2+3\right)}=\int\frac{\sqrt{3}\sec^2\theta d\theta}{3\tan^2\theta+3}=\tfrac{\sqrt{3}}{3}\int\frac{\sec^2\theta}{\sec^2\theta}d\theta=\tfrac{1}{\sqrt{3}}\int d\theta=\tfrac{1}{\sqrt{3}}\theta=\tfrac{1}{\sqrt{3}}\arctan\left(\tfrac{x}{\sqrt{3}}\right)$$

The second integral is solvable by a mere U-substitution:

$$\int\frac{x}{\left(x^2+3\right)^2}dx=\int\frac{\frac{1}{2}dU}{U^2}=\tfrac{1}{2}\int\frac{dU}{U^2}=-\tfrac{1}{2}U^{-1}+C=-\tfrac{1}{2}\left(x^2+3\right)^{-1}+C$$

Hence combining:

$$\int\frac{x^2+x+3}{x^4+6x^2+9}dx=\int\left[\frac{1}{\left(x^2+3\right)}+\frac{x}{\left(x^2+3\right)^2}\right]dx=\tfrac{1}{\sqrt{3}}\arctan\left(\tfrac{x}{\sqrt{3}}\right)-\frac{1}{2\left(x^2+3\right)}+C$$

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[1] Or one could use methods from §8.6: $$\int\frac{dx}{\left(x^2+\left(\sqrt{3}\right)^2\right)}=\tfrac{1}{\sqrt{3}}\arctan\left(\tfrac{x}{\sqrt{3}}\right)$$

- Example (39, § 9.5)

$$\int \frac{3\cos x}{\sin^2 x + \sin x - 2}dx \qquad u = \sin x \Rightarrow du = \cos x dx$$

$$\therefore \int \frac{3\cos x}{\sin^2 x + \sin x - 2}dx = \int \frac{3du}{u^2 + u - 2} = 3\int \frac{du}{(u+2)(u-1)}$$

This integral is Strategy A case:

$$\frac{1}{(u+2)(u-1)} = \frac{A_1}{(u+2)} + \frac{A_2}{(u-1)} \Rightarrow 1 = A_1(u-1) + A_2(u+2)$$

Set $u = 1$: $\quad 1 = A_1(1-1) + A_2(1+2) \Rightarrow 1 = 3A_2 \Rightarrow A_2 = \frac{1}{3}$

Set $u = -2$: $\quad 1 = A_1(-2-1) + A_2(-2+2) \Rightarrow 1 = -3A_1 \Rightarrow A_1 = -\frac{1}{3}$

So: $$\frac{1}{(u+2)(u-1)} = \frac{A_1}{(u+2)} + \frac{A_2}{(u-1)} = \frac{1}{3}\left\{\frac{-1}{(u+2)} + \frac{1}{(u-1)}\right\}$$

Check:

$$\frac{1}{3}\left\{\frac{-1}{(u+2)} + \frac{1}{(u-1)}\right\} = \frac{1}{3}\left\{\frac{-u+1+u+2}{(u+2)(u-1)}\right\} = \frac{1}{3}\left\{\frac{3}{(u+2)(u-1)}\right\} = \frac{1}{(u+2)(u-1)}$$

So:

$$\therefore \int \frac{3\cos x}{\sin^2 x + \sin x - 2}dx = \int \frac{3du}{u^2 + u - 2} = 3\int \frac{du}{(u+2)(u-1)} = \frac{3}{3}\int \left[\frac{-1}{(u+2)} + \frac{1}{(u-1)}\right]du$$
$$= -\int \frac{du}{(u+2)} + \int \frac{du}{(u-1)} = -\ln|u+2| + \ln|u-1| + C = \ln\left|\frac{u-1}{u+2}\right| + C = \ln\left|\frac{\sin x - 1}{\sin x + 2}\right| + C$$

- Example (42, § 9.5)

$$\int \frac{e^x}{(e^{2x}+1)(e^x-1)}dx \Rightarrow u = e^x, du = e^x dx \Rightarrow \int \frac{du}{(u^2+1)(u-1)}$$

