MA262 **Calculus II: Notes for Feb 7**
Spring 2008 Kallfelz

Note: Regarding the extension of the deadline of Assignment I, I will subtract late points if you hand it in after Tuesday, Feb. 12th. If you'd like it returned and graded on Tuesday's class (Feb. 12) please leave it my mailbox (faculty lounge) no later than 5:00pm, Monday (Feb. 11).

- ***IMPROPER INTEGRALS (CONT.)***

Recall:

Case 1c.
$$\int_{-\infty}^{\infty} f(x)dx \Rightarrow \lim_{b\to-\infty}\int_{b}^{a} f(x)dx + \lim_{b\to\infty}\int_{a}^{b} f(x)dx$$
(for any real number a)

Perhaps you may be wondering why the improper integral just couldn't be written in the following form: $\lim_{b\to\infty}\int_{-b}^{b} f(x)dx$ instead. Answer: You could (in the divergent case) introduce a $\infty-\infty$ indeterminate form, which requires the use of **L'Hopital's Rule.**

(See http://www.glue.umd.edu/~wkallfel/MA261-2/Oct30notesb.pdf for more details on L'Hopitals' Rule and a review/refresher). This would introduce unnecessary extra work—so what seems like a shortcut actually creates more of a hassle)

- Example (#15, § 9.7)

$$\int_{-\infty}^{\infty}\frac{1}{1+x^2}dx$$

Pick the simplest constant c possible when using the case 1.c formula, assuming the integrand won't diverge at that point**.** In this case, $c = 0$ is the most suitable.

$$\int_{-\infty}^{\infty}\frac{1}{1+x^2}dx = \lim_{b\to\infty}\int_{-b}^{0}\frac{1}{1+x^2}dx + \lim_{b\to\infty}\int_{0}^{b}\frac{1}{1+x^2}dx$$

$$x=\tan\theta \Rightarrow \theta = \arctan x, dx = \sec^2\theta d\theta$$

$$\int_{-\infty}^{\infty}\frac{1}{1+x^2}dx=\lim_{b\to\infty}\int_{-b}^{0}\frac{1}{1+x^2}dx+\lim_{b\to\infty}\int_{0}^{b}\frac{1}{1+x^2}dx=\lim_{\beta\to \mathit{artcan}(\infty)}\int_{\beta}^{\arctan(0)}\frac{\sec^2\theta}{1+\tan^2\theta}d\theta$$

$$+\lim_{\beta\to\arctan(\infty)}\int_{\arctan 0}^{\beta}\frac{\sec^2\theta}{1+\tan^2\theta}d\theta$$

$$=\lim_{\beta\to\frac{\pi}{2}}\int_{-\beta}^{0}\frac{\sec^2\theta}{\sec^2\theta}d\theta+\lim_{\beta\to\frac{\pi}{2}}\int_{0}^{\beta}\frac{\sec^2\theta}{\sec^2\theta}d\theta=\lim_{\beta\to\frac{\pi}{2}}\left\{\int_{-\beta}^{0}d\theta+\int_{0}^{\beta}d\theta\right\}$$

$$=\lim_{\beta\to\frac{\pi}{2}}\left\{\theta\Big|_{-\beta}^{0}+\theta\Big|_{0}^{\beta}\right\}=0-\lim_{\beta\to\frac{\pi}{2}}(-\beta)+\lim_{\beta\to\frac{\pi}{2}}\beta-0=\tfrac{\pi}{2}+\tfrac{\pi}{2}=\pi$$

- Example (#17, § 9.7)

$$\int_{0}^{\infty}\frac{dx}{e^x+e^{-x}}=\lim_{b\to\infty}\int_{0}^{b}\frac{1}{e^x\left(1+e^{-2x}\right)}dx=\lim_{b\to\infty}\int_{0}^{b}\frac{e^{-x}}{\left(1+e^{-2x}\right)}dx$$

$$u=e^{-x}\Rightarrow du=-e^{-x}dx$$

$$\therefore\lim_{b\to\infty}\int_{0}^{b}\frac{e^{-x}dx}{\left(1+e^{-2x}\right)}=\lim_{b\to\infty}\int_{u(0)}^{u(b)}\frac{du}{1+u^2}=-\lim_{b\to\infty}\int_{1}^{e^{-b}}\frac{du}{1+u^2}$$

