MA262 KEY ASSIGNMENT II

I.a) (1) $1, x, \frac{x^2}{2}, \frac{x^3}{6}, \frac{x^4}{24}, \frac{x^5}{120}, \ldots$

$$= \frac{x^0}{0!}, \frac{x^1}{1!}, \frac{x^2}{2!}, \frac{x^3}{3!}, \frac{x^4}{4!}, \frac{x^5}{5!}, \ldots \quad \therefore \quad a_n = \frac{x^n}{n!} \qquad (n \geq 0)$$

I.b) (2) $a_n = \frac{n}{\sqrt{n^2-1}} \Leftrightarrow f(x) = \frac{x}{\sqrt{x^2-1}} \Rightarrow \lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{x}{\sqrt{x^2-1}} = \lim_{x\to\infty} \frac{1}{\sqrt{1-x^{-2}}} = 1$

$\therefore a_n \to 1$

I.c) (4) $\lim_{n\to\infty} (\sqrt{n^2+n} - n) \Leftrightarrow \lim_{x\to\infty} (\sqrt{x^2+x} - x)$ $\infty - \infty$ (Indeterminate)

$$\overset{\text{L'Hopital's Rule}}{=} \lim_{x\to\infty} \frac{x(x^2+x) - x^2(x^2+x)^{1/2}}{x\sqrt{x^2+1}}$$

$$= \lim_{x\to\infty} \frac{(x^3+x^2) - x^2(x^2+x)^{1/2}}{x\sqrt{x^2+1}}$$

Alternatively: $\lim_{x\to\infty} (\sqrt{x^2+x} - x) \cdot \frac{(\sqrt{x^2+x} + x)}{\sqrt{x^2+x} + x}$

$$= \lim_{x\to\infty} \frac{x^2+x-x^2}{\sqrt{x^2+x}+x} = \lim_{x\to\infty} \frac{x}{\sqrt{x^2+x}+x} \overset{LHR}{=} \lim_{x\to\infty} \frac{1}{\frac{1}{2}(x^2+x)^{-1/2}(2x)+1}$$

$$= \lim_{x\to\infty} \frac{1}{\frac{x}{\sqrt{x^2+x}}+1} = \lim_{x\to\infty} \frac{1}{(\sqrt{1+1/x})^{-1}+1} = \frac{1}{1+1} = \frac{1}{2}$$

$$\therefore \lim_{n\to\infty} (\sqrt{n^2+n} - n) = \lim_{x\to\infty} (\sqrt{x^2+x} - x) = \frac{1}{2}$$

I.d) (8) $a_n = \int_{-1+1/n}^{1-1/n} \frac{dx}{\sqrt{1-x^2}}$ $\quad x = \sin\theta \quad dx = \cos\theta\, d\theta$

$$= \int_{\arcsin(-(1-1/n))}^{\arcsin(1-1/n)} \frac{\cos\theta\, d\theta}{\cos\theta} = \theta \Big|_{\arcsin(-(1-1/n))}^{\arcsin(1-1/n)}$$

$$= \arcsin(1-1/n) - \arcsin(-(1-1/n))$$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left\{\arcsin(1-1/n) - \arcsin(-(1-1/n))\right\} = \arcsin(1) - \arcsin(-1) = \frac{\pi}{2} - -\frac{\pi}{2} = \pi$$

$\therefore\ a_n \to \pi$

I.e) $\quad a_n = a_{n-1} + a_{n-2}$

(10) $\quad a_{n+1} = a_n + a_{n-1}$

$$b_n = \frac{a_{n+1}}{a_n} = \frac{a_n + a_{n-1}}{a_n} = 1 + \frac{a_{n-1}}{a_n} = 1 + \frac{1}{a_n/a_{n-1}} = 1 + \frac{1}{b_{n-1}}$$

$$\frac{a_{n+1}}{a_n} = \frac{a_n + a_{n-1}}{a_{n-1} + a_{n-2}}$$

$$\lim_{n\to\infty} b_n \equiv \rho = \lim_{n\to\infty}\left(1 + \frac{1}{b_{n-1}}\right) = 1 + \frac{1}{\lim_{n\to\infty} b_{n-1}} = 1 + \frac{1}{\rho}$$

