MA262 **Calculus II: Notes for Feb 5**
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- ***(From Q & A): AN EXAMPLE OF A DEFINITE INTEGRAL WHICH IS A TRIG SUBST.:***

Consider: $\int_0^{\sqrt{2}} \frac{(x^2+2)}{x^2}dx$ Hence: $x = \sqrt{2}\tan\theta \Rightarrow dx = \sqrt{2}\sec^2\theta d\theta$
$\Rightarrow \theta = \arctan\left(\frac{x}{\sqrt{2}}\right)$

$$\therefore \int_0^{\sqrt{2}} \frac{(x^2+2)^{3/2}}{x^2}dx = \int_{\arctan 0}^{\arctan(\sqrt{2}/\sqrt{2})} \frac{(2\tan^2\theta+2)}{2\tan^2\theta}\sqrt{2}\sec^2\theta d\theta = \sqrt{2}\int_0^{\arctan(1)}(1+\cot^2\theta)\sec^2\theta d\vartheta$$

$$= \sqrt{2}\int_0^{\pi/4}(\sec^2\theta+\csc^2\theta)d\theta = \sqrt{2}[\tan\theta-\cot\theta]_0^{\pi/4} = \sqrt{2}[(1-1)-(0-\infty)] = \infty$$

(I.e., the integral diverges.)

- *RATIONALIZING SUBSTITUTIONS (Cont.)*

Recall (see p.11 for their derivations in Jan. 31 notes):

$$du = \frac{2du}{u^2+1} \qquad \sin x = \frac{2u}{u^2+1} \qquad \cos x = \frac{1-u^2}{u^2+1}$$

- <u>Example (#68, § 9.6)</u>

$$\int \frac{d\theta}{\sec\theta - \tan\theta}$$

This integral can be solved in a number of ways. Consider:

Method 1: $\int \frac{d\theta}{\sec\theta-\tan\theta} = \int \frac{d\theta}{\frac{1}{\cos\theta}-\frac{\sin\theta}{\cos\theta}} = \int \frac{\cos\theta}{1-\sin\theta}d\theta$

At this point, one can evaluate the integral as a simple *u*-substitution:

$$U = 1-\sin\theta \Rightarrow dU = -\cos\theta d\theta \Rightarrow -\int\frac{dU}{U} = -\ln|U| + C = \ln\left|\frac{1}{1-\sin\theta}\right| + C$$

Method 2:

$$\int \frac{d\theta}{\sec\theta-\tan\theta}\cdot\frac{\sec\theta+\tan\theta}{\sec\theta+\tan\theta}=\int\frac{\sec\theta+\tan\theta}{\sec^{2}-\tan^{2}\theta}d\theta=\int(\sec\theta+\tan\theta)d\theta$$

$$=\int\sec\theta d\theta+\int\tan\theta d\theta=\ln|\sec\theta+\tan\theta|-\ln|\cos\theta|$$

$$=\ln\left|\frac{\sec\theta+\tan\theta}{\cos\theta}\right|=\ln\left|\frac{\frac{1}{\cos\theta}+\frac{\sin\theta}{\cos\theta}}{\cos\theta}\right|=\ln\left|\frac{1+\sin\theta}{\cos^{2}\theta}\right|=\ln\left|\frac{1+\sin\theta}{1-\sin^{2}\theta}\right|=\ln\left|\frac{1+\sin\theta}{(1+\sin\theta)(1-\sin\theta)}\right|$$

$$\ln\left|\frac{1}{1-\sin\theta}\right|+C$$

Method 3: Adopt the reationalizing substitution

$$\int\frac{d\theta}{\sec\theta-\tan\theta}=\int\frac{d\theta}{\frac{1}{\cos\theta}-\frac{\sin\theta}{\cos\theta}}=\int\frac{\cos\theta}{1-\sin\theta}d\theta=\int\frac{\frac{1-u^{2}}{1+u^{2}}}{1-\frac{2u}{1+u^{2}}}\cdot\frac{2du}{1+u^{2}}$$

$$=2\int\frac{1-u^{2}}{1+u^{2}-2u}\cdot\frac{du}{1+u^{2}}=2\int\frac{(1-u)(1+u)}{(1-u)^{2}}\cdot\frac{du}{1+u^{2}}=2\int\frac{1+u}{(1+u^{2})(1-u)}du$$

