MA262 **Calculus II: Notes for Jan24**
Spring 2008 Kallfelz

- ***POWERS OF TRIG. FUNCTIONS (cont.)***

Recall the table from Jan. 22 notes:

Type	*Strategy*
$\sin^n x\cos^m x$ *n* or *m* odd **(Case 1)**	Reduce one to first power, use the Pythagorean Identiy : $\sin^2 x + \cos^2 x = 1$ to express in terms of powers of sine or cosine. The first power term is a "du" term . Can be converted to a simple *u*-substitution procedure, without the need for integrating by parts.
$\sin^n x\cos^m x$ ***n* and *m* both even** **(Case 2)**	**Use**: $\sin 2x = 2\sin x\cos x$ $\sin^2 x = {}^1/_2\,(1 - \cos 2x)$, $\cos^2 x = {}^1/_2\,(1 + \cos 2x)$ ***This procedure may have to be repeated more than once!***
$\tan^n x$, $\cot^n x$ (*n* odd) **(Case 3)**	Use: $\tan^2 x + 1 = \sec^2 x$ or $\cot^2 x + 1 = \csc^2 x$ (the second and third Pythagorean Identities). Can be converted to a simple *u*-substitution procedure, without the need for integrating by parts.
sin(mx)cos(nx) **(Case 4)**	**Use the Sum-Product Identities:** $\sin A\cos B = {}^1/_2[\sin(A-B)+\sin(A+B)]$ $\sin A\sin B = {}^1/_2[\cos(A-B) - \cos(A+B)]$ $\cos A\cos B = {}^1/_2[\cos(A-B)+\cos(A+B)]$ (No integration by parts necessary)
$\tan^n x$, $\cot^n x$ (*n* even) **(Case 5)**	Use: $\tan^2 x + 1 = \sec^2 x$ or $\cot^2 x + 1 = \csc^2 x$ (the second and third Pythagorean Identities.) **Integration by Parts is necessary**.
$\sec^n x$, $\csc^n x$ (*any n*) **(Case 6)**	Use: $\tan^2 x + 1 = \sec^2 x$ or $\cot^2 x + 1 = \csc^2 x$ (the second and third Pythagorean Identities.) **Integration by Parts is necessary**. Usually the Integration by Parts is laborious, as demonstrated above. Reduction Formulae are far more efficient. For derivations of some of them, see pp. 5-7, **<u>Nov 27</u> class notes,** as well as exercises 75-78, p. 517

Also, as the problem in pp. 4-7 (Jan 22 notes) indicates, reduction formulae are useful labor-saving devices for the inevitably intricate integrations in **Case 6** scenarios. As indicated in exercises 75-78 (p. 517), however, there are reduction formulae covering **Cases 1 & 2** as well:

A. $\int \sin^n x dx = -\frac{1}{n}\sin^{n-1} x\cos x + \frac{n-1}{n}\int \sin^{n-2} x dx$

B. $\int \cos^n x dx = \frac{1}{n}\cos^{n-1} x\sin x + \frac{n-1}{n}\int \cos^{n-2} x dx$

C. $\int \cos^m x\sin^n x dx = -\frac{1}{m+n}\sin^{n-1} x\cos^{m+1} x + \frac{n-1}{m+n}\int \cos^m x\sin^{n-2} x dx$

It's unclear, however, in the relative simple cases (1 & 2) that the above three formulae save you much time and effort (as opposed to just integrating form first principles, based on the above strategic points). For example, *Example* 1 (p. 8 **Jan. 22 notes**) can be integrated via formula C.:

$$\int \cos^3 x \sin^2 x dx = -\tfrac{1}{5}\sin x \cos^4 x + \tfrac{1}{5}\int \cos^3 x \sin^0 x dx = -\tfrac{1}{5}\sin x \cos^4 x + \tfrac{1}{5}\int \cos^3 x dx$$

…and then formula B. could be applied to the integral on the right hand side:

$$\int \cos^3 x dx = \tfrac{1}{3}\cos^2 x \sin x + \tfrac{2}{3}\int \cos x dx = \tfrac{1}{3}\cos^2 x \sin x + \tfrac{2}{3}\sin x + C$$

…and combining:

$$\begin{aligned}
&\int \cos^3 x \sin^2 x dx = -\tfrac{1}{5}\sin x \cos^4 x + \tfrac{1}{5}\left(\tfrac{1}{3}\cos^2 x \sin x + \tfrac{2}{3}\sin x\right) + C \\
&= -\tfrac{1}{5}\sin x\left(1-\sin^2 x\right)^2 + \tfrac{1}{15}\left(1-\sin^2 x\right)\sin x + \tfrac{2}{15}\sin x + C \\
&= -\tfrac{1}{5}\sin x\left(1 - 2\sin^2 x + \sin^4 x\right) + \tfrac{1}{15}\sin x - \tfrac{1}{15}\sin^3 x + \tfrac{2}{15}\sin x + C \\
&= -\tfrac{1}{5}\sin x + \tfrac{2}{5}\sin^3 x - \tfrac{1}{5}\sin^5 x + \tfrac{1}{15}\sin x - \tfrac{1}{15}\sin^3 x + \tfrac{2}{15}\sin x + C \\
&= -\tfrac{1}{5}\sin^5 x + \left(\tfrac{2}{5} - \tfrac{1}{15}\right)\sin^3 x + \left(-\tfrac{1}{5} + \tfrac{1}{15} + \tfrac{2}{15}\right)\sin x + C \\
&= -\tfrac{1}{5}\sin^5 x + \tfrac{1}{3}\sin x + C
\end{aligned}$$

…agreeing with our previous answer obtained in page 8 (**Jan. 22 notes**). But note all the extra work involved here! Using first principles, on the other hand, required only about 1-2 lines worth of calculation.