$$=-\lim_{b\to\infty}\int_{\arctan(1)}^{\arctan(e^{-b})}\frac{\sec^2\theta}{1+\tan^2\theta}d\theta=-\lim_{b\to\infty}\theta\Big|_{\arctan(1)=\pi/4}^{\arctan\left(e^{-b}\right)}=\tfrac{\pi}{4}-\lim_{b\to\infty}\arctan\left(e^{-b}\right)$$

$$=\tfrac{\pi}{4}-\arctan(0)=\tfrac{\pi}{4}$$

- Example (#25, § 9.7)

$$\int_{0}^{1}x\ln x dx=\lim_{b\to0}\int_{b}^{1}x\ln x dx\Rightarrow u=\ln x,du=\frac{dx}{x},dv=xdx,v=\tfrac{1}{2}x^2$$

$$=\lim_{b\to0}\left\{\tfrac{1}{2}x^2\ln x\Big|_{b}^{1}-\tfrac{1}{2}\int_{b}^{1}xdx\right\}=\lim_{b\to0}\left\{\tfrac{1}{2}x^2\ln x\Big|_{b}^{1}-\tfrac{1}{4}x^2\Big|_{b}^{1}\right\}=\tfrac{1}{2}\left(0-\lim_{b\to0}b^2\ln b\right)-\tfrac{1}{4}\left(1-\lim_{b\to0}b^2\right)$$

$$=-\tfrac{1}{2}\lim_{b\to0}\frac{\ln b}{\frac{1}{b^2}}-\tfrac{1}{4}+0=-\tfrac{1}{2}\lim_{b\to0}\frac{\frac{1}{b}}{-\frac{2}{b^3}}-\tfrac{1}{4}=\tfrac{1}{2}\lim_{b\to0}\frac{b^2}{2}-\tfrac{1}{4}=-\tfrac{1}{4}$$

Note: L'Hopital's Rule was adopted in the $b^2\ln b$, since that produces a $0\cdot\infty$ indeterminacy.

- Example (#27, § 9.7)

$$\int_0^{\frac{\pi}{2}} \tan\theta d\theta = \lim_{b\to\frac{\pi}{2}} \int_0^b \frac{\sin\theta}{\cos\theta}d\theta = \lim_{b\to\frac{\pi}{2}} -\ln|\cos\theta|\Big|_0^b = \ln|\cos(0)| - \lim_{b\to\frac{\pi}{2}} \ln|\cos b|$$

$$= \ln 1 - \lim_{u\to 0} \ln|u| = 0 - \infty = -\infty$$

Diverges

- Example (#32, § 9.7)

$$\int_0^2 \int_0^2 \frac{dx}{(x-1)^{4/3}} = \int_{u(0)}^{u(2)} \frac{du}{u^{4/3}} = \int_{-1}^{1} \frac{du}{u^{4/3}} = \lim_{b\to 0} \int_{-1}^{b} u^{-4/3}du + \lim_{b\to 0} \int_{b}^{1} u^{-4/3}du$$

$$= \lim_{b\to 0}\left\{-3u^{-1/3}\Big|_{-1}^{b} + -3u^{-1/3}\Big|_{b}^{1}\right\} = -3\left\{\left[\lim_{b\to 0} b^{-1/3} - (-1)^{-/3}\right] + \left[1 - \lim_{b\to 0} b^{-1/3}\right]\right\}$$

$$= -3\left\{2 - \lim_{b\to 0}\left(\frac{1}{\sqrt[3]{b}} - \frac{1}{\sqrt[3]{b}}\right)\right\} = -6$$

Note: In this case we're subtracting identical terms in the limit expression ($b^{-1/3}$), so the ∞ - ∞ indeterminacy is essentially trivial (no need to use L'Hopital's Rule).