$$\Rightarrow \rho = 1 + \frac{1}{\rho} \Rightarrow \rho^2 = \rho + 1 \Rightarrow \rho^2 - \rho - 1 = 0$$

$$\rho_{1,2} = \frac{+1 \pm \sqrt{1+4}}{2} = \frac{1}{2}(1 \pm \sqrt{5}) = \phi_{1,2} \quad \text{(Golden Ratio)}$$

Note that $\phi_{1,2}$ enters into the Fibonacci Sequence in the following fashion:

The general (n-th term) formula for $a_n = \frac{1}{\sqrt5}\left[\phi_1^n - \phi_2^n\right]$ (using methods in Discrete Mathematics, one can derive this formula for a_n)

Observe: $a_1 = \frac{1}{\sqrt5}\left[\phi_1^1 - \phi_2^1\right] = \frac{1}{\sqrt5}\left[\left(\frac{1+\sqrt5}{2}\right)^1 - \left(\frac{1-\sqrt5}{2}\right)^1\right] = \frac{\sqrt5}{\sqrt5} = 1$

$$a_2 = \frac{1}{\sqrt5}\left[\phi_2^2 - \phi_c^2\right] = \frac{1}{\sqrt5}\left[\frac{1}{4}(1+2\sqrt5+5) - \frac{1}{4}(1-2\sqrt5+5)\right]$$

$$= \frac{1}{\sqrt5}\left[\frac{4}{4}\sqrt5\right] = 1$$

$$a_3 = \frac{1}{\sqrt5}\left[\phi_1^3 - \phi_2^3\right] = \frac{1}{\sqrt5}\left[\frac{1}{4}(6+2\sqrt5)\cdot\frac{1}{2}(1+\sqrt5) - \frac{1}{4}(6-2\sqrt5)\cdot\frac{1}{2}(1-\sqrt5)\right]$$

$$= \frac{1}{\sqrt5}\left[\frac{1}{8}(16+8\sqrt5) - \frac{1}{8}(16-8\sqrt5)\right] = \frac{1}{\sqrt5}\cdot 16\sqrt5 = 2, \text{ etc.}$$

II, a) (2) $$S = \sum_{n=5}^{\infty} 2\left(-\tfrac{3}{4}\right)^n = 2\sum_{n=5}^{\infty}\left(\frac{-3}{4}\right)^n = 2\sum_{j=0}^{\infty}\left(\frac{-3}{4}\right)^{j+5}$$

$$= 2\cdot\left(-\tfrac{3}{4}\right)^5 \sum_{j=0}^{\infty}\left(\frac{-3}{4}\right)^j = 2\cdot\left(-\tfrac{3}{4}\right)^5\left[\frac{1}{1+3/4}\right]$$

$$= \frac{2\cdot(-3/4)^5}{7/4} = -\frac{4}{7}\cdot 2\cdot\frac{3^5}{4^5} = -\frac{2\cdot 3^5}{7\cdot 4^4} = -\frac{2\cdot 243}{7\cdot 256} = \frac{-243}{7\cdot 128} = -\frac{243}{896}$$

b) $$\sum_{n=1}^{\infty}\frac{1}{(2n+1)(2n+3)}$$
(2)

$$\frac{1}{(2n+1)(2n+3)} = \frac{A_1}{(2n+1)} + \frac{A_2}{(2n+3)} \Rightarrow 1 = A_1(2n+3) + A_2(2n+1)$$

$$n = \tfrac{1}{2} \Rightarrow 1 = 2A_1 \Rightarrow A_1 = \tfrac{1}{2}$$

$$n_2 = -\tfrac{3}{2} \Rightarrow 1 = -2A_2 = A_2 = -\tfrac{1}{2}$$

$$\therefore \sum_{n=1}^{\infty}\frac{1}{(2n+1)(2n+3)} = \sum_{n=1}^{\infty}\frac{1}{2}\left\{\frac{1}{(2n+1)} - \frac{1}{(2n+3)}\right\}$$

