MA262 **Calculus II: Notes for Jan29**
Spring 2008 Kallfelz

- ***HINTS ON SOME EXERCISES (ASSIGNMENT I)***

I.b) (Exercise 18, p. 506)

$$\int \frac{x^3 e^{x^2}}{\left(x^2+1\right)^2} dx$$

- **Method 1**: Let $u = x^2$, and simplify via a few substitutions **before** integrating by parts:

$$\int \frac{x^3 e^{x^2}}{\left(x^2+1\right)^2} dx = \int \frac{x^2 x e^{x^2}}{\left(x^2+1\right)^2} dx = \tfrac{1}{2}\int \frac{u e^u}{(u+1)^2} du$$

Let $w = u + 1 \Rightarrow du = dw$, $u = w - 1$:

$$\int \frac{x^3 e^{x^2}}{\left(x^2+1\right)^2} dx = \tfrac{1}{2}\int \frac{u e^u}{(u+1)^2} du = \tfrac{1}{2}\int \frac{(w-1)}{w^2} e^{(w-1)} dw = \tfrac{1}{2e}\int \left(w^{-1} - w^{-2}\right) e^w dw$$
$$= \tfrac{1}{2e}\left\{\int w^{-1} e^w dw - \int w^{-2} e^{-w} dw\right\}$$

Integrate by parts for the first integral, letting (according to suggested strategy) $dV = e^w dw$ and $U = w^{-1}$. You'll find that a nice cancellation occurs (with the second integral). **Make sure, however, to express your final answer in terms of *x* (the original variable of integration.)**

- **Method 2:** Integrate by parts without any simplifying substitutions as below:

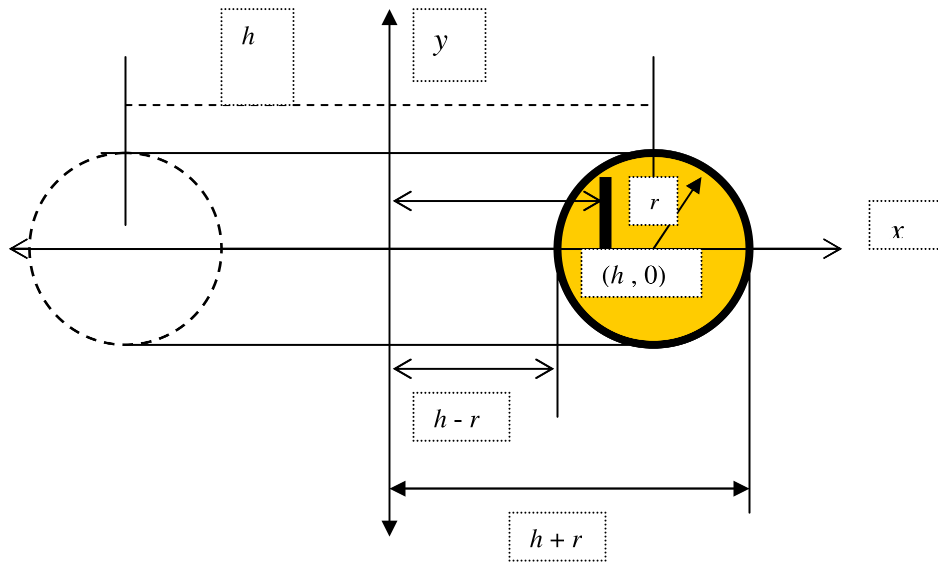
$$\int \frac{x^3 e^{x^2}}{\left(x^2+1\right)^2} dx = \int x^2 e^{x^2} \frac{x}{\left(x^2+1\right)} dx$$

Let $u = x^2 e^{x^2} \Rightarrow du = 2xe^{x^2} + x^2\left(2xe^{x2}\right) = 2xe^{x^2}\left(x^2+1\right)dx$

$$dv = \frac{x}{\left(x^2+1\right)^2} dx = \frac{\tfrac{1}{2} dU}{U^2} \Rightarrow v = -\tfrac{1}{2} U^{-1} = -\frac{1}{2\left(x^2+1\right)}$$

…you see that terms cancel nicely in the second integral (upon integrating by parts)

II.d) (Exercise 54, p. 526) See figure below:



Use method of shells (or washers[1]).