The Laplace Transform: This operator enables one to solve complicated differential equations—there's an entire course covering this topic, which I taught at Capitol College Spring 2006, **MA 360** (For more information, see http://www.wam.umd.edu/%7Ewkallfel/MA360/

Its definition:

$$L[f(t)] = F(s) = \lim_{b\to\infty} \int_0^b e^{-st} f(t)dt$$

(Note typographical error in book: the coefficient in the exponential term should be s, not a)

- Example (#51, § 9.7)

$$L[\cos at] = \lim_{b\to\infty}\int_0^b e^{-st}\cos at\,dt = \lim_{b\to\infty}\left\{\frac{e^{-st}}{a^2+s^2}[-s\cos at + a\sin at]\right\}\Bigg|_0^b$$

$$= \lim_{b\to\infty}\frac{e^{-sb}}{a^2+s^2}[-s\cos ab + a\sin ab] - \frac{e^0}{a^2+s^2}(-s+0)$$

$$= \frac{s}{a^2+s^2}$$

(The first limit → 0 since the term e^{-as} term dominates. This can be shown rigorously using a comparison test[1], the 'Sandwich Theorem':

$$\lim_{b\to\infty}\frac{e^{-sb}}{a^2+s^2}(-s\cos ab + a\sin ab) = \frac{1}{a^2+s^2}\lim_{b\to\infty}\left(\frac{-s\cos ab + a\sin ab}{e^{sb}}\right)$$

$$\Rightarrow \frac{1}{a^2+s^2}\lim_{b\to\infty}\left(\frac{-|s\cos ab + a\sin ab|}{e^{sb}}\right) \le \frac{1}{a^2+s^2}\lim_{b\to\infty}\left(\frac{-s\cos ab + a\sin ab}{e^{sb}}\right) \le$$

$$\frac{1}{a^2+s^2}\left(\frac{|s\cos ab + a\sin ab|}{e^{sb}}\right)$$

Now observe that the numerator terms on the left and to the right are (respectively) greater than or equal to -2 and less than or equal to 2 (why?)

Hence:

$$\frac{-2}{a^2+s^2}\lim_{b\to\infty} e^{-sb} \le \frac{1}{a^2+s^2}\lim_{b\to\infty}\left(\frac{-s\cos ab + a\sin ab}{e^{sb}}\right) \le \frac{2}{a^2+s^2}\lim_{b\to\infty} e^{-sb}$$

The left and the right expressions obviously converge to 0, so the middle term must also.

- <u>Example</u> (#53, § 9.7)

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[1] L'Hopital's Rule won't assist much in the way of getting a simple answer, though if you apply it several times, you'll see that the denominator term gets increasingly dominant, compared to the numerator term.

$$L[\cosh at] = \lim_{b\to\infty}\int_0^b e^{-st}\cosh at\,dt = \lim_{b\to\infty}\int_0^b e^{-st}\cdot\tfrac{1}{2}\left(e^{at}+e^{-at}\right)dt$$

$$= \tfrac{1}{2}\lim_{b\to\infty}\int_0^b \left(e^{(a-s)t}+e^{-(a+s)t}\right)dt = \tfrac{1}{2}\lim_{b\to\infty}\left\{\frac{e^{(a-s)t}}{(a-s)}-\frac{e^{-(a+s)t}}{(a+s)}\right\}\Bigg|_0^b$$

$$= \tfrac{1}{2}\lim_{b\to\infty}\left\{\frac{e^{a}e^{-sb}}{(a-s)}-\frac{e^{-a}e^{-sb}}{(a+s)}\right\} - \tfrac{1}{2}\left\{\frac{1}{a-s}-\frac{1}{a+s}\right\} = 0+\tfrac{1}{2}\left\{\frac{a+s-(a-s)}{(a-s)(a+s)}\right\}$$

$$= 0+\tfrac{1}{2}\cdot\frac{2s}{a^2-s^2} = \frac{s}{a^2-s^2}$$

Note how the limits of the first two converge to 0 straightforwardly.

