MA262 **Calculus II: Notes for Jan22**
Spring 2008 Kallfelz

Welcome to Calculus II! Since the course begins in the midst of rather heady material (chapter 9) which will involve advanced integration techniques, as I emphasized in the first two weeks (by email), brushing up on some Calc I material is not only helpful, but a must, in many instances. To facilitate a targeted review, I've briefly listed here some of the topics and the textbook section with the accompanying links[1] to the class notes from the website. **I strongly recommend looking at the Links to the Class Notes in right collum, as you'll see plenty of worked-out solutions to problems in text, with notes and comments, as well as derivations and explanations to material and formulae left unexplained in the text.**

Topic	Textbook Section	Links to Class notes
Exponential and Logarithm functions)	7.1-7.5 (selections)	• See pp. 8- 12 in **Notes on Centroids and The Calculus of Exp and Log functions (Oct 18 class)** http://www.glue.umd.edu/%7Ewkallfel/MA261-2/Oct15notesb.pdf • See pp. 1-5 in **Notes on Exponentials/Logs (Oct 23 class)** http://www.glue.umd.edu/%7Ewkallfel/MA261-2/Oct22notesa.pdf • See pp. 1-4 in: **Notes on Exponentials/Logs/ Logarithmic differentiation (Oct 25 class)** http://www.glue.umd.edu/%7Ewkallfel/MA261-2/Oct22notesb.pdf • See pp. 6-8 in **Notes on Exponentials/Logs/ Logarithmic differentiation (Oct 25 class)** http://www.glue.umd.edu/%7Ewkallfel/MA261-2/Oct22notesb.pdf
Logarithmic Differentiation	7.5	• See pp. 4-6 in

[1] To access these links, simply point to the link and right click or do a **CTRL and then Right –Click** (if you don't see the index finger icon on the URL). You may also access them from the course website http://www.glue.umd.edu/%7Ewkallfel/MA261-2/index.html by clicking on the link for **Calc II Weekly Materials,** or simply scrolling down the page to the appropriate section.

	(selections)	**Notes on Exponentials/Logs/ Logarithmic differentiation (Oct 25 class)** http://www.glue.umd.edu/%7Ewkallfel/MA261-2/Oct22notesb.pdf
Trigonometric functions: integration and differentiation	8.1-8.4	• See pp. 1-7 **Trig Review and Trig Limits (Nov. 6 class)** http://www.glue.umd.edu/%7Ewkallfel/MA261-2/trig.pdf • See pp. 1-2 **Notes on derivatives of the inverse trig functions + examples from 8.5 (Nov. 13 class)** http://www.glue.umd.edu/%7Ewkallfel/MA261-2/trig.pdf
Inverse Trigonometric Functions (differentiation and integration)	8.5, 8.6	• See pp. 2-16 **Notes on derivatives of the inverse trig functions + examples from 8.5 (Nov. 13 class)** http://www.glue.umd.edu/%7Ewkallfel/MA261-2/trig.pdf • See pp. 1-6 **Notes on integrals of the inverse trig functions + calculus of hyperbolic functions + examples (Nov. 15 class)** http://www.glue.umd.edu/%7Ewkallfel/MA261-2/Nov15notes.pdf
Hyperbolic Trig Functions	8.7	• See pp.6-19 **Notes on integrals of the inverse trig functions + calculus of hyperbolic functions + examples (Nov. 15 class)** http://www.glue.umd.edu/%7Ewkallfel/MA261-2/Nov15notes.pdf
Applications of above topics via integration	9.1	• See pp. 1-3 **Notes on Sections 9.1, 9.2, 9.3 (Nov. 27 class)** http://www.glue.umd.edu/%7Ewkallfel/MA261-2/Nov27notes.pdf
Integrating by parts:	9.2, 9.3	• See pp. 3-7

Applications to Powers of Trig functions		**Notes on Sections 9.1, 9.2, 9.3 (Nov. 27 class)** http://www.glue.umd.edu/%7Ewkallfel/MA261-2/Nov27notes.pdf • See pp. 1-4 **Notes on Sections 9.2, 9.3& 9.4 (selections) (Nov. 29 class)** http://www.glue.umd.edu/%7Ewkallfel/MA261-2/Nov29notes.pdf

…If you'd like to review any other topics from Calculus I not covered in the table above, simply refer to the appropriate posted notes on the course website.