Using Strategies A and B:

$$\frac{1}{(u^2+1)(u-1)} = \frac{A_1u+B_1}{(u^2+1)} + \frac{A_2}{(u-1)} \Rightarrow 1 = (A_1u+B_1)(u-1) + A_2(u^2+1)$$

Inserting $u = 1$: $\quad 1 = (A_1 \cdot 1 + B_1)(1-1) + A_2(1^2+1) \Rightarrow 1 = 2A_2 \Rightarrow A_2 = \frac{1}{2}$

So:

$$\begin{aligned} &1 = (A_1u+B_1)(u-1) + A_2(u^2+1) = (A_1u+B_1)(u-1) + \tfrac{1}{2}u^2 + \tfrac{1}{2} \\ &\Rightarrow \tfrac{1}{2} = (A_1 + \tfrac{1}{2})u^2 + (B_1 - A_1)u - B_1 \end{aligned}$$

Or:

$$\begin{aligned} x^2 &: \quad 0 = A_1 + \tfrac{1}{2} \Rightarrow A_1 = -\tfrac{1}{2} \\ x^1 &: \quad 0 = B_1 - A_1 = B_1 + \tfrac{1}{2} \Rightarrow B_1 = -\tfrac{1}{2} \\ x^0 &: \quad \tfrac{1}{2} = -(-\tfrac{1}{2}) = \tfrac{1}{2} \end{aligned}$$

(The last equation was used as a check.)

So: $$\frac{1}{(u^2+1)(u-1)} = \frac{A_1u+B_1}{(u^2+1)} + \frac{A_2}{(u-1)} = \frac{1}{2}\left\{\frac{-u-1}{(u^2+1)} + \frac{1}{(u-1)}\right\} = \frac{1}{2}\left\{\frac{-1}{(u^2+1)} + \frac{1}{(u-1)}\right\}$$

Hence:

$$\begin{aligned} &\int \frac{e^x}{(e^{2x}+1)(e^x-1)}dx = \int \frac{du}{(u^2+1)(u-1)} = \frac{1}{2}\left\{\int \frac{-1}{(u^2+1)}du + \int \frac{du}{(u-1)}\right\} \\ &= \tfrac{1}{2}\{-\arctan u + \ln|u-1|\} = \tfrac{1}{2}\{\ln|e^x - 1| - \arctan(e^x)\} + C \end{aligned}$$

- <u>Example (46, § 9.5)</u> Derive the formula:

$$\int \frac{dx}{x^2(a+bx)} = -\frac{1}{ax} - \frac{b}{a^2}\ln\left|\frac{x}{a+bx}\right| + C$$

As in example 26 discussed above, two different methods can be applied: the quadratic term can be factored into repeating linear terms (hence applying Strategy C or left alone and hence applying Strategy D.) I'll use strategy C (and you're welcome to apply strategy D as a separate exercise)

$$\frac{1}{x \cdot x \cdot (a+bx)} = \frac{A_1}{x} + \frac{A_2}{x^2} + \frac{A_3}{(a+bx)} \Rightarrow 1 = A_1 x(a+bx) + A_2(a+bx) + A_3 x^2$$

Inserting $x = 0$: $1 = aA_2 \Rightarrow A_2 = \frac{1}{a}$

Inserting $x = -\frac{a}{b}$: $1 = A_3 \frac{a^2}{b^2} \Rightarrow A_3 = \frac{b^2}{a^2}$

So: $1 = A_1 x(a+bx) + A_2(a+bx) + A_3 x^2 = A_1 x(a+bx) + \frac{1}{a}(a+bx) + \frac{b^2}{a^2} x^2$

Equating in terms of powers of x:

$$\begin{aligned}
x^2 &: \quad 0 = A_1 b + \tfrac{b^2}{a^2} \Rightarrow A_1 = -\tfrac{b}{a^2} \\
x^1 &: \quad 0 = A_1 a + \tfrac{d}{a} = -\tfrac{b}{a} + \tfrac{b}{a} = 0 \\
x^0 &: \qquad\quad 1 = \tfrac{1}{a} a = 1
\end{aligned}$$

(The last two equations were used as a check.)