$$= \frac{1}{2}\sum_{n=1}^{\infty}\left\{\frac{1}{2n+1} - \frac{1}{2n+3}\right\} = \frac{1}{2}\left\{\left(\frac{1}{3} - \cancel{\frac{1}{5}}\right) + \left(\cancel{\frac{1}{5}} - \cancel{\frac{1}{7}}\right) + \left(\cancel{\frac{1}{7}} - \frac{1}{9}\right) + \dots + \cancel{\frac{1}{2n+1}} - \underbrace{\frac{1}{2n+3}}_{\to\, 0\ (\lim n\to\infty)}\right\}$$

$$= \frac{1}{6}\Big]$$

c) (2) $$0.23\overline{23}\ldots = \frac{23}{100} + \frac{23}{10^4} + \frac{23}{10^6} + \dots = \frac{23}{100}\left(1 + \frac{1}{10^2} + \frac{1}{10^4} + \dots\right)$$

$$= \frac{23}{100}\left(\sum_{j=0}^{\infty}\left(\frac{1}{10^2}\right)^j\right) = \frac{23}{100}\cdot\left\{\frac{1}{1-\frac{1}{100}}\right\} = \frac{23}{100}\cdot\frac{100}{99} = \frac{23}{99}$$

d) (3) $$\sum_{n=1}^{\infty}\frac{1}{n(n+3)} \qquad \frac{1}{n(n+3)} = \frac{A_1}{n} + \frac{A_2}{(n+3)} \Rightarrow 1 = A_1(n+3) + A_2 n$$

$$n = 0 \Rightarrow A_1 = \tfrac{1}{3}$$

$$n = -3 \Rightarrow A_2 = -\tfrac{1}{3}$$

$$\sum_{n=1}^{\infty}\frac{1}{n(n+3)} = \frac{1}{3}\sum_{n=1}^{\infty}\left\{\frac{1}{n} - \frac{1}{n+3}\right\} = \frac{1}{3}\left\{\left(1 - \cancel{\tfrac{1}{4}}\right) + \left(\tfrac{1}{2} - \cancel{\tfrac{1}{5}}\right) + \left(\tfrac{1}{3} - \cancel{\tfrac{1}{6}}\right) + \left(\cancel{\tfrac{1}{4}} - \tfrac{1}{7}\right)\right.$$
$$\left. + \left(\cancel{\tfrac{1}{5}} - \tfrac{1}{8}\right) + \left(\cancel{\tfrac{1}{6}} - \tfrac{1}{9}\right) + \dots\right\}$$

$$= \frac{1}{3}\left[1 + \frac{1}{2} + \frac{1}{3}\right] + \lim_{n\to\infty} \frac{-1}{n+3} = \frac{1}{3}\left[\frac{6+3+2}{6}\right] = \frac{11}{18}\Big]$$

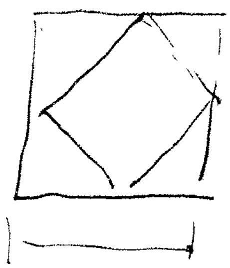
Note also, by SCT: $n^2 + 3n > n^2$ for all $n \geq 1$

$$\Rightarrow \frac{1}{n^2+3n} < \frac{1}{n^2}$$

$\therefore \Downarrow$ $\qquad \Downarrow \sum \frac{1}{n^2}$ (convergent p series)

$\therefore \sum \frac{1}{n(n+3)}$ is converges.