In addition to formulae A.-C. above, there are also the following reduction formulae:

D. $\int \sec^n x dx = \tfrac{1}{n-1}\sec^{n-2} x \tan x + \tfrac{n-2}{n-1}\int \sec^{n-2} x dx$

E. $\int \csc^n x dx = -\tfrac{1}{n-1}\csc^{n-2} x \cot x + \tfrac{n-2}{n-1}\int \csc^{n-2} x dx$

F. $\int \tan^n x dx = \tfrac{1}{n-1}\tan^{n-1} x - \int \tan^{n-2} x dx$

G. $\int \cot^n x dx = -\tfrac{1}{n-1}\cot^{n-1} x - \int \cot^{n-2} x dx$

…which can all be obtained via the strategies suggested in the above Table. For example, Formula F. is obtained via **Case 3.** procedure (i.e. one requiring implementing a Pythagorean identity, but not requiring integration by parts):

$$\int \tan^n x dx = \int \tan^{n-2} x \tan^2 x dx = \int \tan^{n-2} x\left(\sec^2 x - 1\right)dx = \int \tan^{n-2} x \sec^2 x - \int \tan^{n-2} x dx$$
$$= \int U^{n-2} dU - \int \tan^{n-2} x dx = \tfrac{1}{n-1}U^{n-1} - \int \tan^{n-2} x dx = \tfrac{1}{n-2}\tan^{n-1} x - \int \tan^{n-2} x dx$$

The derivation of formula G. works in a virtually identical matter (you're welcome to try it). Formula D. on the other hand falls under the strategy in Case 6., *requiring* the use of integration by parts:

$$\int \sec^n x dx = \int \sec^{n-2} x \sec^2 x dx = \int \sec^{n-2} x(\tan^2 x + 1)dx = \int \sec^{n-2} \tan^2 x dx + \int \sec^{n-2} x dx$$
$$= \int \sec^{n-3} x(\sec x \tan x)\tan x dx + \int \sec^{n-2} x dx$$

The first integral on the right hand side must be integrated by parts:

$$dv = \sec^{n-3} x(\sec x \tan x dx) \Rightarrow v = \tfrac{1}{n-2}\sec^{n-2} x$$
$$u = \tan x \Rightarrow du = \sec^2 x dx$$
$$\therefore \int \sec^n x dx = \left\{\tfrac{1}{n-2}\sec^{n-2} x \tan x - \tfrac{1}{n-2}\int \sec^{n-2} x \sec^2 x dx\right\} + \int \sec^{n-2} x dx$$
$$\int \sec^n x dx = -\tfrac{1}{n-2}\int \sec^n x dx + \tfrac{1}{n-2}\sec^{n-2} x \tan x + \int \sec^{n-2} x dx$$
$$\left(1 + \tfrac{1}{n-2}\right)\int \sec^n x dx = \tfrac{n-1}{n-2}\int \sec^n x dx = \tfrac{1}{n-2}\sec^{n-2} x \tan x + \int \sec^{n-2} x dx$$
$$\therefore \int \sec^n x dx = \tfrac{n-2}{n-1}\left\{\tfrac{1}{n-2}\sec^{n-2} x \tan x + \int \sec^{n-2} x dx\right\} = \tfrac{1}{n-1}\sec^{n-2} x \tan x + \tfrac{n-2}{n-1}\int \sec^{n-2} x dx$$

The strategy of deriving formula D. works virtually the same way.[1]

And as the procedure in page 7 (**Jan 22 notes**) indicates, adopting a reduction formula (D.) is far more economical than trying to resolve the integral from using first principles in a **Case 6** scenario, which is in direct opposition to the above example involving **Case 1** scenario versus reduction formulae A., B., C.

So when *do* the reduction formulae A. – G. become a time-saving procedure, in comparison to integrating from first principles adopting the strategies suggested for Cases 1.-6.?

- **Answer: i) Use Formulae D. – G. when encountering a Case 5 or Case 6 scenario. ii.) Use first principles suggested by the strategies in the above table for Cases 1 and 2.[2] iii.) For relatively small *n* either approach will end up costing you about the same amount of work, for Case 3. However, for *n* > 7, integrating from first principles using strategy suggested in Case 3 is more efficient.**

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[1] ..and was done in in pp. 5-6 **Nov. 27 notes (Calc 1) :** http://www.glue.umd.edu/%7Ewkallfel/MA261-2/Nov27notes.pdf

[2] An exception might be if you're trying to write an algorithm or you have some aversion to using the double angle formulae in Case 2 (which as shown in Example 2, p. 8 **Jan 22 notes** often has to be used more than once.)