In the case of the shell method: $$V = 2\times 2\pi \int_{(h-r)}^{(h+r)} x\sqrt{r^2 + (x-h)^2}\,dx$$

- **Note1:** The term in the radical was constructed via the equation of the circle (of radius r centered at $(h , 0)$): $(x-h)^2 + y^2 = r^2$ (solving for y, the height of the rectangle)
- **Note 2:** Observe that the expression is doubled, since the inscribed rectangle (of width dx) starts at the x-axis, hence has height y. Of course, you could have perfectly well extended the rectangle down to $-y$, producing a rectangle of height $2y$ and this eliminating the need to double the answer.

Then use the following substitution: $u = x - h$. (So $x = u + h$). Rewrite integral in terms of u—**make sure you adjust the limits of the integral accordingly!—**this will produce a term $(u + h)$ in the integrand. Split the integral into the sum of two, according to the latter term. You'll notice one of the integrals can be directly evaluated by a simple U-substitution. The other one, of course, requires a trigonometric substitution.

[1] For extra credit, try both!

- *MORE ON TRIGONOMETRIC SUBSTITUTIONS*

- <u>Exercise: Derive the formulae in Thm 9.2 (p. 524)</u>

$$\int \sqrt{u^2 \pm a^2}^{1} du = \tfrac{1}{2}\left\{u\sqrt{u^2 \pm a^2} \pm \ln\left|u + \sqrt{u^2 \pm a^2}\right|\right\} + C$$

We'll work it out here for the "-" case (you can complete it for the "+" case, as an exercise)

$$\int \sqrt{u^2 - a^2}^{1} du \Rightarrow u = a\sec\theta \rightarrow du = a\sec\theta\tan\theta d\theta$$

$$\therefore \int \sqrt{a^2\sec^2\theta - a^2}\, a\sec\theta\tan\theta d\theta = \int \sqrt{a^2\tan^2\theta}\, a\sec\theta\tan\theta d\theta$$

$$= \int a^2\tan^2\theta\sec\theta d\theta = a^2\int\left(\sec^2\theta - 1\right)\sec\theta d\theta = a^2\left\{\int\sec^3\theta d\theta - \int\sec\theta d\theta\right\}$$

The first integral can be performed via the reduction formula (derived in Jan. 24 notes):

$$\int \sec^n x dx = \tfrac{1}{n-1}\sec^{n-2} x\tan x + \tfrac{n-2}{n-1}\int\sec^{n-2} x dx$$

So: $\int \sec^3\theta d\theta = \tfrac{1}{2}\sec\theta\tan\theta + \tfrac{1}{2}\int\sec\ \theta d\theta$

Inserting:

$$\int \sqrt{u^2 - a^2}^{1} du = a^2\left\{\int\sec^3\theta d\theta - \int\sec\theta d\theta\right\} = a^2\left\{\tfrac{1}{2}\sec\theta\tan\theta + \tfrac{1}{2}\int\sec\theta d\theta\right\} - a^2\int\sec\theta d\theta$$

$$= a^2\left\{\tfrac{1}{2}\sec\theta\tan\theta + \tfrac{1}{2}\int\sec\theta d\theta - \int\sec\theta d\theta\right\} = \tfrac{a^2}{2}\left\{\sec\theta\tan\theta - \int\sec\theta d\theta\right\}$$

$$= \tfrac{a^2}{2}\left\{\sec\theta\tan\theta - \ln\left|\sec\theta + \tan\theta\right|\right\} + C$$

Now, $u = a\sec\theta$, so:

Method 1 (Using a Pythagorean Identity):

$$\sec^2\theta - 1 = \tan^2\theta \Rightarrow \tan\theta = \sqrt{\sec^2\theta - 1} = \sqrt{\frac{a^2\sec^2\theta - a^2}{a^2}} = \tfrac{1}{a}\sqrt{a^2\sec^2\theta - a^1}$$

$$= \tfrac{1}{a}\sqrt{u^2 - a^2}$$

Method 2 (Using Right Triangles):

$$u = a\sec\theta \Rightarrow \sec\theta = \tfrac{u}{a} = \frac{HYP}{ADJ} \Rightarrow OPP = \sqrt{HYP^2 - ADJ^2} = \sqrt{u^2 - a^2}$$

$$\therefore \tan\theta = \frac{OPP}{ADJ} = \frac{\sqrt{u^2 - a^2}}{a} = \tfrac{1}{a}\sqrt{u^2 - a^2}$$