II.e)
(5)



$\ell = 4$

Square 1: $A = \ell^2 = 16$

Square 2 $= \left[\frac{1}{2}\ell\sqrt{2}\right]^2 = \left[\frac{1}{\sqrt{2}}\ell\right]^2 = \frac{1}{2}\ell^2 = 8$

Square 3 $= \left[\frac{1}{\sqrt{2}}\ell_2\right]^2 = \frac{1}{2}\ell_2^2 = \frac{1}{2}\cdot\frac{1}{2}\ell^2 = \frac{1}{4}16 = 4$

$\vdots$

$$\sum \text{areas} = 16 + 8 + 4 + \dots$$
$$= 16\left[1 + \tfrac{1}{2} + \tfrac{1}{4} + \dots\right]$$
$$= 16 \sum_{k=0}^{\infty} (1/2)^k = 16 \cdot \left[\frac{1}{1 - 1/2}\right] = 32$$

II.f) (3) $\displaystyle\sum_{n=1}^{\infty} \frac{n}{\sqrt{n^2+1}}$

$\Updownarrow$

$$\int_1^{\infty} \frac{x\,dx}{\sqrt{x^2+1}} = \lim_{b\to\infty} \int_1^b \frac{x\,dx}{\sqrt{x^2+1}} \qquad u = x^2+1,\quad du = 2x\,dx$$

$$= \lim_{b\to\infty} \int_2^{b} \frac{du}{u^{1/2}} = \lim_{b\to\infty} 2u^{1/2}\Big|_2^b = 2\lim_{b\to\infty} b^{1/2} - 1 = \infty \text{ (diverges)}$$

II.g.) (3) $\displaystyle\sum_{n=2}^{\infty} \frac{\ln n}{n^3} \Leftrightarrow \int_2^{\infty} \frac{\ln x}{x^3}\,dx = \lim_{b\to\infty} \int_2^b x^{-3}\ln x\,dx$

$u = \ln x \qquad dv = x^{-3}dx$

$du = \frac{dx}{x} \qquad v = -\frac{1}{2}x^{-2}$

$$= \lim_{b\to\infty} \left\{ \ln x\left(-\tfrac{1}{2}x^{-2}\right)\Big|_2^b + \frac{1}{2}\int_2^b x^{-3}dx \right\}$$

$$= \lim_{b\to\infty} \left\{ -\frac{\ln x}{2x^2}\Big|_2^b - \frac{1}{4x^2}\Big|_2^b \right\} = \lim_{b\to\infty} -\frac{\ln b}{2b^2} + \frac{\ln 2}{8} - \lim_{b\to\infty}\frac{1}{4b^2} + \frac{1}{16}$$

$\overset{LHR}{=} \lim_{b\to\infty} \frac{-11b}{4b} + \frac{\ln 2}{8} - 0 + \frac{1}{16} = \frac{1}{8}\ln 2 + \frac{1}{16}$ converges.

II.h) (2) $\sum_{n=1}^{\infty} \frac{1}{3\sqrt[4]{n}-1}$ $\quad 3n^{1/4}-1 < 3n^{1/4}$

$\therefore \frac{1}{3n^{1/4}-1} > \frac{1}{3}\frac{1}{n^{1/4}}$

$\sum_{n=1}^{\infty}\frac{1}{n^{1/4}}$ diverges (p series, $p = 1/4 \leq 1$) $\therefore \sum_{n=1}^{\infty}\frac{1}{3n^{1/4}-1}$ diverg

II.i) (3) $\sum_{n=1}^{\infty} \frac{n}{(n+1)2^{n-1}}$

$a_n = \frac{n}{(n+1)2^{n-1}} = \frac{1}{(1+1/n)2^{n-1}}$ $\qquad b_n = \frac{1}{2^{n-1}}$

$\lim_{n\to\infty} b_n/a_n = \lim_{n\to\infty} \frac{1/\cancel{2^{n-1}}}{1/[(1+1/n)\cancel{2^{n-1}}]} = \lim_{n\to\infty}\frac{1}{1+1/n} = 1$

certainly $0 < \lim_{n\to\infty} b_n/a_n < \infty$

Now, by integral test: $\sum_{n=1}^{\infty} b_n \Longleftrightarrow \lim_{b\to\infty}\int_1^b \frac{dx}{2^{x-1}} = \frac{1}{2}\lim_{b\to\infty}\int_1^b 2^{-x}dx$

$= \frac{1}{2}\lim_{b\to\infty}\int_1^{-b} 2^u du = \frac{1}{2}(\ln 2)^{-1}\lim_{b\to\infty} 2^u\Big|_1^{-b} = \frac{1}{2\ln 2}\left\{\lim_{b\to\infty} 2^{-b} - 2\right\}$

$= \frac{1}{2\ln 2}\{0 - 2\} < \infty$ $\therefore \sum b_n$ converges $\therefore \sum_{n=1}^{\infty} a_n$ converges by L.C.T.