- ***TRIGONOMETRIC SUBSTITUTIONS***

Recall from Calculus I the definition of arc-length:[3]

$$L = \int_a^b \sqrt{1+\left(f'(x)\right)^2}\,dx = \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

…for a parameterization of a function $y = f(x)$.[4] However, in the simple case of evaluating arc lengths of parabolae and circles introduces integrals of the form:

$$L = \int_a^b \sqrt{1+a^2x^2}\,dx \qquad , \qquad L = \int_a^b \frac{dx}{\sqrt{r^2-x^2}}$$

…for parabolae and circles, respectively. Such expressions *cannot* (up until now) be evaluated by exact means.[5]

This is just the tip of the iceberg. Generally, binomial expressions of half integer power, i.e. expressions of the form $\left(a \pm u(x)\right)^{\frac{n}{2}}$ (where $u(x)$ is some function) pose non-trivial technical challenges.[6] Basically, the **method of trigonometric substitutions** eliminates such undesirable cases, by reducing a half-integer powered binomial expression $\left(a^2 \pm u(x)^2\right)^{\frac{n}{2}}$ into a single (trigonometric) function.

The procedure can be summarized via the following recipe:

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[3] For a refresher, see § 6.4, text, or **Oct 16** course notes: http://www.glue.umd.edu/%7Ewkallfel/MA261-2/Oct15notes.pdf

[4] One could, as in the case of evaluating volumes, areas, also parameterize in terms of y, i.e. $x(y) = f^{-1}(y)$, where f^{-1} is the inverse function. Then the above formula would take on form:

$$L = \int_{y_1=f^{-}(a)}^{y_2=f^{-1}(b)} \sqrt{1+\left(f^{-1'}(y)\right)^2}\,dy = \int_{y_1=f^{-1}(a)}^{y_2=f^{-1}(b)} \sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy$$

[5] For an example involving an approximate procedure, namely, the trapezoidal rule, applied to a problem for calculating the arc-length of a segment of a circle of radius 3, see pp. 2-3, **Oct 16** course notes http://www.glue.umd.edu/%7Ewkallfel/MA261-2/Oct15notes.pdf

[6] Later in this course we'll view binomial expansion in terms of a power-series according to the Binomial Theorem: $(a \pm b)^p = \sum_{k=o}^{p} (\pm 1)^k \binom{p}{k} a^{n-k} b^k$, where: $\binom{p}{k} = \frac{p!}{k!(p-k)!}$.

Binomial Expressions	*Substitute*
$\left(a^2 - u(x)^2\right)^{\frac{n}{2}}$	$u(x) = a\sin\theta$
$\left(a^2 + u(x)^2\right)^{\frac{n}{2}}$	$u(x) = a\tan\theta$
$\left(u(x)^2 - a^2\right)^{\frac{n}{2}}$	$u(x) = a\sec\theta$

…which collapses the above binomial expressions to a single trigonometric function, by virtue of the Pythagorean Identities:

PI-1: $\sin^2\theta + \cos^2\theta = 1 \Rightarrow 1 - \sin^2\theta = \cos^2\theta$
PI-2: $\tan^2\theta + 1 = \sec^2\theta \Rightarrow \sec^2\theta - 1 = \tan^2\theta$
PI-3: $\cot^2\theta + 1 = \csc^2\theta \Rightarrow \csc^2\theta - 1 = \cot^2\theta$

..because:

$$\left(a^2 - u(x)^2\right)^{\frac{n}{2}} = \left(a^2 - a^2\sin^2\theta\right)^{\frac{n}{2}} = \left(a^2\left(1 - \sin^2\theta\right)\right)^{\frac{n}{2}} = \left(a^2\cos^2\theta\right)^{\frac{n}{2}} = (a\cos\theta)^n$$
$$\left(a^2 + u(x)^2\right)^{\frac{n}{2}} = \left(a^2 + a^2\tan^2\theta\right)^{\frac{n}{2}} = \left(a^2\left(1 + \tan^2\theta\right)\right)^{\frac{n}{2}} = \left(a^2\sec^2\theta\right)^{\frac{n}{2}} = (a\sec\theta)^n$$
$$\left(u(x)^2 - a^2\right)^{\frac{n}{2}} = \left(a^2\sec^2\theta - a^2\right)^{\frac{n}{2}} = \left(a^2\left(\sec^2\theta - 1\right)\right)^{\frac{n}{2}} = \left(a^2\tan^2\theta\right)^{\frac{n}{2}} = (a\tan\theta)^n$$

…adopting PI-1, PI-2 respectively.

- **Note:** In principle, one could just as well substitute $u(x) = a\cos\theta$, $u(x) = a\cot\theta$, $u(x) = a\csc\theta$ (using PI-3 for the latter two cases) for the above three cases and achieve the same result. However, when examining: $\frac{du}{d\theta} = \left(\frac{du}{dx}\right)\left(\frac{dx}{d\theta}\right) \Rightarrow dx = \left(\frac{du}{d\theta}\right)\left(\frac{d\theta}{dx}\right)d\theta$, the disadvantage is that extra minus signs are introduced in the expression for *dx*, introducing the increased possibility for making sign errors. For example: $dx = \frac{d}{d\theta}(a\sin\theta)d\theta = a\cos\theta d\theta$, but: $dx = \frac{d}{d\theta}(a\cos\theta)d\theta = -a\sin\theta d\theta$. So by convention, the recipes are suggested in the table above.[7]