So:

$$\int \sqrt{u^2 - a^2}^{\,1}\, du = \tfrac{a^2}{2}\left\{\sec\theta\tan\theta - \ln\left|\sec\theta + \tan\theta\right|\right\} + C$$

$$= \tfrac{a^2}{2}\left\{\left(\frac{u}{a}\right)\left(\frac{\sqrt{u^2 - a^2}}{a}\right) - \ln\left|\left(\frac{u}{a}\right) + \left(\frac{\sqrt{u^2 - a^2}}{a}\right)\right|\right\} + C$$

$$= \tfrac{1}{2}\left\{a^2 \frac{u\sqrt{u^2 - a^2}}{a^2} - a^2 \ln\left|\frac{1}{a}\left(u + \sqrt{u^2 - a^2}\right)\right|\right\} + C$$

$$= \tfrac{1}{2}\left\{u\sqrt{u^2 - a^2} - a^2 \ln\left|u + \sqrt{u^2 - a^2}\right| + \ln|a|\right\} + C$$

$$\tfrac{1}{2}\left\{u\sqrt{u^2 - a^2} - a^2 \ln\left|u + \sqrt{u^2 - a^2}\right|\right\} + C$$

- **Note 3**: the ln(|1/*a*|) term was factored out as –ln|*a*| and absorbed into the indefinite constant term *C*

<u>**Exercise**: Derive a reduction formula for:</u> $\displaystyle\int \frac{du}{\left(u^2 + a^2\right)^n}$

$$u = a\tan\theta \Rightarrow du = a\sec^2\theta d\theta$$

Hence:

$$\int \frac{du}{\left(u^2 + a^2\right)^n} = \int \frac{a\sec^2\theta d\theta}{\left(a^2\tan^2\theta + a^2\right)^n} = \int \frac{a\sec^2\theta}{a^{2n}\sec^{2n}\theta} d\theta = \frac{1}{a^{2n-1}} \int \frac{d\theta}{\sec^{2n-2}\theta}$$

$$= \frac{1}{a^{2n-1}} \int \cos^{2(n-1)}\theta d\theta$$

Explicitly stated in terms of *u*:

$$u = a\tan\theta \Rightarrow \tan\theta = \frac{u}{a} = \frac{OPP}{ADJ} \Rightarrow HYP = \sqrt{OPP^2 + ADJ^2} = \sqrt{u^2 + a^2} \Rightarrow \cos\theta = \frac{ADJ}{HYP} = \frac{a}{\sqrt{u^2 + a^2}}$$

$$\int \frac{du}{\left(u^2+a^2\right)^n} = \frac{1}{a^{2n-1}} \int \cos^{2(n-1)} \theta d\theta = \frac{1}{a^{2b-1}} \int \left(\frac{a}{\sqrt{u^2+a^2}} \right)^{2n-2} du = \int \frac{a^{2n-2}}{a^{2n-1}\left(u^2+a^2\right)^{n-1}} du$$

$$= \tfrac{1}{a} \int \frac{du}{\left(u^2+a^2\right)^{n-1}}$$

- ***METHOD OF PARTIAL FRACTIONS (PART II)***

Recall (Jan 24th notes) that in the case of *distinct* (non-repeating) linear and quadratic irreducible factors, respectively:

Strategy A: $q(x) = \dfrac{u(x)}{(x-x_1).(x-x_2)...(x-x_m)} = \dfrac{A_{11}}{(x-x_1)} + \dfrac{A_2}{(x-x_2)} + \ldots + \dfrac{A_m}{(x-x_m)}$

Strategy B:

$$q(x) = \frac{u(x)}{\left(\alpha_1 x^2 + \beta_1 x + \gamma_1\right)...\left(\alpha_m x^2 + \beta_m x + \gamma_m\right)} = \frac{A_1 x + B_1}{\left(\alpha_1 x^2 + \beta_1 x + \gamma_1\right)} + \frac{A_2 x + B_2}{\left(\alpha_2 x^2 + \beta_2 x + \gamma_2\right)} + \ldots + \frac{A_m x + B_m}{\left(\alpha_m x^2 + \beta_m x + \gamma_m\right)}$$