- *INTEGRATING BY PARTS*

Recall Ricky Smith's question concerning the basic kinds of integrands and the strategies for using integration by parts. As mentioned in the **Nov. 29** class notes, the basic situation are:

1. Integrals involving $x^n e^{ax}$, $x^n \sin ax$, $x^n \cos ax$ are *recursive* (the parts strategy involves iteration). **Choose $u = x^n$ in these three cases, since vdu will involve a sinusoidal or exponential term, with a simpler x^{n-1} term.**
2. Integrals involving $x^n \ln x$, $x^n \arctan x$, $x^n \arcsin x$ aren't *recursive* **Choose $dv = x^n$ in these three cases.**
3. Integrals involving $e^{ax} \cos bx\, x$, $e^{ax} \sin bx$ aren't *recursive*. **One can choose u to be either the exponential or the sinusoidal term, it makes no difference in the amount of work.**

Detailed worked-out examples of all three cases (**1., 2., 3.)** are found on pages 1-4 of the **Nov. 29** class notes. The example I brought up in class covering case **1**. was the following:

$$\int (\pi x)^3 e^{3x} dx = \pi^3 \int x^3 e^{3x} dx$$

Cases pertaining to 1. are generally *recursive* ("the crank must be turned more than once," i.e. you're faced with integrating by parts more than once, generating a nested procedure). But because the $u = x^n$ term converges to 0, a useful time-saving device of *tabular integration* is applicable:

u	dv
x^3 (+)	e^{3x}
$3x^2$ (-)	$\frac{1}{3}e^{3x}$
$6x$ (+)	$\frac{1}{9}e^{3x}$
6 (-)	$\frac{1}{27}e^{3x}$
0	$\frac{1}{81}e^{3x}$

Hence reading from the table, our answer is:

$$\int (\pi x)^3 e^{3x}dx = \pi^3 \int x^3 e^{3x}dx = \pi^3 \left\{ \tfrac{1}{3}x^3 e^{3x} - \tfrac{3}{9}x^2 e^{3x} + \tfrac{6}{27}e^{3x} - \tfrac{6}{81}e^{3x} \right\}$$
$$= \pi^3 e^{3x}\left(\tfrac{1}{3}x^3 - \tfrac{1}{3}x^2 + \tfrac{2}{9}x - \tfrac{2}{27} \right)$$

…which can be checked by differentiating (through the use of the product rule):

$$\frac{d}{dx}\pi^3 e^{3x}\left(\tfrac{1}{3}x^3 - \tfrac{1}{3}x^2 + \tfrac{2}{9}x - \tfrac{2}{27} \right) = \pi^3\left(\frac{d}{dx}e^{3x} \right)\left(\tfrac{1}{3}x^3 - \tfrac{1}{3}x^2 + \tfrac{2}{9}x - \tfrac{2}{27} \right) + \pi^3 e^{3x}\frac{d}{dx}\left(\tfrac{1}{3}x^3 - \tfrac{1}{3}x^2 + \tfrac{2}{9}x - \tfrac{2}{27} \right)$$
$$= \pi^3 3e^{3x}\left(\tfrac{1}{3}x^3 - \tfrac{1}{3}x^2 + \tfrac{2}{9}x - \tfrac{2}{27} \right) + \pi^3 e^{3x}\left(x^2 - \tfrac{2}{3}x + \tfrac{2}{9} \right)$$
$$= \pi^3 e^{3x}\left[\left(x^3 - x^2 + \tfrac{2}{3}x - \tfrac{2}{9} \right) + \left(x^2 - \tfrac{2}{3}x + \tfrac{2}{9} \right) \right] = \pi^3 e^{3x}x^3 = (\pi x)^3 e^{3x}$$

Just one look at pages 1-2 of the **Nov. 29** class notes indicates how much lengthier the procedure would have been, by just integrating straight from the parts rule, without the use of tabular integration!

However, please keep in mind that tabular integration is only applicable in Case 1., when one of the integrands $u = x^n$, wher $n > 0$. Why? Because the procedure has to terminate. For instance, one could *not* apply the shortcut in the case:

$$\int (\pi x)^{-3} e^{3x}dx = \pi^{-3}\int x^{-3}e^{3x}dx$$

See pages 2-4, **Nov. 29** class notes for applications of **Cases 2., 3.** One I discussed in class (but didn't get to finish) is the example (which brings in issues from section 9.3):

$$\int \csc^5 x dx$$

- **Method 1:** Factor $\csc^5 x = \csc^3 x \csc^2 x$ and use a Pythagorean Identity and Integrate by parts:

$$\int \csc^5 x dx = \int \csc^3 x \csc^2 x dx = \int \csc^3 x(\cot^2 x + 1)dx = \int \csc^3 x \cot^2 x dx + \int \csc^3 x dx$$
$$= \int [\csc^2 x(\csc x \cot x)]\cot x dx + \int \csc^3 x dx$$