[7] Recall another instance of such a convention: when constructing anti-derivative formulae for the expressions involving inverse trigonometric functions. (For further details see § 8.5-8.6 in text or pp. 2-14, **Nov. 13 notes** http://www.glue.umd.edu/%7Ewkallfel/MA261-2/Nov13notes.pdf and pp. 2-6, **Nov. 15th** notes http://www.glue.umd.edu/%7Ewkallfel/MA261-2/Nov15notes.pdf

- Example (# 19, § 9.4)

$$\int_0^{\sqrt{3}/2} \frac{t^2}{\left(1-t^2\right)^{3/2}} dt$$

Choose:
$$t(\theta) = \sin\theta \Rightarrow \theta = \arcsin t$$
$$\Rightarrow dt = \cos\theta d\theta$$

Substituting:

$$\int_0^{\sqrt{3}/2} \frac{t^2}{\left(1-t^2\right)^{3/2}} dt = \int_{\arcsin 0}^{\arcsin\left(\sqrt{3}/2\right)} \sin^2\theta\left(1-\sin^2\theta\right)^{-3/2}\cos\theta d\theta = \int_0^{\pi/3} \frac{\sin^2\theta}{\left(\cos^2\theta\right)^{3/2}}\cos\theta d\theta$$
$$= \int_0^{\pi/3} \frac{\sin^2\theta}{\cos^3\theta}\cos\theta d\theta = \int_0^{\pi/3} \frac{\sin^2\theta}{\cos^2\theta} d\theta = \int_0^{\pi/3} \tan^2\theta d\theta = \int_0^{\pi/3}\left(\sec^2\theta - 1\right)d\theta = \left(\tan\theta - \theta\right)\Big|_0^{\pi/3}$$
$$= \left(\tan\tfrac{\pi}{3} - \tfrac{\pi}{3}\right) - \left(\tan 0 - 0\right) = \sqrt{3} - \tfrac{\pi}{3} \approx 0.684853$$

- Note how the use of the strategy from Case 5 (Table in page 1 of these notes) was adopted, though integration by parts wasn't necessary since the power of secant was only $n = 2$.

- Example (# 31, § 9.4)

$$\int \frac{x}{\sqrt{x^2+4x+8}} dx$$

Completing the square in the denominator term:

$$\int \frac{x}{\sqrt{x^2+4x+8}} dx = \int \frac{xdx}{\sqrt{x^2+4x+4-4+8}} = \int \frac{xdx}{\sqrt{(x+2)^2+2^2}}$$

Let $u = x + 2 = 2\tan\theta$. Then:
$$du = dx = 2\sec^2\theta d\theta$$
$$x = u - 2 = 2\tan\theta - 2 = 2(\tan\theta - 1)$$
$$\theta = \arctan\left(\frac{u}{2}\right) = \arctan\left(\frac{x+2}{2}\right)$$

Substituting:

$$\int \frac{xdx}{\sqrt{(x+2)^2+2^2}} = \int \frac{2(\tan\theta-1)2\sec^2\theta d\theta}{\sqrt{4\tan^2\theta+4}} = 2\int \frac{(\tan\theta-1)\sec^2\theta d\theta}{\sqrt{\tan^2\theta+1}} = 2\int \frac{(\tan\theta-1)\sec^2\theta d\theta}{\sec\theta}$$

$$= 2\int (\tan\theta - 1)\sec\theta d\theta = 2\int \sec\theta\tan\theta d\theta - 2\int \sec\theta d\theta = 2\sec\theta - 2\ln\left|\sec\theta + \tan\theta\right| + C$$

OK, so now the integration is completed, however, we still haven't answered the question! Recall this is an *indefinite integral* in terms of a function (or dummy variable) with respect to independent variable x, not θ. So we must likewise express the above answer in terms of x, not θ.

Recall that $\theta = \arctan\left(\dfrac{u}{2}\right) = \arctan\left(\dfrac{x+2}{2}\right)$, so obviously: $\tan\theta = \dfrac{u}{2} = \frac{1}{2}(x+2)$. But the above answer is also expressed in terms of $\sec\theta$, which must be expressed in terms of x as well.:

Method 1: (Algebraic). Use Pythagorean Identity PI-2:

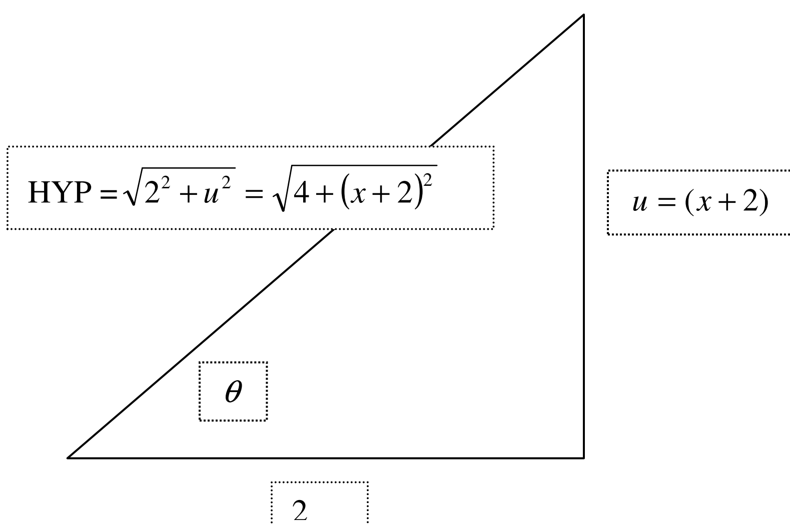
$$\tan^2\theta + 1 = \sec^2\theta \Rightarrow \sec^2\theta - 1 = \tan^2\theta \Rightarrow \sec\theta = \sqrt{\tan^2\theta + 1}$$

