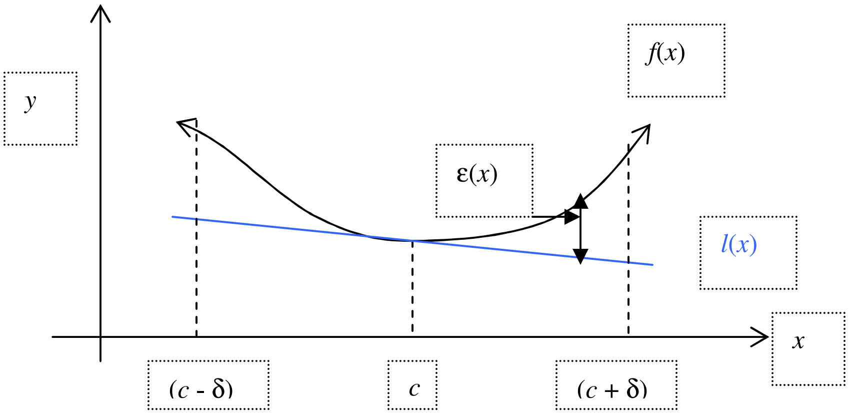
MA262 **Calculus II: Notes for Feb 28**
Spring 2008 Kallfelz

- *TAYLOR POLYNOMIALS AND TAYLOR'S THEOREM*

Recall already from Calculus I that there were occasions when you approximated a function $f(x)$ with a line (i.e. first degree polynomial) tangent to it, meeting the function at point c:



As the above picture shows, the tangent line $l(x)$ represents a linear approximation to $f(x)$ on the interval $I_\delta(c) = (c-\delta, c+\delta) = \{x \mid (c-\delta) < x < (c+\delta)\}$. The 'error' or overshoot $\varepsilon(x) = |f(x) - l(x)|$ is a measure of the inaccuracy of the linear approximation on the interval $I_\delta(c) = (c-\delta, c+\delta)$, which obviously grows if we increase δ. However, note that:

$$f(c) = l(c) \qquad f'(c) = l'(c) = m$$

Of course, from the point-slope formula[1], it's easy to construct the equation for the linear approximation:

$$(l(x) - f(c)) = m(x-c) \Rightarrow (l(x) - f(c)) = f'(c)(x-c) \Rightarrow l(x) = f(c) + f'(c)(x-c)$$

Linear approximations are somewhat crude. Certainly $f(x)$ could be more accurately approximated by a parabolic (or second-degree polynomial) $p(x)$ where:

$$f(c) = p(c) \qquad f'(c) = p'(c) \qquad f''(c) = p''(c)$$

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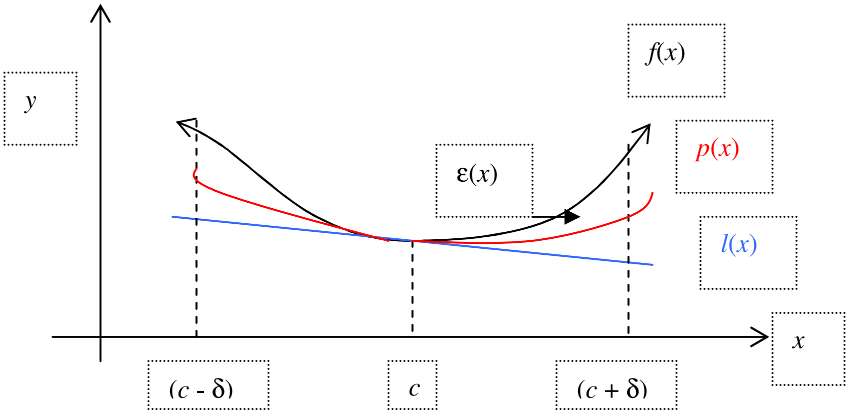
[1] $(y - y_0) = m(x - x_0)$, for any point (x_0, y_0) on a line with slope m.