The first integral can be evaluated by parts in the following way:

$$u = \cot x \Rightarrow du = -\csc^2 x dx$$
$$dv = \csc^2 x(\csc x \cot x) \Rightarrow v = -\tfrac{1}{3}\csc^3 x$$

(Recall, as discussed today in class, that $\frac{d}{dx}\csc x = -\csc x \cot x$, so:

$$v = \int \csc^2 x(\csc x \cot x dx) = \int U^2(-dU) = -\int U^2 du = -\tfrac{1}{3}U^3 = -\tfrac{1}{3}\csc^3 x$$

…To eliminate ambiguity, I adopted a capitalized U for the above substitution to obtain v, so there's no confusion with the original u in the above expression).

Hence:

$$\int u dv = uv - \int v du = -\tfrac{1}{3}\cot x \csc^3 x - \tfrac{1}{3}\int \csc^3 x \csc^2 x dx$$
$$= -\tfrac{1}{3}\cot x \csc^3 x - \tfrac{1}{3}\int \csc^5 x dx$$

So:

$$\int \csc^5 x dx = -\tfrac{1}{3}\cot x \csc^3 x - \tfrac{1}{3}\int \csc^5 x dx + \int \csc^3 x dx$$
$$\therefore \tfrac{4}{3}\int \csc^5 x dx = -\tfrac{1}{3}\cot x \csc^3 x + \int \csc^3 x dx$$
$$\therefore \int \csc^5 x dx = -\tfrac{1}{4}\cot x \csc^3 x + \tfrac{3}{4}\int \csc^3 x dx = -\tfrac{1}{4}\cot x \csc^3 x + \tfrac{3}{4}\int \csc^3 x dx$$

Note how the 'unknown' term $\int \csc^5 x dx$ appearing on the right side as well was brought over to the left side and isolated algebraically.

Of course the integral on the left hand side still remains to be done, but note that the power of csc$x$ decreased to 3 for this expression, which means we're on the right track! (Integrating by parts is a divide and conquer procedure: you should tend up with an integral—and even more hopefully a complete expression—on the left hand side that's less complex than what you originally started off with.)

Adopting a similar procedure:

$$\int \csc^3 x dx = \int \csc x \csc^2 x dx = \int \csc x(\cot^2 x + 1)dx = \int \csc x \cot^2 x dx + \int \csc x dx$$
$$= \int [(\csc x \cot x)]\cot x dx + \int \csc^3 x dx$$

$$u = \cot x \Rightarrow du = -\csc^2 x dx$$
$$dv = (\csc x \cot x) \Rightarrow v = -\csc\ \ x$$

$$\int \csc^3 x dx = -\cot x \csc x - \int \csc^3 x dx + \int \csc x dx$$
$$\therefore 2\int \csc^3 x dx = -\cot x \csc^2 x + \int \csc x dx$$
$$\therefore \int \csc^3 x dx = -\tfrac{1}{2}\cot x \csc^2 x + \tfrac{1}{2}\int \csc x dx$$

Now: $\int \csc x dx = -\ln|\csc x + \cot x| + C = \ln|\csc x - \cot x| + C$

(For details, see page 5 **<u>Nov. 8th Notes</u>**, as well as pp. 1-2 **<u>Nov. 13th Notes</u>**.[2])

So: $\int \csc^3 x dx = -\tfrac{1}{2}\cot x \csc^2 x + \tfrac{1}{2}\int \csc x dx = -\tfrac{1}{2}\cot x \csc^2 x + \tfrac{1}{2}\ln|\csc x - \cot x| + C$

Therefore:

$$\therefore \int \csc^5 x dx = -\tfrac{1}{4}\cot x \csc^3 x + \tfrac{3}{4}\int \csc^3 x dx$$
$$= -\tfrac{1}{4}\cot x \csc^3 x + \tfrac{3}{4}\left[-\tfrac{1}{2}\cot x \csc^2 x + \tfrac{1}{2}\ln|\csc x - \cot x|\right] + C$$
$$= -\tfrac{1}{4}\cot x \csc^3 x - \tfrac{3}{8}\cot x \csc^2 x + \tfrac{3}{8}\ln|\csc x - \cot x| + C$$

This is obviously a very laborious procedure! (Though a valuable demonstration of integrating by parts). It's even *more* laborious if we adopt the method below which I started in class:

- **Method 2:**

[2] Page 2 of **<u>Nov. 13th</u>** notes shows why the two expressions are equal:

$$\int \csc x dx = -\ln|\csc x + \cot xx| + C = \ln\left|\frac{1}{\csc x + \cot x}\right| + C = \ln\left|\frac{1}{\csc x + \cot x}\cdot\frac{\csc x - \cot x}{\csc x - \cot x}\right| + C$$
$$= \ln\left|\frac{\csc x - \cot x}{\csc^2 x - \cot^2 x}\right| + C = \ln\left|\frac{\csc x - \cot x}{\csc^2 x - (\csc^2 - 1)}\right| + C = \ln|\csc x - \cot x| + C$$

$$\int \csc^5 x dx = \int \csc^4 x \csc x dx = \int \left(\cot^2 x + 1\right)^2 \csc x dx = \int \left(\cot^4 x + 2\cot^2 x + 1\right)\csc x dx$$
$$= \int \cot^4 x \csc x dx + 2\int \cot^2 x \csc x dx + \int \csc x dx = \int \cot^3 x \cot x \csc x dx + 2\int \cot x \cot x \csc x + \int \csc x dx$$

…**and which you are welcome to finish ☺. The question becomes: is there a more *direct* and systematic approach. The answer is yes! For powers of csc*x* and sec*x*, one can derive the following *reduction* formulae using integration by parts. (For details of the derivations, see pp. 5-6 <u>November 27th</u> notes):**

$$\int \csc^n x dx = -\frac{\csc^{n-2} x \cot x}{(n-1)} + \frac{(n-2)}{(n-1)}\int \csc^{n-2} x dx$$

To apply the formula in the above case, we start off with *n* = 5:

$$\int \csc^5 x dx = -\frac{\csc^3 x \cot x}{4} + \frac{3}{4}\int \csc^3 x dx$$

…which is the same answer derived above.

Now repeat for *n* = 3 to reduce the integral on the left hand side:

$$\int \csc^3 x dx = -\frac{\csc^2 x \cot x}{2} + \frac{1}{2}\int \csc x dx = -\tfrac{1}{2}\csc^2 x \cot x + \tfrac{1}{2}\ln\left|\csc x - \cot x\right| + C$$

So:

$$\int \csc^5 x dx = -\frac{\csc^3 x \cot x}{4} + \frac{3}{4}\int \csc^3 x dx = -\tfrac{1}{4}\csc^3 x \cot x + \tfrac{3}{4}\left[-\tfrac{1}{2}\csc^2 x \cot x + \tfrac{1}{2}\ln\left|\csc x - \cot x\right|\right] + C$$
$$= -\tfrac{3}{8}\csc^3 x \cot x - \tfrac{3}{8}\csc^2 x \cot x + \tfrac{3}{8}\ln\left|\csc x - \cot x\right| + C$$

Certainly much more efficient!

As discussed in class, there are six major cases to consider (from section 9.3) regarding integrating powers of trigonometric functions. The strategy for dealing with them is summarized in the table below

Type	*Strategy*
$\sin^n x\cos^m x$ *n* or *m* odd **(Case 1)**	Reduce one to first power, use the Pythagorean Identiy : $\sin^2 x + \cos^2 x = 1$ to express in terms of powers of sine or cosine. The first power term is a "du" term . Can be converted to a simple *u*-substitution procedure, without the need for integrating by parts.

$\sin^n x\cos^m x$ ***n* and *m* both even** **(Case 2)**	**Use**: sin2x = 2sinxcosx $\sin^2 x = {}^1/_2\ (1 - \cos 2x)$, $\cos^2 x = {}^1/_2\ (1 + \cos 2x)$ ***This procedure may have to be repeated more than once!***
$\tan^n x$, $\cot^n x$ (*n* odd) **(Case 3)**	Use: $\tan^2 x + 1 = \sec^2 x$ or $\cot^2 x + 1 = \csc^2 x$ (the second and third Pythagorean Identities). Can be converted to a simple *u*-substitution procedure, without the need for integrating by parts.
sin(mx)cos(nx) **(Case 4)**	**Use the Sum-Product Identities:** sinAcosB = ${}^1/_2$[sin(A-B)+sin(A+B)] sinAsinB = ${}^1/_2$[cos(A-B) - cos(A+B)] cosAcosB = ${}^1/_2$[cos(A-B)+cos(A+B)] (No integration by parts necessary)
$\tan^n x$, $\cot^n x$ (*n* even) **(Case 5)**	Use: $\tan^2 x + 1 = \sec^2 x$ or $\cot^2 x + 1 = \csc^2 x$ (the second and third Pythagorean Identities.) **Integration by Parts is necessary**.
$\sec^n x$, $\csc^n x$ (*any n*) **(Case 6)**	Use: $\tan^2 x + 1 = \sec^2 x$ or $\cot^2 x + 1 = \csc^2 x$ (the second and third Pythagorean Identities.) **Integration by Parts is necessary**. Usually the Integration by Parts is laborious, as demonstrated above. Reduction Formulae are far more efficient. For derivations of some of them, see pp. 5-7, **<u>Nov 27</u> class notes,** as well as exercises 75-78, p. 517