So:

$$\sec\theta = \sqrt{\tan^2\theta + 1} = \sqrt{\tfrac{1}{4}u^2 + 1} = \sqrt{\tfrac{1}{4}(x+2)^2 + 1} = \tfrac{1}{2}\sqrt{(x+2)^2 + 4} = \tfrac{1}{2}\sqrt{x^2 + 4x + 8}$$

Method 2: (Geometric). $\tan\theta = \dfrac{u}{2} = \frac{1}{2}(x+2) = \dfrac{OPP}{ADJ}$

Hence using the Pythagorean Theorem to one can specify all three sides of the triangle below:



$$\text{Hence: } \sec\theta = \frac{HYP}{ADJ} = \frac{\sqrt{2^2 + u^2}}{2} = \tfrac{1}{2}\sqrt{x^2 + 4x + 8}$$

Hence:

$$\int \frac{xdx}{\sqrt{(x+2)^2+2^2}} = 2(\sec\theta - \ln|\sec\theta + \tan\theta|) + C = 2\left\{\tfrac{1}{2}\sqrt{x^2+4x+8} - \ln\left|\tfrac{1}{2}\sqrt{x^2+4x+8} + \tfrac{1}{2}(x+2)\right|\right\} + C$$
$$= \sqrt{x^2+4x+8} - 2\ln\left|\tfrac{1}{2}\left(\sqrt{x^2+4x+8} + (x+2)\right)\right| + C = \sqrt{x^2+4x+8} - 2\ln\left|\sqrt{x^2+4x+8} + (x+2)\right| - 2\ln 2 + C$$
$$= \sqrt{x^2+4x+8} - 2\ln\left|\sqrt{x^2+4x+8} + (x+2)\right| + C$$

(Recall **Note1** on page 1 of the handwritten posted answers of the selected exercises in sections 9.1-9.3: Here we used the property of ln(ab) = ln$a$ + ln$b$ to pull out the constant term 2ln(1/2), which got absorbed in the indeterminate integration constant term C.)

- Example (# 33, § 9.4)

$$\int e^x\sqrt{1-e^{2x}}\,dx = \int e^x\sqrt{1-(e^x)^2}\,dx$$

Let: $e^x = \sin\theta \Rightarrow x = \ln|\sin\theta| \Rightarrow dx = \frac{d}{d\theta}\ln|\sin\theta|\,d\theta = \dfrac{\cos\theta}{\sin\theta}d\theta = \cot\theta d\theta$

$\Rightarrow \theta = \arcsin(e^x)$

Hence:

$$\int e^x\sqrt{1-(e^x)^2}\,dx = \int \sin\theta\sqrt{1-\sin^2\theta}\cot\theta d\theta = \int \sin\theta\cos\theta\cot\theta d\theta = \int \cos^2\theta d\theta$$
$$= \int \tfrac{1}{2}(1+\cos 2\theta)d\theta = \tfrac{1}{2}\theta + \tfrac{1}{2}\int \cos 2\theta d\theta = \tfrac{1}{2}\theta + \tfrac{1}{4}\sin 2\theta + C$$

Now to substitute to express the answer in terms of x, note the in the above we have the expression $\sin 2\theta = 2\sin\theta\cos\theta$ (simplifying via the double angle formula). As in the above, one can use either method (algebraic or geometric) to express $\cos\theta$ in terms of x:

Method 1: Use PI-1:

$$\sin^2\theta + \cos^2\theta = 1 \Rightarrow 1 - \sin^2\theta = \cos^2\theta \Rightarrow \cos\theta = \sqrt{1-\sin^2\theta} = \sqrt{1-e^{2x}}$$

Method 2:

$$e^x = \sin\theta = \frac{OPP}{HYP} = \frac{e^x}{1},$$

hence according to Pythagorean Thm: $ADJ = \sqrt{HYP^2 - OPP^2} = \sqrt{1-e^{2x}}$

So: $\cos\theta = \dfrac{ADJ}{HYP} = \dfrac{\sqrt{1-e^{2x}}}{1} = \sqrt{1-e^{2x}}$

Hence:

$$\int e^x \sqrt{1-(e^x)^2}\,dx = \tfrac{1}{2}\theta + \tfrac{1}{4}\sin 2\theta + C = \tfrac{1}{2}(\theta + \sin\theta\cos\theta) + C$$
$$= \tfrac{1}{2}\left(\arcsin(e^x) + e^x\sqrt{1-e^{2x}}\right) + C$$

- *Example*

$$\int (9+x^2)^{3/2}\,dx = \int (3^2+x^2)^{3/2}\,dx$$

$$x(\theta) = 3\tan\theta \Rightarrow dx = 3\sec^2\theta d\theta$$

$$\int (9+x^2)^{3/2}\,dx = \int (3^2+x^2)^{3/2}\,dx = \int (3^2+3^2\tan^2\theta)3\sec^2\theta d\theta$$

$$= \int (3^2\sec^2\theta)^{3/2} 3\sec^2\theta d\theta = 3^4\int \sec^3\theta\sec^2\theta d\theta = 81\int \sec^5\theta d\theta$$