As the above drawing suggests, the accuracy is improved, relative to the linear approximation. According to Simpson's Rule, the equation for $p(x)$ can be constructed to yield:

$$p(x) = f(c) + f'(c)(x-c) + \tfrac{1}{2} f''(c)(x-c)^2$$

In general, in the interval $I_\delta(c) = (c-\delta, c+\delta)$, for any function $f(x)$ differentiable to order n or higher, $f(x)$ can be approximated by some n-th degree polynomial $P_n(x)$, where:

$$P_n(c) = f(c), P_n'(c) = f'(c), \quad P_n''(c) = f''(c), \ldots, P_n^{(n)}(c) = f^{(n)}(c) \qquad \text{(A)}$$

$$P_n(x) = a_0 + a_1(x-c) + a_2(x-c)^2 + \ldots + a_n(x-c)^n = \sum_{k=0}^{n} a_k (x-c)^k \qquad \text{(B)}$$

The matching conditions (A) can be used to solve for the coefficients a_k in (B) (for $0 \le k \le n$), since:

$$P_n'(x) = \frac{d}{dx}(a_0 + a_1(x-c) + \ldots + a_n(x-c)^n) = \frac{d}{dx}\sum_{k=0}^{n} a_k (x-c)^k = \sum_{k=1}^{n} a_k k (x-c)^{k-1}$$

$$\therefore P_n'(c) = f'(c) = \sum_{k=1}^{n} a_k k (c-c)^{k-1} = a_1 \cdot 1 \Rightarrow a_1 = f'(c)$$

$$P_n''(x) = \frac{d^2}{dx^2}(a_0 + a_1(x-c) + \ldots + a_n(x-c)^n) = \frac{d^2}{dx^2}\sum_{k=0}^{n} a_k (x-c)^k = \sum_{k=2}^{n} a_k k(k-1)(x-c)^{k-2}$$

$$\therefore P_n''(c) = f''(c) = \sum_{k=2}^{n} a_k k(k-1)(c-c)^{k-2} = a_2 \cdot 2 \cdot 1 \Rightarrow a_2 = \tfrac{1}{2!} f''(c)$$

$$P_n^{(3)}(x)=\frac{d^3}{dx^3}(a_0+a_1(x-c)+\ldots+a_n(x-c)^n)=\frac{d^3}{dx^3}\sum_{k=0}^{n}a_k(x-c)^k=\sum_{k=3}^{n}a_k k(k-1)(k-2)(x-c)^{k-3}$$

$$\therefore P_n^{(3)}(c)=f^{(3)}(c)=\sum_{k=2}^{n}a_k k(k-1)(k-2)(c-c)^{k-3}=a_3\cdot 3\cdot 2\cdot 1\Rightarrow a_3=\tfrac{1}{3!}f''(c)$$

$$\vdots$$

$$P_n^{(n-1)}(x)=\frac{d^{(n-1)}}{dx^{(n-1)}}(a_0+a_1(x-c)+\ldots+a_n(x-c)^n)=\frac{d^{(n-13}}{dx^{(n-1)}}\sum_{k=0}^{n}a_k(x-c)^k=$$

$$=\sum_{k=n-1}^{n}a_k k(k-1)\ldots(k-n+1)(x-c)^{k-n+1}\ \therefore P_n^{(n-1)}(c)=f^{(n-1)}(c)=$$

$$\sum_{k=n-1}^{n}a_k k(k-1)\ldots(k-n+1)(c-c)^{k-n+1}=a_{n-1}\cdot(n-1)\cdot\ldots\cdot 2\cdot 1\Rightarrow a_{(n-1)}=\tfrac{1}{(n-1)!}f^{(n-1)}(c)$$

$$P_n^{(n)}(x)=\frac{d^{(n)}}{dx^{(n)}}(a_0+a_1(x-c)+\ldots+a_n(x-c)^n)=\frac{d^{(n3}}{dx^{(n)}}\sum_{k=0}^{n}a_k(x-c)^k=$$

$$=\sum_{k=n}^{n}a_k k(k-1)\ldots(k-n)(x-c)^{k-n}\ \therefore P_n^{(n)}(c)=f^{(n)}(c)=$$

$$\sum_{k=n}^{n}a_k k(k-1)\ldots(k-n)(c-c)^{k-n}=a_n\cdot n\cdot(n-1)\cdot\ldots\cdot 2\cdot 1\Rightarrow a_n=\tfrac{1}{n!}f^{(n)}(c)$$