So: $$\frac{1}{x \cdot x \cdot (a+bx)} = \frac{A_1}{x} + \frac{A_2}{x^2} + \frac{A_3}{(a+bx)} = -\frac{b}{a^2 x} + \frac{1}{ax^2} + \frac{b^2}{a^2(a+bx)}$$

$$\begin{aligned}
&\therefore \int \frac{dx}{x^2(a+bx)} = \int \left\{ \frac{1}{a} x^{-2} - \frac{b}{a} x^{-1} + \frac{b^2}{a^2}(a+bx)^{-1} \right\} dx \\
&= \frac{1}{a}\int \frac{dx}{x^2} - \frac{b}{a^2}\int \frac{dx}{x} + \frac{b^2}{a^2}\int \frac{dx}{(a+bx)} = -\frac{1}{ax} - \frac{b}{a^2}\ln|x| + \frac{b^2}{a^2}\int \frac{\frac{1}{b}dU}{U} \\
&= -\frac{1}{ax} - \frac{b}{a^2}\ln|x| + \frac{b}{a^2}\int \frac{dU}{U} = -\frac{1}{ax^2} - \frac{b}{a^2}\ln|x| + \frac{b}{a^2}\ln|a+bx| + C \\
&= -\frac{1}{ax^2} + \frac{b}{a^2}\ln\left|\frac{a+bx}{x}\right| + C
\end{aligned}$$

- ***USING THE INTEGRAL TABLE***

Referring to pages A14 - A19 in **Appendix C** of the text you'll encounter a standard integral table for a typical 200-level calculus text.[2] The table consists of 91 formulae, classified into eleven major categories. Interestingly, *all* of these formulae you can derive from first principles, based on the methods we've

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[2] There are quite a few more comprehensive integral tables for more advanced levels of mathematics and mathematical physics. The CRC manuals provide such a comprehensive compendium.

covered thus far. (In fact, in these and in previous course notes, I have derived several of the integration formulae in Appendix C. Try to find them!)

The eleven major categories are arranged as follows:

Category	***Fundamental Integrand Form***	***Formulae***
I	u^n	1., 2.
II	$a+bu$	3.-13.
III	$\sqrt{a+bu}$	16.-22.
IV	$a^2 \pm u^2$, $a>0$	23. – 25., 14.-15. [3]
V	$\sqrt{u^2 \pm a^2}$, $a>0$	26. -36.
VI	$\sqrt{a^2 - u^2}$, $a>0$	37. -45.
VII	sin$u$ or cos$u$	46. -58.
VIII	tan$u$ or cot$u$ or sec$u$ or csc$u$	59. -74.
IX	Inverse trigonometric functions (arcsin, arcos, etc.)	75. -80.
X	e^u	81.-86.
XI	ln$u$	87. -91.

The table is *comprehensive,* which means that *any* integral found in this textbook or equivalently any sophomore Calculus text can be resolved by applying the appropriate integral formula in Appendix C. However, using the table requires some experience in integration, since usually at least one u-substitution or simplification is necessary before comparing the integral expression in Appendix C.

The strategy however is straightforward:

1. Examine the part of the integrand that makes the integral difficult (or seemingly impossible) to perform. Compare the form of that part to any of the above fundamental integral forms to determine which category the integrand belongs to. If you can't find an equivalent form in any of the eleven cases, then a u-substitution is required to simplify.
2. Once you've ascertained which category the integrand belongs to, search through the formulae in that category for the one that matches the integran form exactly. Note that the formulae are arranged in increasing order of complexity.