Adopting reduction formula D. (mentioned in p. 2 above):

$$\int \sec^n x dx = \tfrac{1}{n-1}\sec^{n-2} x\tan x + \tfrac{n-2}{n-1}\int \sec^{n-2} x dx$$

$$\int \sec^5\theta d\theta = \tfrac{1}{4}\sec^3\theta\tan\theta + \tfrac{3}{4}\int \sec^3\theta d\theta \qquad (n=5)$$
$$\int \sec^3\theta d\theta = \tfrac{1}{2}\sec\theta\tan\theta + \tfrac{1}{2}\int \sec\theta d\theta = \tfrac{1}{2}\left(\sec\theta\tan\theta + \ln|\sec\theta + \tan\theta|\right) + C \qquad (n=3)$$

So combining:

$$\int \sec^5\theta d\theta = \tfrac{1}{4}\sec^3\theta\tan\theta + \tfrac{3}{4}\int \sec^3\theta d\theta = \tfrac{1}{4}\left\{\sec^3\theta\tan\theta + \tfrac{3}{2}\left[\sec\theta\tan\theta + \ln|\sec\theta+\tan\theta|\right]\right\} + C$$
$$= \tfrac{1}{8}\left\{2\sec^3\theta\tan\theta + 3\sec\theta\tan\theta + \ln|\sec\theta+\tan\theta|\right\} + C$$

To find $\sec\theta$ in terms of *x*:

Method 1: $\tan^2\theta + 1 = \sec^2\theta \Rightarrow \sec\theta = \sqrt{\tan^2\theta+1} = \sqrt{\tfrac{x^2}{3^2}+1} = \tfrac{1}{3}\sqrt{x^2+9}$

Method 2:

$$\tan\theta = \frac{x}{3} = \frac{OPP}{ADJ} \Rightarrow HYP = \sqrt{OPP^2 + ADJ^2} = \sqrt{3^2 + x^2} = \sqrt{x^2+9}$$

$$\Rightarrow \sec\theta = \frac{HYP}{ADJ} = \frac{\sqrt{x^2+9}}{3} = \tfrac{1}{3}\sqrt{x^2+9}$$

Hence: $\int \left(9 + x^2\right)^{3/2} dx = \int \left(3^2 + x^2\right)^{3/2} dx =$

$$= \tfrac{1}{8}\left\{2\sec^3\theta\tan\theta + 3\sec\theta\tan\theta + \ln\left|\sec\theta + \tan\theta\right|\right\} + C$$

$$= \tfrac{1}{8}\left\{\tfrac{2}{81}\left(x^2+9\right)^{3/2}x + \left(x^2+9\right)^{1/2}x + \ln\left|\tfrac{1}{3}\left[\left(x^2+9\right)^{1/2} + x\right]\right|\right\} + C$$

$$= \tfrac{1}{8}\left\{\tfrac{2}{81}x\left(x^2+9\right)^{3/2} + \left(x^2+9\right)^{1/2}x + \ln\left|\left(x^2+9\right)^{1/2} + x\right|\right\} + C$$

- *METHOD OF PARTIAL FRACTIONS (Part A: Non-repeating terms)*

Many integrals involving complicated rational expressions can become resolved via this *entirely algebraic* method, which essentially amounts to the reverse of combining rational expressions (i.e. fractions) via finding the common denominator (through finding the LCM, least common multiple).

CaseI: The simplest case is when the rational expression involves a product of m non-repeating linear factors in the denominator, i.e.:

$$q(x) = \frac{u(x)}{(x - x_1).(x - x_2)...(x - x_m)}$$

Then one must decompose $q(x)$ in the following manner:

$$q(x) = \frac{u(x)}{(x - x_1).(x - x_2)...(x - x_m)} = \frac{A_{11}}{(x - x_1)} + \frac{A_2}{(x - x_2)} + ... + \frac{A_m}{(x - x_m)}$$

…and solve for the undetermined constants.

For example, suppose one wished to decompose the expression: $\frac{1}{x(x-1)} = \frac{1}{(x-0)(x-1)}$ into a sum of two simple fractions, then according to the above:

$$q(x) = \frac{1}{x(x-1)} = \frac{A_{11}}{(x-0)} + \frac{A_2}{(x-1)}$$

The above two constants can be easily solved for via multiplying across by the denominator term:

$$q(x) = \frac{1}{x(x-1)} = \frac{A_{11}}{(x-0)} + \frac{A_2}{(x-1)} \Rightarrow 1 = A_1(x-1) + A_2 x$$

- Set $x = x_1 = 0$ in the above equation to solve for A_1: $1 = -A_1 \Rightarrow A_1 = -1$.
- Set $x = x_2 = 1$ in the above equation to solve for A_2: $1 = A_2 \Rightarrow A_2 = 1$.