- *<u>Example 1</u>*

$$\int \sin^2 x \cos^3 x dx$$

This is a **Case 1** example:

$$\int \sin^2 x\cos^3 x dx = \int \sin^2 x\cos^2 x\cos x dx = \int \sin^2 x(1-\sin^2 x)\cos x = \int \sin^2 x\cos x - \int \sin^4 x\cos x dx$$
$$= \int u^2 du - \int u^4 du = \tfrac{1}{3}u^3 - \tfrac{1}{5}u^5 + C = \tfrac{1}{3}\sin^3 x - \tfrac{1}{5}\sin^5 x + C$$

- *<u>Example 2</u>*

$$\int \sin^2 x \cos^2 x dx$$

This is a **Case 2** example:

$$\int \sin^2 x\cos^2 x dx = \int \tfrac{1}{2}(1-\cos 2x)\tfrac{1}{2}(1+\cos 2x)dx = \tfrac{1}{4}\int (1-\cos^2 2x)dx$$
$$= \tfrac{1}{4}x - \tfrac{1}{4}\int \cos^2 2x dx = \tfrac{1}{4}x - \tfrac{1}{4}\int \tfrac{1}{2}(1+\cos 4x) = \tfrac{1}{4}x - \tfrac{1}{8}\int dx - \tfrac{1}{8}\int \cos 4x dx$$
$$= \tfrac{1}{4}x - \tfrac{1}{8}x + \tfrac{1}{32}\sin 4x + C$$

- *Example 3*

$$\int \tan^3 x dx$$

This is a **Case 3** example:

$$\int \tan^3 x dx = \int \tan^2 x \tan x dx = \int \left(\sec^2 x - 1\right)\tan x dx = \int \sec^2 x \tan x dx - \int \tan x dx$$
$$= \int \sec x(\sec x \tan x)dx - \int \tan x dx = \int u du + \ln|\cos x| = \tfrac{1}{2}u^2 + \ln|\cos x| + C = \tfrac{1}{2}\sec^2 x + \ln|\cos x| + C$$

- *Example 4*

$$\int \sin 2x \cos 3x dx$$

This is a **Case 4** example:

$$\int \sin 2x \cos 3x dx = \int [\tfrac{1}{2}\sin(2x - 3x) + \tfrac{1}{2}\sin(2x + 3x)]dx$$
$$= \tfrac{1}{2}\int \sin(-x)dx + \tfrac{1}{2}\int \sin 5x dx = -\tfrac{1}{2}\int \sin x dx + \tfrac{1}{2}\int \sin 5x dx$$
$$= \tfrac{1}{2}\cos x - \tfrac{1}{10}\cos 5x + C$$

- *Example 5*

$$\int \tan^4 x dx$$

This is a **Case 5** example:

$$\int \tan^4 x dx = \int \tan^2 x \tan^2 x dx = \int \left(\sec^2 x - 1\right)\tan^2 x dx = \int \sec^2 x \tan^2 x dx - \int \tan^2 x dx$$
$$= \int \tan^2 x \sec^2 x dx - \int \left(\sec^2 x - 1\right)dx = \int \tan^2 x \sec^2 x dx - \tan x + x$$

The integral on the right hand side must be evaluated by parts:

$$\int \tan^2 x \sec^2 x dx \Rightarrow u = \tan^2 x, du = 2\tan x \sec^2 x dx, dv = \sec^2 x, v = \tan x$$
$$\int u dv = uv - \int v du \Rightarrow \int \tan^2 x \sec^2 x dx = \tan^3 x - 2\int \tan^2 x \sec^2 x dx$$

…we can isolate this ‘unknown’ quantity algebraically:

$$3\int \tan^2 x \sec^2 x dx = \tan^3 x + C \Rightarrow \int \tan^2 x \sec^2 x dx = \tfrac{1}{3}\tan^3 + C$$

So: $\int \tan^4 x dx = \frac{1}{3}\tan^3 x - \tan x + x + C$

Of course, our **Case 6** example was dealt with above in pages 5 and 6 of these notes.