Hence (B) is re-expressed as:

$$P_n(x)=a_0+a_1(x-c)+a_2(x-c)^2+\ldots+a_n(x-c)^n=\sum_{k=0}^{n}a_k(x-c)^k$$

$$=f(c)+f'(c)(x-c)+\tfrac{1}{2!}f''(c)(x-c)^2+\ldots+\tfrac{1}{n!}f^{(n)}(c)(x-c)^n$$

$$=\sum_{k=0}^{n}\tfrac{1}{k!}f^{(k)}(c)(x-c)^k$$

This above polynomial is known as the *n-th degree Taylor Polynomial* (*or series*) *approximation* of *f*(*x*). In the special case where $c = 0$:

$$P_n(x)=\sum_{k=0}^{n}\tfrac{1}{k!}f^{(k)}(0)x^k$$

…is known as the *n-th degree Taylor Polynomial n-th degree McClaurin Polynomial* (*or series*) *approximation* of *f*(*x*). Hence one may write, for any $x\in I_\delta(c)=(c-\delta,c+\delta)$: $\qquad f(x)\approx P_n(x)=\sum_{k=0}^{n}\tfrac{1}{k!}f^{(k)}(c)(x-c)^k$

Or in the special case: $x \in I_\delta(0) = (-\delta, \delta)$: $f(x) \approx P_n(x) = \sum_{k=0}^{n} \tfrac{1}{k!} f^{(k)}(0) x^k$

Thus a function can be *approximated* by *n*-th degree Taylor or McClaurin polynomials, which a *finite* power series. There are also cases in which *infinite* Taylor (or McClaurin) series can *converge* to *f*(*x*), in which case we write:

For any $x \in I_R(c) = (c - R, c + R)$: $f(x) = \sum_{k=0}^{\infty} \tfrac{1}{k!} f^{(k)}(c)(x-c)^k$

Or in the special case of the McClaurin series:

For any $x \in I_R(0) = (-R, R)$: $f(x) = \sum_{k=0}^{\infty} \tfrac{1}{k!} f^{(k)}(0) x^k$

…where *R* is the *radius of convergence*, i.e. the maximum half-width the interval can have around the point *c* such that *f*(*x*) is *exactly equal* to the infinite convergent (Taylor or McClaurin) power series. Much of the methods in the subsequent sections in Chapter 10 deal with fixing such conditions for convergence.

Everything that was established somewhat informally above can be given secure underpinnings via *Taylor's Theorem*:

Given a function *f*(*x*) differentiable at least to order (*n*+1) on interval [*c* , *x*), there exists a $z \in (c, x)$ such that: $f(x) = P_n(x) + R_n(x)$

where: $P_n(x) = \sum_{k=0}^{n} \tfrac{1}{k!} f^{(k)}(c)(x-c)^k$

and: $R_n(x) = \tfrac{1}{n!} \int_c^x f^{(n+1)}(t)(x-t)^n \, dt = \dfrac{f^{(n+1)}(z)}{(n+1)!}(x-c)^{n+1}$

Proof:

In class, iterating the procedure of integration by parts several times, the results for *n* = 1,2,3,4 establishing the above formula were obtained. However, as illuminating as the procedure may have been, this doesn't qualify as a *general* proof. To show the formula holds for *any n* one needs to apply *Mathematical Induction*:

- **Base Step: Show above formula holds in the case of *n* = 1:**

According to the Fundamental Theorem of Calculus, since *f*(*x*) differentiable at least to order (*n*+1) on interval [*c* , *x*). Then:

$$\int_c^x f'(t)dt = f(t)\Big|_c^x = f(x) - f(c) \Rightarrow f(x) = f(c) + \int_c^x f'(t)dt$$

Integrating the last integral by parts, choose: $\begin{array}{l} u(t) = f'(t) \Rightarrow du = f''(t)dt \\ dv(t) = dt \Rightarrow v(t) = -(x-t) \end{array}$