Hence: $\frac{1}{x(x-1)} = \frac{-1}{x} + \frac{1}{(x-1)}$

..which can be easily checked: $\frac{-1}{x} + \frac{1}{(x-1)} = \frac{1-x+x}{x(x-1)} = \frac{1}{x(x-1)}$

- To see an interesting example of the above method at work in an integral application, see the problem discussed on logistical growth (predator-prey model) on pp. 5-8, **Oct 30 notes:**

http://www.glue.umd.edu/%7Ewkallfel/MA261-2/Oct30notes.pdf

CaseII: The next simplest case is when the rational expression involves a product of m non-repeating quadratic irreducible factors[8] in the denominator, i.e.:

$$q(x) = \frac{u(x)}{(\alpha_1 x^2 + \beta_1 x + \gamma_1)(\alpha_2 x^2 + \beta_2 x + \gamma_2)\ldots(\alpha_m x^2 + \beta_m x + \gamma_m)}$$

Then one must decompose $q(x)$ in the following manner:

$$q(x) = \frac{A_1 x + B_1}{(\alpha_1 x^2 + \beta_1 x + \gamma_1)} + \frac{A_2 x + B_2}{(\alpha_2 x^2 + \beta_2 x + \gamma_2)} + \ldots + \frac{A_m x + B_m}{(\alpha_m x^2 + \beta_m x + \gamma_m)}$$

…and solve for the undetermined constants.

[8] Recall that a quadratic irreducible is one that cannot be factored into two linear terms over the real numbers, because it has no real roots. For example: $x^2 + 1$ is a quadratic irreducible. The way to check in any general case for a quadratic expression $ax^2 + bx + c$ is via the *discriminant*: $D = b^2 - 4ac$. If $D < 0$, then it's irreducible.

For example:

$$q(x) = \frac{x^2 + 3x}{(x^2+1)(x^2+2x+3)} = \frac{A_1 x + B_1}{(x^2+1)} + \frac{A_2 x + B_2}{(x^2+2x+3)}$$

To solve for the four undetermined constants, begin by multiplying across the denominator term:

$$q(x) = \frac{x^2 + 3x}{(x^2+1)(x^2+2x+3)} = \frac{A_1 x + B_1}{(x^2+1)} + \frac{A_2 x + B_2}{(x^2+2x+3)}$$
$$\Rightarrow x^2 + 3x = (A_1 x + B_1)(x^2+2x+3) + (A_2 x + B_2)(x^2+1)$$
$$\Rightarrow x^2 + 3x = A_1 x^3 + (B_1 + 2A_1)x^2 + (3B_1 + 2A_1)x + 3B_1 + A_2 x^3 + B_2 x^2 + A_2 x + B_2$$
$$\Rightarrow x^2 + 3x = (A_1 + A_2)x^3 + (B_1 + 2A_1 + B_2)x^2 + (3B_1 + 2A_1 + A_2)x + (3B_1 + B_2)$$

The four constants can be solved by equating the coefficients of the powers of x terms (i.e. extracting four equations from the above, beginning with x^3 terms and ending with x^0) to create the following 4 x 4 system:

$$\begin{aligned}
x^3 &: \quad 0 = A_1 + A_2 \\
x^2 &: \quad 1 = B_1 + 2A_1 + B_2 \\
x^1 &: \quad 3 = 3B_1 + 2A_1 + A_2 \\
x^0 &: \quad 0 = 3B_1 + B_2
\end{aligned}$$

There are fancy matrix methods (Cramer's Rule, etc.) and simple substitution methods to solve for N x N linear systems of equations. In this case, the above can be solved by simple substitution, beginning with the first and last equation:

$$\begin{aligned}
x^3 &: \quad 0 = A_1 + A_2 \Rightarrow A_1 = -A_2 \\
x^2 &: \quad 1 = B_1 + 2A_1 + B_2 \\
x^1 &: \quad 3 = 3B_1 + 2A_1 + A_2 \\
x^0 &: \quad 0 = 3B_1 + B_2 \Rightarrow B_2 = -3B_1
\end{aligned}$$

Hence:

$$\begin{aligned}
x^2 &: \quad 1 = B_1 - 2A_2 - 3B_1 = -2B_1 - 2A_2 \\
x^1 &: \quad 3 = 3B_1 - 2A_2 + A_2 = 3B_1 - A_2
\end{aligned}$$

So[9]: $3\times"x^2"+2\times"x^1": \Rightarrow 9=-8A_2 \Rightarrow A_2=-\frac{9}{8} \Rightarrow B_1=\frac{15}{24}$

Check: $1=-2\left(\frac{15}{24}\right)-2\left(-\frac{9}{8}\right)=-\frac{30}{24}+\frac{18}{8}=\frac{-30+54}{24}=\frac{24}{24}=1$

$3=3\left(\frac{15}{24}\right)-\left(-\frac{9}{8}\right)=\frac{45}{24}+\frac{9}{8}=\frac{45+27}{24}=\frac{72}{24}=3$

Hence: $A_1=-A_2=\frac{9}{8} \qquad B_2=-3B_1=-\frac{45}{24}=-\frac{15}{8}$

So:

$$q(x)=\frac{x^2+3x}{(x^2+1)(x^2+2x+3)}=\frac{\frac{9}{8}x+\frac{15}{24}}{(x^2+1)}+\frac{-\frac{9}{8}x-\frac{15}{8}}{(x^2+2x+3)}$$

$$=\frac{1}{24}\left\{\frac{27x+15}{(x^2+1)}-\frac{27x+45}{(x^2+2x+3)}\right\}$$

Certainly combinations of Cases I & Cases II can occur, like: $q(x)=\dfrac{x}{(x+1)(x^2+1)}$