III.) a) (5) $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges

Consider $\sum_{n=1}^{\infty} a_n$ BY L.C.T. $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{a_n}{1/n} = na_n$

If $\lim_{n\to\infty} na_n \neq 0 \Rightarrow \sum a_n$ diverges.

b) (3) $\sum_{n=2}^{\infty} \frac{1}{\ln n}$ BY L.C.T. $\lim_{n\to\infty} na_n = \lim_{n\to\infty} \frac{n}{\ln n} \overset{LHR}{=} \lim_{n\to\infty} \frac{1}{1/n} = \infty$

Moreover, note that (by S.C.T.) $n\ln n > \ln n$

$\therefore \frac{1}{n\ln n} < \frac{1}{\ln n}$

$$\int_2^{\infty} \frac{1}{x\ln x}\,dx = \lim_{b\to\infty} \int_2^{b} \frac{(1/x)\,dx}{\ln x}$$

$$= \lim_{b\to\infty} \int_{\ln 2}^{\ln b} (u)\,du$$

$$= \lim_{b\to\infty} \ln(u) \Big|_2^b$$

$$= \lim_{b\to\infty} \ln(\ln b) - \ln(\ln 2) = \infty$$

$\therefore \sum_2^{\infty} \frac{1}{n\ln 2}$ diverges, so $\sum_{n=2}^{\infty} \frac{1}{\ln 2}$ diverges

III. c) (2) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n\sqrt{n}} = \sum_{n=1}^{\infty} a_n$

$\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ a converging p-series ($p = 3/2 \geq 1$)

$\therefore \sum_{n=1}^{\infty} a_n$ converges absolutely.

. d) (3) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}$ $\left| \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{N} a_n \right| = |R_N| \leq a_{N+1}$

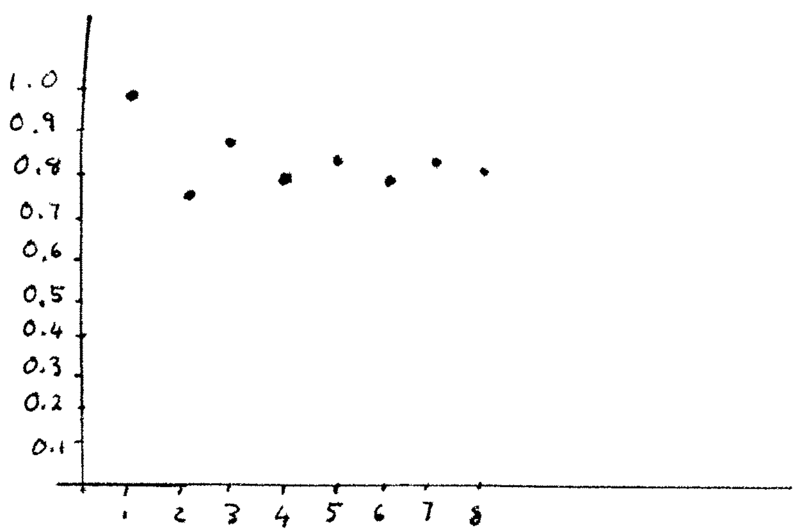
$\therefore\ a_{N+1} = \dfrac{1}{(N+1)^2} \le 10^{-3} \Rightarrow N+1 \ge \lceil \sqrt{10^3} \rceil = 32 \quad \therefore\ \underline{N=31}$

$\therefore\ \displaystyle\sum_{n=1}^{31} \frac{(-1)^{n+1}}{n^2}$ approximates $\dfrac{\pi^2}{12}$ within 10^{-3}