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[3] Note: An extra minor section is introduced by the authors here for formulae 14., 15., in the category of integrals involving the term: $a+bu+cu^2, b^2 \neq 4ac$. However note that upon completing the square for such expressions, the result can be included in Category **IV.**

- Example (10, §9.7) $\int \frac{xdx}{\sqrt{9-x^4}}$

The denominator term $\sqrt{9-x^4}$ is what makes the integral difficult to evaluate. However in this form the $\sqrt{9-x^4}$ doesn't seem to belong to any of the above eleven categories. However note that $\sqrt{9-x^4} = \sqrt{3^2-(x^2)^2}$. So a natural choice is: $u = x^2 \Rightarrow \int \frac{x}{\sqrt{3^2-(x^2)^2}}dx = \int \frac{\frac{1}{2}du}{\sqrt{3^2-u^2}} = \frac{1}{2}\int \frac{du}{\sqrt{3^2-u^2}}$

…which makes it a category **VI.** Integral. In that category, **Formula 41.** is the one to apply:

$$\frac{1}{2}\int \frac{du}{\sqrt{3^2-u^2}} \xrightarrow{41.} \arcsin\left(\frac{u}{3}\right) + C$$

Don't forget however to express the final answer in terms of x!

$$\frac{1}{2}\int \frac{du}{\sqrt{3^2-u^2}} \xrightarrow{41.} \arcsin\left(\frac{u}{3}\right) + C = \arcsin\left(\tfrac{1}{3}x^2\right) + C$$

- Example (18, §9.7) $\int \frac{1}{2x^2(2x-1)^2}dx = \frac{1}{2}\int \frac{1}{x(2x-1)^2}dx$

What makes the integral difficult to perform is the (2x – 1) term in the denominator. This is a Category **II** integral, with $a = -1$, $b = 2$, $u = x$.

Referring to Appendix C., **Formula 12** is the one to apply:

$$\int \frac{1}{2x^2(2x-1)^2}dx = \frac{1}{2}\int \frac{1}{x(2x-1)^2}dx \xrightarrow{12} \frac{1}{2}\left\{\left(\frac{1}{x} - 2\ln\left|\frac{x}{2x-1}\right|\right)\right\} + C$$

(Note **that Formula 12** was derived using partial fractions in the previous exercise in page 7 above!)

- Example (32, §9.7) $\int \frac{\cos^3\sqrt{x}}{\sqrt{x}}dx$

$u = \sqrt{x} = x^{1/2} \Rightarrow du = \frac{1}{2}x^{-1/2}dx \Rightarrow dx = 2\sqrt{x}du$

$\therefore \int \frac{\cos^3 \sqrt{x}}{\sqrt{x}}dx = 2\int \frac{\cos^3 u\sqrt{x}}{\sqrt{x}}du = 2\int \cos^3 u du$ (A category **VII.** Integral). Formula **51** applies (a reduction formula from section 9.3)

$$2\int \cos^3 u du \xrightarrow{51} 2\left\{\frac{1}{3}\cos u \sin u + \frac{2}{3}\int \cos u du\right\} = \frac{2}{3}\left(\cos u \sin u + 2\int \cos u du\right)$$

$$= \frac{2}{3}(\cos u \sin u + 2\sin u) + C = \frac{2}{3}\sin u(\cos u + 2) + C = \frac{2}{3}\sin\sqrt{x}\left(\cos\sqrt{x} + 2\right) + C$$

- Example (39, §9.7) $\int \frac{\ln x}{x(3 + 2\ln x)}dx \Rightarrow u = \ln x, du = \frac{dx}{x} \Rightarrow \int \frac{u}{(3 + 2u)}du$

  (A category **II** integral, with Formula **3**. applicable)

  $$\int \frac{udu}{(3 + 2u)} \xrightarrow{3} \tfrac{1}{2^2}\left(2u - 3\ln|3 + 2u|\right) + C = \tfrac{1}{4}\left(2\ln x - 3\ln|3 + 2\ln x|\right) + C$$

- *ANOTHER TRIGONOMETRIC SUBSTITUTION*

Integrals involving rational expressions with trigonometric functions can often, despite all the techniques we've covered thus far, still be quite cumbersome and sometimes seemingly impossible to evaluate. Even worse, they often have no Appendix C equivalent form. Examples would include: $\int \frac{\cos x + 2\sin x}{\sin x - \cos x}dx$, etc.