…which gets resolved via the procedure:

$$q(x)=\frac{x}{(x+1)(x^2+1)}=\frac{A_1}{x+1}+\frac{A_2x+B_2}{x^2+1}$$

- <u>Example (# 9, §9.5)</u>

$$\int\frac{2x^3-4x^2-15x+5}{x^2-2x-8}dx$$

Before you start to use partial fractions, note that the integrand can be reduced (since the degree of the numerator > degree of the denominator). A cumbersome way to reduce it would be via polynomial long division. There's a simpler way, however, in this case, which involves teasing out the denominator term in the first three terms of the numerator:

$$\frac{2x^3-4x^2-15x+5}{x^2-2x-8}=\frac{2x^3-4x^2-16x+x+5}{x^2-2x-8}=\frac{2x(x^2-2x-8)+x+5}{x^2-2x-8}$$

$$=\frac{2x(x^2-2x-8)}{x^2-2x-8}+\frac{x+5}{x^2-2x-8}=2x+\frac{x+5}{x^2-2x-8}$$

So now the attention is focused on the much easier second fraction term, which can be resolved via partial fractions:

[9] I.e multiplying the first equation by 3 and adding it to the second equation multiplied by 2.

$$\frac{x+5}{x^2-2x-8}=\frac{x+5}{(x-4)(x+2)}=\frac{A_1}{(x-4)}+\frac{A_2}{(x+2)}\Rightarrow x+5=A_1(x+2)+A_2(x-4)$$

Setting $x = 4$: $\quad 9 = 6A_1 \Rightarrow A_1 = \frac{9}{6} = \frac{3}{2}$

Setting $x = -2$: $\quad 3 = -6A_2 \Rightarrow A_2 = -\frac{3}{6} = -\frac{1}{2}$

Therefore: $\dfrac{2x^3-4x^2-15x+5}{x^2-2x-8}=2x+\dfrac{x+5}{x^2-2x-8}=2x+\tfrac{1}{2}\left(\dfrac{3}{x-4}-\dfrac{1}{x+2}\right)$

Hence the integral becomes:

$$\int\frac{2x^3-4x^2-15x+5}{x^2-2x-8}dx=\int\left[2x+\tfrac{1}{2}\left(\frac{3}{x-4}-\frac{1}{x+2}\right)\right]dx$$

$$=\int 2x\,dx+\tfrac{3}{2}\int\frac{dx}{x-4}-\tfrac{1}{2}\int\frac{dx}{x+2}=x^2+\tfrac{3}{2}\ln|x-4|-\tfrac{1}{2}\ln|x+2|+C$$

…which is how the authors present the answer. However note that the logarithm terms can be combined via the properties of natural log:

$$\int\frac{2x^3-4x^2-15x+5}{x^2-2x-8}dx=\int 2x\,dx+\tfrac{3}{2}\int\frac{dx}{x-4}-\tfrac{1}{2}\int\frac{dx}{x+2}=x^2+\tfrac{3}{2}\ln|x-4|-\tfrac{1}{2}\ln|x+2|+C$$

$$=x^2+\ln\left(|x-4|\right)^{3/2}-\ln\left(|x+2|\right)^{1/2}+C=x^2+\ln\sqrt{\frac{|x-4|^3}{|x+2|}}+C$$

- Note: The above two integrals were obtained via a simple *u*-substitution. In the first case, $u = x - 4$, and in the second case $u = x + 2$

- <u>Example</u>

$$\int\frac{4x^2+2x-1}{x^3+x}dx$$

This is an example of a mixture of Cases I, II:

$$\frac{4x^2+2x-1}{x(x^2+1)}=\frac{A_1}{x}+\frac{A_2x+B_2}{x^2+1}\Rightarrow 4x^2+2x-1=A_1(x^2+1)+(A_2x+B_2)x$$

$$\Rightarrow 4x^2+2x-1=(A_1+A_2)x^2+B_2x+A_1$$

Hence equating coefficients via powers of x:

$x^2: \quad 4 = A_1 + A_2$

$x^1: \quad 2 = B_2$

$x^0: \quad -1 = A_1$

Hence: $A_2 = 5$

$$\int \frac{4x^2+2x-1}{x^3+x}dx = \int\left(-\frac{dx}{x}+\frac{5x+2}{x^2+1}\right)dx = -\ln|x| + \int\frac{5x+2}{x^2+1}dx$$

So: $= -\ln|x| + 5\int\frac{x}{x^2+1}dx + 2\int\frac{dx}{x^2+1} = -\ln|x| + \frac{5}{2}\int\frac{du}{u} + 2\arctan x + C$

$= -\ln|x| + \frac{5}{2}\ln(x^2+1) + 2\arctan x + C$

- Note: The last integral involves inverse trig functions (covered in Section 8.6 of the text and also in **Nov 13 notes** beginning on page 2 of

  http://www.glue.umd.edu/%7Ewkallfel/MA261-2/Nov13notes.pdf

You of course could have integrated it using a trig substitution as well! In this regard, section 9.5 is a generalization of section 8.6 in the text.