III. e) (7) $\quad S = \displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}$

$S_N = \displaystyle\sum_{n=1}^{N} \frac{(-1)^{n+1}}{n^2}$

N	S_N
1	1
2	0.75
3	$0.86\overline{1}$
4	$0.7986\overline{1}$
5	$0.8386\overline{1}$
6	$0.8108\overline{3}$
7	0.831241497
8	0.815616497



$E_8 = |S - S_8| = \left|\dfrac{\pi^2}{12} - S_8\right| \doteq 6.851 \times 10^{-3}$

$a_{8+1} = \dfrac{1}{9^2} \doteq 0.01235 \quad$ certainly $E_8 = |S - S_8| \le a_{8+1}$, confirming Thm 10.15

$|S - S_8| \approx 6.9 \times 10^{-3}$

$a_9 \approx 1.2 \times 10^{-2}$

$$\text{III. f) (2)} \qquad S = \sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2}\left(\frac{3}{2}\right)^n$$

$$\underline{\text{RoT}}: \quad r = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{(3/2)^{n+1}}{(n+1)^2}\cdot\frac{n^2}{(3/2)^n} = \frac{3}{2}\lim_{n\to\infty}\frac{n^2}{(n+1)^2}$$

$$= \frac{3}{2}\lim_{n\to\infty}\left[\frac{n}{n+1}\right]^2 = \frac{3}{2}\lim_{n\to\infty}\left[\frac{1}{1+1/n}\right]^2 = \frac{3}{2} > 1 \Rightarrow \text{diverges.}$$

$$\text{III. g) (3)} \qquad S = \sum_{n=1}^{\infty} (n^{1/n} - 1)^n$$

$$\underline{\text{RoT}}: \quad \rho = \lim_{n\to\infty} (|a_n|)^{1/n} = \lim_{n\to\infty} \sqrt[n]{|n^{1/n} - 1|^n} = \lim_{n\to\infty} |n^{1/n} - 1|$$

$$= \left|\lim_{n\to\infty} n^{1/n} - 1\right|$$

$$\Longrightarrow \ln\left(\lim_{n\to\infty} n^{1/n}\right) = \lim_{n\to\infty}\frac{\ln(n)}{n} \overset{\text{LHR}}{=} \lim_{n\to\infty}\frac{1/n}{1} = 0$$

$$\lim_{n\to\infty} n^{1/n} = e^0 = 1$$

$$\therefore \quad \rho = |e^0 - 1| = 0 < 1 \Rightarrow \underline{\text{CONVERGES}}$$

$$\text{IV. a) (3)} \qquad f(x) = x^2 e^{-x}$$

$$f'(x) = -x^2 e^{-x} + 2x e^{-x} = x e^{-x}(2-x)$$

$$f^{(2)}(x) = e^{-x}(2-x) - x e^{-x}(2-x) - x e^{-x}$$

$$= e^{-x}[2 - x - 2x + x^2 - x] = e^{-x}[2 - 4x + x^2]$$

$$f^{(3)}(x) = e^{-x}[-4 + 2x] - e^{-x}[2 - 4x + x^2]$$

$$= e^{-x}[-4 + 2x - 2 + 4x - x^2] = e^{-x}[-6 + 6x - x^2]$$

$$f^{(4)}(x) = -e^{-x}[-6 + 6x - x^2] + e^{-x}[6 - 2x]$$

$$= e^{-x}[6 - 6x + x^2 + 6 - 2x] = e^{-x}[12 - 8x + x^2]$$

$$\therefore \quad a_0 = \frac{1}{0!}f^{(0)}(0) = 0 = \frac{1}{1!}f^{(1)}(0) = a_1$$

$$a_2 = \frac{1}{2!}f^{(2)}(0) = \frac{1}{2!}e^0[2 - 0 + 0] = \frac{2}{2!} = 1$$

$$a_3 = \frac{1}{3!}f^{(3)}(0) = \frac{1}{3!}e^0(-6 + 0) = -6/6 = -1$$

$$a_4 = \frac{1}{4!}f^{(4)}(0) = \frac{1}{4!}e^0[12] = \frac{12}{24} = \frac{1}{2}$$

$$\therefore P_4(x) = \sum_{k=0}^{4} \frac{1}{k!} f^{(k)}(0) x^k = 0 + 0\cdot x + 1\cdot x^2 - 1\cdot x^3 + \frac{1}{2}\cdot x^4$$
$$= x^2 - x^3 + \frac{1}{2}x^4$$