In such cases, the following u –substitution proves handy;

$u = \tan\frac{x}{2}$

Then observe: $du = \frac{1}{2}\sec^2\frac{x}{2}dx \Rightarrow dx = \frac{2du}{\sec^2\frac{x}{2}} = \frac{2du}{1 + \tan^2\frac{x}{2}} = \frac{2du}{u^2 + 1}$

Furthermore:

$$u = \tan\tfrac{x}{2} = \frac{\sin\frac{x}{2}}{\cos\frac{x}{2}} = \frac{\sqrt{\frac{1}{2}(1-\cos x)}}{\sqrt{\frac{1}{2}(1+\cos x)}} = \sqrt{\frac{1-\cos x}{1+\cos x}} = \frac{\sqrt{(1-\cos x)(1+\cos x)}}{\sqrt{(1+\cos x)^2}}$$

$$= \frac{\sqrt{1-\cos^2 x}}{1+\cos x} = \frac{\sin x}{1+\cos x}$$

Hence:

$$u = \frac{\sin x}{1+\cos x} \Rightarrow u^2 = \frac{\sin^2 x}{(1+\cos x)^2} = \frac{1-\cos^2 x}{(1+\cos x)^2} = \frac{(1-\cos x)(1+\cos x)}{(1+\cos x)^2} = \frac{1-\cos x}{1+\cos x}$$

$$\Rightarrow (1+\cos x)u^2 = 1-\cos x \Rightarrow u^2 + u^2\cos x = 1-\cos x \Rightarrow u^2 - 1 = -\cos x\left(1+u^2\right)$$

$$\Rightarrow \cos x = \frac{1-u^2}{1+u^2}$$

Also:

$$\sin^2 x + \cos^2 x = 1 \Rightarrow \sin^2 x = 1-\cos^2 x = 1 - \frac{\left(1-u^2\right)^2}{\left(1+u^2\right)^2} = \frac{\left(1+u^2\right)^2 - \left(1-u^2\right)^2}{\left(1+u^2\right)^2}$$

$$= \frac{1+2u^2+u^4-\left(1-2u^2+u^4\right)}{\left(1+u^2\right)^2} = \frac{4u^2}{\left(1+u^2\right)^2} \Rightarrow \sin x = \sqrt{\frac{4u^2}{\left(1+u^2\right)^2}} = \frac{2u}{\left(1+u^2\right)}$$

Therefore, when adopting the substitution: $u = \tan\tfrac{x}{2}$, the following results:

$$du = \frac{2du}{u^2+1} \qquad \sin x = \frac{2u}{u^2+1} \qquad \cos x = \frac{1-u^2}{u^2+1}$$

- Example (61, §9.7) $\displaystyle\int_0^{\pi/2} \frac{1}{1+\sin\theta+\cos\theta}d\theta$

Adopt the substitution:

$$u = \tan\tfrac{\theta}{2} \Rightarrow u_1 = \tan\left(\frac{0}{2}\right) = \tan 0 = 0, u_2 = \tan\left(\frac{\frac{\pi}{2}}{2}\right) = \tan\frac{\pi}{4} = 1$$

(adjusting the limits of integration)

So:

$$\int_0^{\pi/2} \frac{1}{1+\sin\theta+\cos\theta}d\theta = \int_0^1 \frac{1}{1+\frac{2u}{u^2+1}+\frac{1-u^2}{1+u^2}} \cdot \frac{2du}{u^2+1}$$

$$\int_0^1 \frac{1}{1+\frac{2u}{u^2+1}+\frac{1-u^2}{1+u^2}}\cdot\frac{2du}{u^2+1}=2\int_0^1 \frac{du}{\left(\frac{u^2+1+2u+1-u^2}{u^2+1}\right)\left(u^2+1\right)}$$

$$=2\int_0^1 \frac{1}{2+2u}=\frac{2}{2}\int_0^1 \frac{du}{1+u}=\int_1^2 \frac{dU}{U}=\ln U\Big|_1^2=\ln 2-\ln 1=\ln 2$$