So:

$$f(x) = f(c) + \int_c^x f'(t)dt = f(c) + \left\{ -f'(t)(x-t)\Big|_c^x + \int_c^x f''(t)(x-t)dt \right\}$$

$$= f(c) + f'(c)(x-c) + \int_c^x f''(t)(x-t)dt$$

By inspection, the above formula has been established for the case $n = 1$, since:

$$P_1(x) = \sum_{k=0}^{1} \tfrac{1}{k!} f^{(k)}(c)(x-c)^k = \tfrac{1}{0!} f^{(0)}(c)(x-c)^0 + \tfrac{1}{1!} f^{(1)}(c)(x-c)^1$$
$$= f(c) + f'(c)(x-c)$$

$$R_1(x) = \tfrac{1}{1!}\int_c^x f^{(1+1)}(t)(x-t)^1\,dt = \int_c^x f''(t)(x-t)dt$$

- **Induction Step: Assume the above formula holds in the case of *k* : (where 0 < *k* < *n*):**

$$f(x) = P_k(x) + R_k(x) \qquad \text{where:}$$

$$P_k(x) = \sum_{j=0}^{k} \tfrac{1}{j!} f^{(j)}(c)(x-c)^j \qquad R_k(x) = \tfrac{1}{k!}\int_c^x f^{(k+1)}(t)(x-t)^k\,dt$$

...And use the assumption to *prove* that the above formula holds in the case of *k* + 1:

$$f(x) = P_{k+1}(x) + R_{k+1}(x) \qquad \text{where:}$$

$$P_{k+1}(x) = \sum_{j=0}^{k+1} \tfrac{1}{j!} f^{(j)}(c)(x-c)^j \quad R_{k+1}(x) = \tfrac{1}{(k+1)!}\int_c^x f^{(k+2)}(t)(x-t)^{k+1}\,dt$$

Since we can *assume* that:

$$f(x) = P_k(x) + R_k(x) = \sum_{j=0}^{k} \tfrac{1}{j!} f^{(j)}(c)(x-c)^j + \tfrac{1}{k!}\int_c^x f^{(k+1)}(t)(x-t)^k\, dt$$

Then adopt the same procedure of integration by parts:

$$u(t) = f^{(k+1)}(t) \Rightarrow du = f^{(k+2)}(t)dt$$
$$dv(t) = (x-t)^k \Rightarrow v(t) = -\tfrac{1}{(k+1)}(x-t)^{(k+1)}$$

Hence:

$$f(x) = \sum_{j=0}^{k} \tfrac{1}{j!} f^{(j)}(c)(x-c)^j + \tfrac{1}{k!}\int_c^x f^{(k+1)}(t)(x-t)^k\, dt$$

$$= \sum_{j=0}^{k} \tfrac{1}{j!} f^{(j)}(c)(x-c)^j + \tfrac{1}{k!}\left\{ - f^{(k+1)}(t)\frac{(x-t)^{(k+1)}}{(k+1)}\Bigg|_c^x + \tfrac{1}{(k+1)}\int_c^x f^{(k+1)}(t)(x-t)^{k+1}\, dt \right\}$$

$$= \sum_{j=0}^{k} \tfrac{1}{j!} f^{(j)}(c)(x-c)^j + \tfrac{1}{k!}\left\{ \tfrac{1}{(k+1)} f^{(k+1)}(c)(x-c)^{(k+1)} + \tfrac{1}{(k+1)}\int_c^x f^{(k+1)}(t)(x-t)^{k+1}\, dt \right\}$$

$$= \sum_{j=0}^{k} \tfrac{1}{j!} f^{(j)}(c)(x-c)^j + \tfrac{1}{(k+1)!} f^{(k+1)}(c)(x-c)^{(k+1)} + \tfrac{1}{(k+1)!}\int_c^x f^{(k+1)}(t)(x-t)^{k+1}\, dt$$

$$\therefore f(x) = \sum_{j=0}^{k+1} \tfrac{1}{j!} f^{(j)}(c)(x-c)^j + \tfrac{1}{(k+1)!}\int_c^x f^{(k+1)}(t)(x-t)^{k+1}\, dt$$