Resolving by partial fractions:

$$\frac{1+u}{(1+u^{2})(1-u)}=\frac{A_{1}u+B_{1}}{u^{2}+1}+\frac{A_{2}}{1-u}\Rightarrow 1+u=(A_{1}u+B_{1})(1-u)+A_{2}(1+u^{3})$$

$$u=1\Rightarrow 2=2A_{2}\Rightarrow A_{2}=1$$

$$u=0\Rightarrow 1=B_{1}+1\Rightarrow B_{1}=0$$

$$1+u=A_{1}u(1-u)+(1+u^{2})\Rightarrow A_{1}u=u\Rightarrow A_{1}=1$$

Hence:

$$2\int\frac{1+u}{(1+u^{2})(1-u)}du=2\left\{\int\frac{udu}{1+u^{2}}+\int\frac{du}{1-u}\right\}=\ln|1+u^{2}|-2\ln|1-u|$$

$$=\ln\left|\frac{1+u^{2}}{(1-u)^{2}}\right|=\ln\left|\frac{1+\frac{\sin^{2}\theta}{(1+\cos\theta)^{2}}}{\left(1-\frac{\sin\theta}{1+\cos\theta}\right)^{2}}\right|=\ln\left|\frac{(1+\cos\theta)^{2}+\sin^{2}\theta}{(1+\cos\theta-\sin\theta)^{2}}\right|$$

$$\ln\left|\frac{1+2\cos\theta+\cos^2\theta+\sin^2\theta}{1+\cos^2\theta+\sin^2\theta+\cos\theta-\sin\theta+\cos\theta-\cos\theta\sin\theta-\sin\theta-\sin\theta\cos\theta}\right|$$

$$==\ln\left|\frac{2(1+\cos\theta)}{2+2\cos\theta-2\sin\theta-2\sin\theta\cos\theta}\right|=\ln\left|\frac{1+\cos\theta}{1+\cos\theta-\sin\theta(1+\cos\theta)}\right|$$

$$=\ln\left|\frac{1+\cos\theta}{(1+\cos\theta)(1-\sin\theta)}\right|=\ln\left|\frac{1}{1-\sin\theta}\right|+C$$

Certainly not the simplest method at all by a long shot, but eventually we recover the answer! (Obviously Method 1was the most straightforward)

OTHER RATIONALIZING SUBSTITUTIONS

- Example: $\int\frac{dx}{1+\sqrt{x}}$ Use: $u^2=x\Rightarrow 2udu=dx$

$$\therefore\int\frac{dx}{1+\sqrt{x}}=\int\frac{2udu}{1+u}=2\int\frac{u}{1+u}du=2\left\{\int\left(1-\frac{1}{u+1}\right)du\right\}=2u-\ln|u+1|+C$$

$$=2\sqrt{x}-\ln\left|\sqrt{x}+1\right|+C$$

- Example: $\int\frac{dx}{\sqrt{x}+\sqrt[3]{x}}$ Use: $u^6=x\Rightarrow 6u^5du=dx$

$$\therefore\int\frac{dx}{\sqrt{x}+\sqrt[3]{x}}=6\int\frac{u^5}{u^3+u^2}du=6\int\frac{u^5}{u^2(u+1)}du=6\int\frac{u^3}{u+1}du=\int\left(u^2-u+1-\frac{1}{u+1}\right)du$$

$$=\tfrac{1}{3}u^3-\tfrac{1}{2}u^2+u-\ln|u+1|+C=\tfrac{1}{3}\sqrt{x}-\tfrac{1}{2}\sqrt[3]{x}+\sqrt[6]{x}+\ln\left(\frac{1}{\sqrt[6]{x}+1}\right)+C$$

Note how the absolute value bars were dropped, since in this case the argument is always positive. Also note the sign change by using property: $\ln\left(\frac{1}{x}\right)=\ln\left(x^{-1}\right)=-\ln x$

- Example:: $\int\sqrt{1+e^x}\,dx$ $u^2=1+e^x\Rightarrow 2udu=e^xdx\Rightarrow dx=\frac{2udu}{e^x}=\frac{2udu}{u^2-1}$

$$\therefore \int \sqrt{1+e^x}\,dx = \int \sqrt{u^2}\,\frac{2udu}{u^2-1} = 2\int \frac{u^2}{u^2-1}du = 2\int\left[1-\frac{1}{u^2-1}\right]du = 2\int du - 2\int \frac{1}{(u+1)(u-1)}du$$

$$= 2u - 2\cdot\left\{\tfrac{1}{2}\int\left(\frac{1}{u-1}-\frac{1}{u+1}\right)du\right\} = 2u - \int\frac{du}{u-1} + \int\frac{du}{u+1} = 2u - \ln|u-1| + \ln|u+1| + C$$

$$= 2u + \ln\left|\frac{u+1}{u-1}\right| + C = 2\sqrt{1+e^x} + \ln\left|\frac{\sqrt{1+e^x}+1}{\sqrt{1+e^x}-1}\right| + C$$

- *IMPROPER INTEGRALS*

Case 1a.) $\int_a^{\infty} f(x)dx \Rightarrow \lim_{b\to\infty}\int_a^b f(x)dx$

Case 1b.) $\int_{-\infty}^{a} f(x)dx \Rightarrow \lim_{b\to-\infty}\int_b^a f(x)dx$

Case 1c.) $\int_{-\infty}^{\infty} f(x)dx \Rightarrow \lim_{b\to-\infty}\int_b^a f(x)dx + \lim_{b\to\infty}\int_a^b f(x)dx$

(for any real number *a*)

Case 2a.)