- Example (#53, § 9.7)

Prove by Induction that $\int_0^\infty x^n e^{-x}dx$ converges for any positive integer[2]

Base (n = 1) $\int_0^\infty x\ e^{-x}dx = \lim_{b\to\infty}\int_0^b xe^{-x}dx \Rightarrow u = x, du = dx, dv = e^{-x}, v = -e^{-x}$

$$\therefore \lim_{b\to\infty}\int_0^b xe^{-x}dx = \lim_{b\to\infty}\left\{-xe^{-x}\Big|_0^b + \int_0^b e^{-x}dx\right\} = -\lim_{b\to\infty}\frac{b}{e^b} + 0 - \lim_{b\to\infty} e^{-x}\Big|_0^b$$

$$= -\lim_{b\to\infty}\frac{1}{e^b} - \lim_{b\to\infty} e^{-b} + 1 = 1$$

(note how L'Hopital's Rule was used in the first term)

Inductive Step: a.) ***Assume*** $\int_0^\infty x^k e^{-x}dx < \infty$ for any $1 < k < \infty$

b.) Given assumption in a.), show that $\int_0^\infty x^{k+1}e^{-x}dx < \infty$

$$\int_0^\infty x^{k+1}e^{-x}dx = \lim_{b\to\infty}\int_0^b x^{k+1}e^{-x}dx \Rightarrow u = x^{k+1}, du = (k+1)x^k dx, dv = e^{-x}, v = -e^{-x}$$

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[2] In class I assumed *n* was non-negative, though the base case is easy to show for 1 as well

$$\therefore \lim_{b\to\infty}\int_0^b x^{k+1}e^{-x}dx = \lim_{b\to\infty}\left\{-\frac{x^{k+1}}{e^x}\bigg|_0^b + (k+1)\int_0^b x^k e^{-x}dx\right\} = 0+\lim_{b\to\infty}\frac{b^{k+1}}{e^b}+(k+1)\int_0^\infty e^{-x}x^k dx$$

Now, the integral in the last expression converges by assumption a.). So the problem reduces itself to applying L'Hopital's Rule successively to the $\frac{b^{k+1}}{e^b}$ term: (k+1) times[3]

$$\lim_{b\to\infty}\frac{b^{k+1}}{e^b}\xrightarrow{LR}\lim_{b\to\infty}\frac{(k+1)b^k}{e^b}\xrightarrow{LR}\ldots\to\lim_{b\to\infty}\frac{(k+1)!}{e^b}=0$$

- Example (#65, § 9.7)

$$P = k\int_1^\infty \frac{1}{\left(a^2+x^2\right)^{3/2}}dx = k\lim_{b\to\infty}\int_1^b \frac{dx}{\left(a^2+x^2\right)^{3/2}} \Rightarrow x = a\tan\theta, dx = a\sec^2\theta d\theta, \theta = \arctan\left(\tfrac{x}{a}\right)$$

$$= k\lim_{b\to\infty}\int_{\arctan\left(\frac{1}{a}\right)}^{\arctan\left(\frac{b}{a}\right)} \frac{a\sec^2\theta}{\left(a^2+a^2\tan^2\theta\right)^{3/2}}d\theta = \frac{k}{a^2}\lim_{\beta\to\frac{\pi}{2}}\int_{\alpha=\arctan\left(\frac{1}{a}\right)}^{\beta}\frac{\sec^2\theta}{\sec^3\theta}d\theta = \frac{k}{a^2}\lim_{\beta\to\frac{\pi}{2}}\int_\alpha^\beta \cos\theta d\theta$$

$$= \frac{k}{a^2}\lim_{\beta\to\frac{\pi}{2}}\sin\theta\bigg|_\alpha^\beta = \frac{k}{a^2}\left\{\lim_{\beta\to\frac{\pi}{2}}\sin\beta-\sin\alpha\right\} = \frac{k}{a^2}\left\{1-\sin\left(\arctan\left(\tfrac{1}{a}\right)\right)\right\}$$

To simplify last expression, note:

$$\alpha = \arctan\left(\tfrac{1}{a}\right) \Rightarrow \tan\alpha = \tfrac{1}{a} = \frac{OPP}{ADJ} \Rightarrow \sin\alpha = \frac{OPP}{HYP} = \frac{1}{\sqrt{a^2+1}}$$

$$\therefore P = \frac{k}{a^2}\left(1-\frac{1}{\sqrt{a^2+1}}\right) = \frac{k}{a^2}\left(\frac{\sqrt{a^2+1}-1}{\sqrt{a^2+1}}\right) = \frac{k\left(\sqrt{a^2+1}-1\right)}{a\sqrt{a^2+1}}$$

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[3] As I mentioned in class, though by informal reasoning you can see that the denominator term dominates over the numerator term, regarding rates of growth.