….thus establishing the formula holds for the case (*k* + 1). QED

Corollary

The non-integral remainder form:

$$R_n(x) = \tfrac{1}{n!}\int_c^x f^{(n+1)}(t)(x-t)^n\, dt = \frac{f^{(n+1)}(z)}{(n+1)!}(x-c)^{n+1}$$

can be established via the Mean Value Theorem:[2]

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[2] For more information, see pp. 18-22 in http://www.glue.umd.edu/%7Ewkallfel/MA261-2/notessept18.pdf

(MVT) There exists a $z \in (c\ ,\ x)$ such that for any $g(t)$ (differentiable on $[c\ ,\ x)$):

$$\int_c^x g(t)dt = g(z)(x-c) = g(z)\int_c^x dt$$

…which has a more general expression, respectively known as the *Weighted Mean Value Theorem* (WMVT):

There exists a $z \in (c\ ,\ x)$ such that for any $g(t)$, $h(t)$ (differentiable on $[c\ ,\ x)$):

$$\int_c^x g(t)h(t)dt = g(z)\int_c^x h(t)dt$$

Applying WMVT to $R_n(x) = \tfrac{1}{n!}\int_c^x f^{(n+1)}(t)(x-t)^n\, dt$, let:

$$g(t) = \tfrac{1}{n!} f^{(n+1)}(t) \qquad h(t) = (x-t)^n$$

Then: $\int_c^x g(t)h(t)dt = g(z)\int_c^x h(t)dt = \tfrac{1}{n!} f^{(n+1)}(z)\int_c^x (x-t)^n\, dt$

However: $\int_c^x (x-t)^n\, dt = -\tfrac{1}{(n+1)}(x-t)^{n+1}\Big|_c^x = \dfrac{(x-c)^{n+1}}{(n+1)}$

Hence: $R_n(x) = \int_c^x g(t)h(t)dt = \tfrac{1}{n!} f^{(n+1)}(z)\int_c^x (x-t)^n\, dt = \dfrac{f^{(n+1)}(z)}{(n+1)!}(x-c)^{(n+1)}$

Taylor Series and power series methods enjoy a variety of applications.[3] In this respect one can think of Taylor's Theorem as a doorway to higher applications (in engineering, applied mathematics, as well as in the physical sciences). Series methods are especially useful in real-world cases, since one can construct (to a degree of precision delimited by R_n) a series representation of functional dependencies among sets of independent, versus dependent variables, *even if you don't know exactly what the functional dependencies are.* To put it another way, the world gives us measured *data* (constants, i.e. numbers). It's up to the scientist/engineer to *interpolate*, i.e. construct a function which best fits the data. This is just one instance of how series methods can prove essential.

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[3] See for instance, **MA355** (Numerical Analysis), just as one example http://www.glue.umd.edu/%7Ewkallfel/MATH355_Capitol%20College/

Other instances occur when one perhaps may know the *differential equations*[4] governing some complicated dynamical phenomena, but such equations may prove difficult if not impossible to solve. One can *approximate* a solution via power series methods, which also give one an insight into *perturbation analysis*—that is to say, how varying certain initial data can affect the behavior of classes of solutions, to certain orders of magnitude.

So the techniques presented here are quite important, basic, and useful!

- Example (§10.7, #15)

Find the McClaurin ($c = 0$) 3rd degree polynomial for $f(x) = \tan x$:

$\boldsymbol{n}$	$\boldsymbol{f^{(n)}(x)}$	$\boldsymbol{f^{(n)}(0)}$
0	$\tan x$	0
1	$\sec^2 x$	1
2	$2\sec x(\sec x\tan x) = 2\sec^2 x\tan x$ $= 2(1+\tan^2 x)\tan x = 2\tan x + 2\tan^3 x$	0
3	$2\sec^2 x + 6\tan^2 x(\sec^2 x) = 2\sec^2 x(1+3\tan^2 x)$	2

Hence:

$$P_3(x) = \sum_{k=0}^{3} \frac{d^k}{dx^k}\left(\tan x\big|_{x=0}\right)x^k = f(0) + f'(0)x + \tfrac{1}{2}f''(0)x^2 + \tfrac{1}{6}f'''(0)x^3$$

$$= 0 + 1\cdot x + 0 + \tfrac{2}{6}x^3 = x + \tfrac{1}{3}x^3$$

- Example (§10.7, #20)