On the other hand, in the case of ***repeated*** linear and quadratic irreducible factors, one must respectively apply:

Strategy C:

$$q(x) = \frac{u(x)}{(x-x_0)^n} = \frac{A_1}{(x-x_0)} + \frac{A_2}{(x-x_0)^2} + \ldots + \frac{A_n}{(x-x_0)^n}$$

Strategy D:

$$q(x) = \frac{u(x)}{\left(\alpha x^2 + \beta x + \gamma\right)^n} = \frac{A_1 x + B_1}{\left(\alpha x^2 + \beta x + \gamma\right)} + \ldots + \frac{A_n x + B_n}{\left(\alpha x^2 + \beta x + \gamma\right)^n}$$

…producing a total of four different cases (distinct linear or irreducible quadratic, repeating linear or irreducible quadratic, respectively.)

- **Note 4:** Of course, *any* combination or mixture of *any* of the above four cases can occur as well.
- **Note 5:** The authors say "quadratic" where I specify "quadratrc irreducible." There's a subtle but important difference. Any quadratic *reducible* expression can be broken down

into the product of two linear factors, i.e. if $\alpha x^2 + \beta x + \gamma$ is **reducible**, then it has two (either repeating or non-repeating) **real** roots x_0 , x_1 and hence may be re-written:

$$\alpha x^2 + \beta x + \gamma = \alpha(x - x_0)(x - x_1)$$

…and hence be treated as a case of the product of linear factors. In the exercise below, I'll work them both ways, if the case of a *reducible* quadratic expression arises. **However, my overall recommendation is to factor a reducible quadratic expression whenever possible, treating it as a product of linear terms. (The algebra is easier in this case). Hence my restricting the strategies C and D to the cases of *irreducible* quadratic expressions.**

- Example (35, § 9.5) Find: $\int_2^3 \frac{x^2 - x + 2}{x^3 - x^2 + x - 1}dx$

Attend to finding the anti-derivative first:

$$\int \frac{x^2 - x + 2}{x^3 - x^2 + x - 1}dx = \int \frac{x^2 - x + 2}{x^2(x-1) + (x-1)}dx = \int \frac{x^2 - x + 2}{(x^2+1)(x-1)}dx$$

The denominator is a product of one (hence distinct) linear factor with one (hence distinct) quadratic irreducible factor. Hence Strategies A and B must be employed:

$$\frac{x^2 - x + 2}{(x^2+1)(x-1)} = \frac{A_1}{(x-1)} + \frac{A_2 x + B_2}{(x^2+1)} \Rightarrow x^2 - x + 2 = A_1(x^2+1) + (A_2 x + B_2)(x-1)$$

Note that by inserting $x = 1$ in the top equation, A_1 is automatically obtained:

$$1^2 - 1 + 2 = A_1(1^2+1) + (A_2 + B_2)(1-1) \Leftrightarrow 2 = 2A_1 \Rightarrow A_1 = 1$$

Hence:

$$x^2 - x + 2 = (x^2+1) + (A_2 x + B_2)(x-1) = (1 + A_2)x^2 + (B_2 - A_2)x + (1 - B_2)$$

Solving by equating coefficients of powers of x:

$$\begin{aligned} x^2 &: \quad 1 = 1 + A_2 \Rightarrow A_2 = 0 \\ x^1 &: \quad -1 = (B_2 - A_2) = B_2 - 0 \Rightarrow B_2 = -1 \\ x^0 &: \quad 2 = 1 - B_2 = 1 + 1 \end{aligned}$$

- **Note 6**: The last equation (for the constant terms x^0) merely serves as a check

So: $$\frac{x^2-x+2}{(x^2+1)(x-1)}=\frac{A_1}{(x-1)}+\frac{A_2x+B_2}{(x^2+1)}=\frac{1}{(x-1)}+\frac{-1}{(x^2+1)}$$

This answer can likewise be checked through recombining via cross-multiplying:

$$\frac{1}{(x-1)}-\frac{1}{(x^2+1)}=\frac{(x^2+1)-(x-1)}{(x-1)(x^2+1)}=\frac{x^2-x+2}{(x-1)(x^2+1)}$$