If on [*a*, *b*], $\lim_{x\to a^+} f(x) = \pm\infty$ for any *x* on (*a*, b] then: $\int_a^b f(x)dx \Rightarrow \lim_{c\to a^+}\int_c^b f(x)dx$

Case 2b.)

If on [*a*, *b*], $\lim_{x\to b^{-+}} f(x) = \pm\infty$ for any *x* on [*a*, b) then: $\int_a^b f(x)dx \Rightarrow \lim_{c\to b^{-+}}\int_a^c f(x)dx$

Case 2b.)

If on [*a*, *b*], $\lim_{x\to c} f(x) = \pm\infty$ for any *x* on [*a*, *b*] and any *c* on (*a*, *b*):

$$\int_a^b f(x)dx \Rightarrow \lim_{d\to c^-}\int_a^d f(x)dx + \lim_{d\to c^+}\int_d^b f(x)dx$$

- <u>Example (Thm 9.3)</u>

$$\int_1^{\infty}\frac{dx}{x^p} = \lim_{b\to\infty}\int_1^b x^{-p}dx = \lim_{b\to\infty}\left.\frac{x^{1-p}}{1-p}\right|_1^b = \lim_{b\to\infty}\frac{1}{1-p}b^{1-p} - \frac{1}{1-p}$$

$$= \frac{1}{1-p}\left\{\lim_{b\to\infty}\frac{b}{b^p}-1\right\}$$

Case 1: $p > 1 \Rightarrow \lim_{p\to\infty}\frac{b}{b^p} = \lim_{b\to\infty}\frac{1}{b^k} = 0$ (where $k = p - 1 > 0$)

Case 2: $p = 1 \Rightarrow \lim_{b\to\infty}\int_1^b \frac{dx}{x} = \lim_{b\to\infty}\ln x\Big|_1^b = \lim_{b\to\infty}\ln b = \infty$

Case 3: $p < 1 \Rightarrow \lim_{p\to\infty}\frac{b}{b^p} = \lim_{b\to\infty} b^n = \infty$ (where $n = 1 - p > 0$)

- Example:

$$\int_0^2 \frac{dx}{x^2-1} = \lim_{b\to 1^-}\int_0^b \frac{dx}{x^2-1} + \lim_{b\to 1^+}\int_b^2 \frac{dx}{x^2-1}$$

Method 1: Partial Fractions

$$\int_0^2 \frac{dx}{x^2-1} = \lim_{b\to 1^-}\int_0^b \frac{dx}{x^2-1} + \lim_{b\to 1^+}\int_b^2 \frac{dx}{x^2-1}$$

$$= \lim_{b\to 1^-}\tfrac{1}{2}\int_0^b\left(\frac{1}{x-1}-\frac{1}{x+1}\right)dx + \lim_{b\to 1^+}\tfrac{1}{2}\int_b^2\left(\frac{1}{x-1}-\frac{1}{x+1}\right)dx$$

$$= \tfrac{1}{2}\lim_{b\to 1^-}\ln\left|\frac{x-1}{x+1}\right|\Bigg|_0^b + \tfrac{1}{2}\lim_{b\to 1^+}\ln\left|\frac{x-1}{x+1}\right|\Bigg|_b^2$$

$$= \tfrac{1}{2}\lim_{b\to 1^-}\ln\left|\frac{b-1}{b+1}\right| - 0 + 0 - \tfrac{1}{2}\lim_{b\to 1^+}\ln\left|\frac{b-1}{b+1}\right|$$

(divergent)

Method 2: (Trig subst) $x = \sec\theta \Rightarrow dx = \sec\theta\tan\theta$

$$\therefore \int\frac{dx}{x^2-1} = \int\frac{\sec\theta\tan\theta}{\sec^2\theta-1}d\theta = \int\frac{\sec\theta}{\tan\theta}d\theta = \int\csc\theta d\theta = \ln\left|\csc\theta-\cot\theta\right|$$

Hence:

$$\int_0^2 \frac{dx}{x^2-1} = \lim_{b\to 1^-}\int_0^b \frac{dx}{x^2-1} + \lim_{b\to 1^+}\int_b^2 \frac{dx}{x^2-1}$$

$$= \lim_{b\to 1^-} \ln\left|\csc\theta - \cot\theta\right|\Big|_{arc\sec 0}^{arc\sec b} + \lim_{b\to 1^+} \ln\left|\csc\theta - \cot\theta\right|\Big|_{arc\sec b}^{arc\sec 2}$$

Divergent, arcsec0 out of range