(This can be checked via: $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} \quad \therefore \quad e^{-x} = \sum_{k=0}^{\infty} (-1)^k \frac{x^k}{k!}$

$$\therefore f(x) = x^2 e^{-x} = x^2 \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} x^k = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} x^{k+2} = \underbrace{x^2 - x^3 + \frac{1}{2!} x^4}_{P_4(x)} + \ldots$$

IV.b) (5) $\quad f(x) = \sec x$

$$f'(x) = \sec x \tan x$$

$$f^{(2)}(x) = \sec^3 x + \sec x \tan^2 x$$

$$a_0 = f^{(0)}(0) = \sec 0 = 1$$
$$a_1 = \frac{1}{1!} f'(0) = \sec 0 \tan 0 = 0$$
$$a_2 = \frac{1}{2!} f^{(2)}(0) = \frac{1}{2!}\left[\sec^3 0 + \sec 0 \tan^2 0\right] = \frac{1}{2!}$$

$$\therefore P_2(x) = a_0 + a_1 x + a_2 x^2 = 1 + \frac{1}{2} x^2$$

IV.c) (2) $\quad \sum_{n=0}^{\infty} \frac{(-1)^n n! (x-4)^n}{3^n}$

RaT.: $\quad r(x) = \lim_{n\to\infty} \frac{|a_{n+1}(x)|}{|a_n(x)|} = \lim_{n\to\infty} \frac{(n+1)!}{3^{n+1}} \frac{|x-4|^{n+1}}{n!} \cdot \frac{3^n}{|x-4|^n}$

$$= \frac{|x-4|}{3} \lim_{n\to\infty} \frac{(n+1)!}{n!} = \frac{|x-4|}{3} \lim_{n\to\infty} (n+1) = \infty \text{ for all } x \neq 4$$

$$\therefore R = 0, \quad I_4^0 = \{4\}$$

IV.d) (5) $\quad f(x) = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!} \qquad g(x) = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$

for $f(x)$

(4)

(10)

RaT : $r(x) = \lim_{n\to\infty} \frac{|a_{n+1}(x)|}{|a_n(x)|} = \lim_{n\to\infty} \frac{|x|^{2n+3}}{(2n+3)!} \cdot \frac{(2n+1)!}{|x|^{2n+1}}$

$= \lim_{n\to\infty} |x|^2 \cdot \frac{\cancel{(2n+1)!}}{(2n+3)(2n+2)\cancel{(2n+1)!}} = |x|^2 \lim_{n\to\infty} \cdot \frac{1}{(2n+3)(2n+2)}$

$= |x|^2 \lim_{n\to\infty} \frac{1}{4n^2+10n+6} = |x|^2 \lim_{n\to\infty} \frac{1/4n^2}{\left[1+\frac{5}{2n}+\frac{3}{2n^2}\right]} = |x|^2 \cdot 0$

$\therefore \; R_f = \infty \Rightarrow I_0^\infty = (-\infty, \infty)$

for $g(x)$ $\quad r(x) = \lim_{n\to\infty} \frac{|a_{n+1}(x)|}{|a_n(x)|} = \lim_{n\to\infty} \frac{|x|^{2n+2}}{(2n+2)!} \cdot \frac{(2n)!}{|x|^{2n}}$

$= |x|^2 \lim_{n\to\infty} \frac{\cancel{(2n)!}}{(2n+2)(2n+1)\cancel{(2n)!}} = |x|^2 \lim_{n\to\infty} \frac{1}{(4n^2+6n+2)}$