Therefore:

$$\int\frac{x^2-x+2}{x^3-x^2+x-1}dx=\int\frac{x^2-x+2}{(x^2+1)(x-1)}dx=\int\left(\frac{1}{x-1}-\frac{1}{x^2+1}\right)dx=\int\frac{dx}{x-1}-\int\frac{dx}{x^2+1}=\ln|x-1|-\arctan x+C$$

So:

$$\int_2^3\frac{x^2-x+2}{x^3-x^2+x-1}dx=\left(\ln|x-1|-\arctan x\right)\Big|_2^3=\ln 2-\ln 1-\arctan 3+\arctan 2=\ln 2-\arctan 3+\arctan 2$$

- <u>Example:</u> Suppose the integrand in the above example was slightly modified:

$$\int\frac{x^2-x+2}{x^3-x^2-x+1}dx=\int\frac{x^2-x+2}{x^2(x-1)-(x-1)}dx=\int\frac{x^2-x+2}{(x^2-1)(x-1)}dx$$

Now we have a quadratic *reducible* factor in the denominator. Adopting my suggestion in **Note 5** means it should be factored, thus producing a mixture of strategies A and C:

<u>Method 1:</u>

$$\frac{x^2-x+2}{(x-1)(x+1)(x-1)}=\frac{x^2-x+2}{(x+1)(x-1)^2}=\frac{A_1}{(x+1)}+\frac{A_2}{(x-1)}+\frac{A_3}{(x-1)^2}$$
$$\Rightarrow x^2-x+2=A_1(x-1)^2+A_2(x+1)(x-1)+A_3(x+1)$$

Inserting $x=1$ gives us automatically A_3:[2]

$$1^2-1+2=A_1(1-1)^2+A_2(1+1)(1-1)+A_3(1+1)\Leftrightarrow 2=2A_3\Rightarrow A_3=1$$

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[2] Note that in class, I solved the system in a more cumbersome but more general fashion of equating powers of coefficients of x on both sides. Be familiar with both approaches!

Inserting x = -1 gives us automatically A_1:

$$(-1)^2-(-1)+2=A_1(-1-1)^2+A_2(-1+1)(-1-1)+A_3(-1+1)\Leftrightarrow 4=4A_1\Rightarrow A_1=1$$

Hence:

$$x^2-x+2=A_1(x-1)^2+A_2(x+1)(x-1)+A_3(x+1)=(x-1)^2+A_2(x+1)(x-1)+(x+1)$$
$$\Rightarrow x^2-x+2=(x^2-2x+1+x+1)+A_2(x^2-1)\Rightarrow x^2-x+2=x^2-x+2+A_2(x^2-1)$$
$$\Rightarrow 0=A_2(x^2-1)\Rightarrow A_2=0$$

So: $$\frac{x^2-x+2}{(x-1)(x+1)(x-1)}=\frac{x^2-x+2}{(x+1)(x-1)^2}=\frac{1_1}{(x+1)}+\frac{0}{(x-1)}+\frac{1}{(x-1)^2}=\frac{1}{(x+1)}+\frac{1}{(x\ \ -1)^2}$$

Check: $$\frac{1}{(x+1)}+\frac{1}{(x-1)^2}=\frac{(x-1)^2+(x+1)}{(x+1)(x-1)^2}=\frac{x^2-x+2}{(x+1)(x-1)^2}$$

Therefore:

$$\int\frac{x^2-x+2}{x^3-x^2-x+1}dx=\int\frac{x^2-x+2}{(x^2-1)(x-1)}dx=\int\left(\frac{1}{x+1}+\frac{1}{(x-1)^2}\right)dx$$
$$=\int\frac{dx}{x+1}+\int\frac{dx}{(x-1)^2}=\ln|x+1|-\frac{1}{(x-1)}+C$$

<u>**Method 2**</u>: Strategy A and B could be adopted instead (i.e., by not factoring the quadratic expression). **I don't however recommend this approach: factor whenever possible.**

$$\frac{x^2-x+2}{(x^2-1)(x-1)}=\frac{A_1x+B_1}{(x^2-1)}+\frac{A_2}{(x-1)}$$
$$\Rightarrow x^2-x+2=(A_1x+B_1)(x-1)+A_2(x^2-1)=(A_1+A_2)x^2+(B_1-A_1)x-(B_1+A_2)$$

x^2 : $\quad 1=A_1+A_2\Rightarrow A_1=1-A_2$
x^1 : $\quad -1=B_1-A_1=B_1-(1-A_2)\Rightarrow 0=B_1+A_2\Rightarrow B_1=-A_2$
x^0 : $\qquad 2=-(B_1+A_2)$