Find the Taylor ($c = \pi$) 2rd degree polynomial for $f(x) = x^2\sec x$:

$\boldsymbol{n}$	$\boldsymbol{f^{(n)}(x)}$	$\boldsymbol{f^{(n)}(\pi)}$
0	$x^2\sec x$	$-\pi^2$
1	$2x\sec x + x^2\sec x\tan x = x\sec x(2 + x\tan x)$	-2π
2	$= \sec x(2 + x\tan x) + x\sec x\tan x(2 + x\tan x)$ $+ x\sec x(\tan x + x\sec^2 x)$	$-2-\pi(\pi) = -(2+\pi^2)$

Hence: $P_2(x) = f(\pi) + f'(\pi)(x-\pi) + \tfrac{1}{2}f''(\pi)(x-\pi)^2$

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[4] I.e., an equation relating an unknown function $y = f(x)$ to an expression involving f's higher order derivatives. **MA340** is devoted to a certain class of differential equations known as *ordinary* (i.e. differential equations involving no partial derivatives—we'll examine the concept of partial differentiation in this course as well as in **Calculus III, MA 263.)**

$$= -\pi^2 - 2\pi(x-\pi) - \tfrac{1}{2}\left(2+\pi^2\right)(x-\pi)^2$$

- Example (§10.7, #31)

For all $x < 0$, what values of x will guarantee that for $f(x) = e^x$, its 3rd degree McClaurin polynomial: $P_3(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} = \sum_{k=0}^{3} \frac{1}{k!} x^k$ is accurate within 10^{-3} ?

$$\left|R_3(x)\right| = \left|f(x) - P_3(x)\right| = \left|\frac{\left.\frac{d^4}{dx^4} e^x\right|_{x=z} x^4}{4!}\right| = \tfrac{1}{4!} e^z |x|^4 < 10^{-3}$$

Note that for all $-\infty < z < 0 : \max_{-\infty < z < 0} e^z = 1$

Hence: $\left|R_3(x)\right| = \frac{1}{4!} e^z |x|^4 \leq \frac{1}{4!} |x|^4 < 10^{-3} \Rightarrow |x| < \sqrt[4]{4! \cdot 10^{-3}} \approx 0.3936$

So $P_3(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} = \sum_{k=0}^{3} \frac{1}{k!} x^k$ is accurate within .001 for all x in the interval: (-0.3936, 0)

- Example

Given the McClaurin Polynomial for sin*x*: $P_n(x) = \sum_{k=0}^{n} (-1)^k \frac{x^{2k+1}}{(2k+1)!}$, find the smallest n to ensure 10^{-5} accuracy for all x in interval: $[\, 0 \,,\, \pi/2]$:

$$\left|R_n(x)\right| = \left|\sin x - P_n(x)\right| = \left|\frac{\left.\frac{d^{n+1}}{dx^{n+1}} \sin x\right|_{x=z} x^{n+1}}{(n+1)!}\right| = \tfrac{1}{(n+1)!} \left|\frac{d^{n+1}}{dx^{n+1}} \left.(\sin x)\right|_{x=z}\right| |x|^{n+1} \leq \tfrac{1}{(n+1)!} |x|^{n+1}$$

$$\leq \tfrac{1}{(n+1)!} \left(\tfrac{\pi}{2}\right)^{n+1} < 10^{-5}$$

The above inequality can be solved for n by trial and error. Using EXCEL:

n	$\frac{1}{(n+1)!}\left(\frac{\pi}{2}\right)^{n+1}$
0	1.570796327
1	1.23370055
2	0.645964098
3	0.253669508
4	0.079692626
5	0.020863481
6	0.004681754

7	0.00091926
8	0.000160441
9	2.5202E-05
10	3.59884E-06
11	4.71087E-07

So by $n = 10$, the McClaurin Polynomial $P_{10}(x)=\sum_{k=0}^{10}(-1)^k \frac{x^{2k+1}}{(2k+1)!}$ is accurate within 10^{-5} for all x in interval: $[\ 0\ ,\ ^{\pi}/_{2}]$.