$= |x|^2 \lim_{n\to\infty} \frac{1/4n^2}{(1+3/2n+1/2n^2)} = 0 \cdot |x|^2 \Rightarrow R_g\, \infty \Rightarrow I_0^\infty = (-\infty, \infty)$

(b) $f'(x) = \frac{d}{dx} \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{(2n+1)x^{2n+1-1}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{\cancel{(2n+1)}x^{2n}}{\cancel{(2n+1)}(2n)!}$

$= \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} = g(x)$

$g'(x) = \frac{d}{dx} \sum_{n=0}^{\infty} \frac{x^{(2n)}}{(2n)!} = \sum_{n=0}^{\infty} \frac{2n x^{2n-1}}{(2n)(2n-1)!} = \sum_{n=0}^{\infty} \frac{x^{2n-1}}{(2n-1)!}$

However, $(-1)!$ not defined

$\sum_{n=0}^{\infty} \frac{x^{2n-1}}{(2n-1)!} = \underset{n=0}{\text{undef.}} + \underset{n=1}{\frac{x^1}{1!}} + \frac{x^3}{3!} + \dots = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!} = f(x)$

$f'(x) = g(x)$ & $g'(x) = f(x) \Rightarrow f(x) = \sinh x \quad g(x) = \cosh x$

Note: $\frac{d}{dx}\operatorname{sech} x = \cosh x \qquad \frac{d}{dx}\cosh x = \operatorname{sech} x$

Check $\operatorname{sech} x = \frac{1}{2}(e^{x} - e^{-x}) = \frac{1}{2}\left\{\sum_{n=0}^{\infty}\frac{x^n}{n!} - \sum_{n=0}^{\infty}\frac{(-1)^n x^n}{n!}\right\}$

$$= \frac{1}{2}\sum_{n=0}^{\infty}\left[\frac{x^n}{n!} + \frac{(-1)^{n+1}}{n!}x^n\right] = \frac{1}{2}\sum_{n=0}^{\infty}\frac{2x^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty}\frac{x^{2n+1}}{(2n+1)!}$$

IV. e) (5) $f(x) = \frac{4}{4+x^2} \qquad c = 0$

$$= \frac{1}{4}\cdot\frac{4}{[1+x^2/4]} = \frac{1}{[1+(x/2)^2]} = \sum_{k=0}^{\infty}(-1)^k\left(x^2/2^2\right)^k = \sum_{k=0}^{\infty}\frac{(-1)^k}{4^k}x^{2k}$$

I.C. : $|x^2/4| < 1 \Rightarrow -1 \leqslant \left|\frac{x^2}{4}\right| \leqslant 1$

$-4 \leqslant x^2 \leqslant 4$

$-2 \leqslant x \leqslant 2 \Rightarrow$ Int. of abs. convergence: $(-2, 2)$

Check endpts: $x = \pm 2 \Rightarrow \sum_{k=0}^{\infty}(-1)^k$ div $\Rightarrow I_o^2 = (-2, 2) = \{x \mid -2 < x < 2\}$

IV. f) $f(x) = \arctan 2x$

(5) $\frac{d}{dx}\arctan x = \frac{1}{1+x^2} = \sum_{k=0}^{\infty}(-1)^k x^{2k}$

$$\therefore \arctan x = \int\sum_{k=0}^{\infty}(-1)^k x^{2k}\,dx = \sum_{k=0}^{\infty}(-1)^k\int x^{2k}\,dx$$

$$\arctan x = \sum_{k=0}^{\infty}(-1)^k\frac{x^{2k+1}}{(2k+1)} \qquad -1 \leq x \leq 1$$

$$\arctan(2x) = \sum_{k=0}^{\infty}(-1)^k\frac{(2x)^{2k+1}}{(2k+1)} = \sum_{k=0}^{\infty}(-1)^k\frac{2^{2k+1}}{(2k+1)}x^{2k+1}$$

$-1 \leq 2x \leq 1 \Rightarrow \boxed{-\frac{1}{2} \leq x \leq \frac{1}{2}}$