Inserting $B_1 = -A_2$ produces the contradictory result that 2 = 0, which means that the above family of equations has no solutions, i.e. there exist no real valued constants A_1, A_2, A_3 such that: $\frac{x^2 - x + 2}{(x^2-1)(x-1)} = \frac{A_1 x + B_1}{(x^2-1)} + \frac{A_2}{(x-1)}$. This is because the bottom two terms contain a common factor $(x - 1)$. **Hence only Method 1 applies.**

- Example (24, § 9.5)

$$\int \frac{x^3}{(x^2-4)^2} dx$$

Method 1 (Factor and apply strategies A and C)

$$\frac{x^3}{(x-2)^2(x+2)^2} = \frac{A_1}{(x-2)} + \frac{A_2}{(x-2)^2} + \frac{A_3}{(x+2)} + \frac{A_4}{(x+2)^2}$$
$$\Rightarrow x^3 = A_1(x-2)(x+2)^2 + A_2(x+2)^2 + A_3(x+2)(x-2)^2 + A_4(x-2)^2$$

Inserting $x = 2$:

$$2^3 = A_2(2+2)^2 \Rightarrow 8 = 16A_2 \Rightarrow A_2 = \tfrac{1}{2}$$

Inserting $x = -2$:

$$(-2)^3 = A_4(-2-2)^2 \Rightarrow -8 = 16A_4 \Rightarrow A_4 = -\tfrac{1}{2}$$

So:

$$x^3 = A_1(x-2)(x+2)^2 + A_2(x+2)^2 + A_3(x+2)(x-2)^2 + A_4(x-2)^2$$
$$= A_1(x-2)(x+2)^2 + \tfrac{1}{2}(x+2)^2 + A_3(x+2)(x-2)^2 - \tfrac{1}{2}(x-2)^2$$
$$= A_1(x-2)(x+2)^2 + A_3(x+2)(x-2)^2 + \tfrac{1}{2}\left(x^2 + 4x + 4 - (x^2 - 4x + 4)\right)$$
$$= A_1(x-2)(x+2)^2 + A_3(x+2)(x-2)^2 + 4x$$
$$= (x-2)(x+2)[A_1(x+2) + A_3(x-2)] + 4x \Rightarrow x^3 = (x^2-4)[(A_1 + A_3)x + 2(A_1 - A_3)] + 4x$$
$$\Rightarrow x^3 = (A_1 + A_3)x^3 + 2(A_1 - A_3)x^2 - 4(A_1 + A_3 - 1) - 8(A_1 - A_3)$$

Equating in terms of powers of x:

$$
\begin{aligned}
x^3 &: \quad 1 = A_1 + A_3 \Rightarrow A_3 = 1 - A_1 \\
x^2 &: \quad 0 = 2(A_1 - A_3) = 2(2A_1 - 1) \Rightarrow A_1 = \tfrac{1}{2}, A_3 = 1 - \tfrac{1}{2} = \tfrac{1}{2} \\
x^1 &: \quad 0 = -4(A_1 + A_3 - 1) = -4\left(\tfrac{1}{2} + \tfrac{1}{2} - 1\right) = 0 \\
x^0 &: \quad 0 = -8(A_1 - A_3) = -8\left(\tfrac{1}{2} - \tfrac{1}{2}\right) = 0
\end{aligned}
$$

(The last two equations were used as a check)

Hence: $A_1 = A_2 = A_3 = \tfrac{1}{2}, A_4 = -\tfrac{1}{2}$

So:

$$
\begin{aligned}
\frac{x^3}{(x-2)^2(x+2)^2} &= \frac{A_1}{(x-2)} + \frac{A_2}{(x-2)^2} + \frac{A_3}{(x+2)} + \frac{A_4}{(x+2)^2} \\
&= \tfrac{1}{2}\left[\frac{1}{x-2} + \frac{1}{(x-2)^2} + \frac{1}{x+2} - \frac{1}{(x+2)^2}\right]
\end{aligned}
$$

Hence:

$$
\int \frac{x^3}{(x^2-4)^2}\,dx = \tfrac{1}{2}\left\{\int \frac{dx}{x-2} + \int \frac{dx}{(x-2)^2} + \int \frac{dx}{x+2} - \int \frac{dx}{(x+2)^2}\right\}
$$

$$
= \tfrac{1}{2}\left\{\ln|x-2| - \frac{1}{(x-2)} + \ln|x+2| + \frac{1}{(x+2)}\right\} + C
$$

This answer can also be combined in a more elegant form:

$$
\int \frac{x^3}{(x^2-4)^2}\,dx = \tfrac{1}{2}\left\{\ln|x-2| - \frac{1}{(x-2)} + \ln|x+2| + \frac{1}{(x+2)}\right\} + C
$$

$$
= \tfrac{1}{2}\left\{\ln|(x-2)(x+2)| + \frac{1}{(x+2)} - \frac{1}{(x-2)}\right\} + C = \ln\sqrt{x^2-4} - \frac{2}{(x^2-4)} + C
$$

Method 2 (Apply strategies B and D)

$$
\frac{x^3}{(x^2-4)^2} = \frac{A_1 x + B_1}{(x^2-4)} + \frac{A_2 x + B_2}{(x^2-4)^2} \Rightarrow x^3 = (A_1 x + B_1)(x^2-4) + (A_2 x + B_2)
$$

Inserting $x = 2$: $2^3 = 2A_2 + B_2 \Rightarrow B_2 = 2(4 - A_2)$

Inserting $x = 0$: $0^3 = -4B_1 + B_2 \Rightarrow B_1 = \tfrac{1}{4}B_2 = \tfrac{1}{2}(4 - A_2)$

Hence:

$$x^3 = (A_1x + B_1)(x^2 - 4) + (A_2x + B_2) = (A_1x - \tfrac{1}{2}A_2 + 2)(x^2 - 4) + (A_2x - 2A_2 + 8)$$
$$x^3 = A_1x^3 + (2 - \tfrac{1}{2}A_2)x^2 + (A_2 - 4A_1)x + 2A_2 - 2A_2 = A_1x^3 + (2 - \tfrac{1}{2}A_2)x^2 + (A_2 - 4A_1)x$$

$$\begin{aligned} x^3 &: \quad 1 = A_1 \\ x^2 &: \quad 0 = (2 - \tfrac{1}{2}A_2) \Rightarrow A_2 = 4 \\ x^1 &: \quad 0 = (A_2 - 4A_1) = 4 - 4 = 0 \\ x^0 &: \quad 0 = 0 \end{aligned}$$

Note: The third equation is used as a check, and the last null equation serves merely in this case as a placeholder.

So: $B_2 = 2(4 - A_2) = 0$, $B_1 = \tfrac{1}{4}B_2 = \tfrac{1}{2}(4-4) = 0$

Hence: $$\frac{x^3}{(x^2-4)^2} = \frac{A_1x + B_1}{(x^2-4)} + \frac{A_2x + B_2}{(x^2-4)^2} = \frac{x}{(x^2-4)} + \frac{4x}{(x^2-4)^2}$$

…which can be checked yet again!

$$\frac{x}{(x^2-4)} + \frac{4x}{(x^2-4)^2} = \frac{x(x^2-4) + 4x}{(x^2-4)^2} = \frac{x^3 - 4x + 4x}{(x^2-4)^2} = \frac{x^3}{(x^2-4)^2}$$

Therefore:

$$\int \frac{x^3}{(x^2-4)^2}dx = \int \frac{x}{(x^2-4)^2}dx + 4\int \frac{x}{(x^2-4)^2}dx = \tfrac{1}{2}\int \frac{dU}{U} + 4 \cdot \tfrac{1}{2}\int \frac{dU}{U^2}$$
$$= \tfrac{1}{2}\ln|U| - 2U^{-1} + C = \tfrac{1}{2}\ln|x^2 - 4| - \frac{2}{(x^2-4)} + C = \ln\sqrt{x^2-4} - \frac{2}{(x^2-4)} + C$$

..thus agreeing with the first answer obtained in Method 1, presented in its more elegant form.