MA262 **Calculus II: Notes for Feb 26**
Spring 2008 Kallfelz

- *THE RATIO AND THE ROOT TESTS*

Consider infinite series like: $\sum_{k=1}^{\infty} \frac{e^k}{k^k}$ and $\sum_{k=1}^{\infty} \frac{\sin k}{k!}$. On the one hand, they seem 'simple' enough that if you were to try a comparison test (either SCT or LCT[1]) you may be hard pressed to think of a simpler candidate (whose converge/divergence is simple to establish) for either direct comparison (in the case of the SCT) or even indirect comparison (in the case of the LCT). On the other hand, they seem hopelessly complicated! The former involves a variable base and exponent in the denominator term, where the latter involves a factorial. Trying some more basic test like the integral test would prove hopelessly messy in the former case and in the latter case, literally impossible.[2]

The above two are ideal tests for the Root and Ratio Tests (Thms. 10.18, 10.17, respectively, text):

Root Test: (RoT)

Let $r = \lim_{k\to\infty} \sqrt[k]{|a_k|} = \lim_{k\to\infty} \left(|a_k|\right)^{1/k}$ for some infinite series $\sum_{k=1}^{\infty} a_k$. Then:

Case 1: $r < 1 \Rightarrow \sum_{k=1}^{\infty} a_k < \infty$ (i.e., series converges)

Case 2: $r > 1 \Rightarrow \sum_{k=1}^{\infty} a_k = \infty$ (i.e., series diverges)

Case 3: $r = 1 \Rightarrow$ Test is inconclusive.

Ratio Test: (RaT)

Let $\rho = \lim_{k\to\infty} \frac{|a_{k+1}|}{|a_k|} = \lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right|$ for some infinite series $\sum_{k=1}^{\infty} a_k$. Then:

Case 1: $\rho < 1 \Rightarrow \sum_{k=1}^{\infty} a_k < \infty$ (i.e., series converges)

Case 2: $\rho > 1 \Rightarrow \sum_{k=1}^{\infty} a_k = \infty$ (i.e., series diverges)

Case 3: $\rho = 1 \Rightarrow$ Test is inconclusive.

---

[1] See Feb. 21 notes, http://www.glue.umd.edu/%7Ewkallfel/MA261-2/Feb21.pdf

[2] Recall (Feb. 14 notes) that the factorial function has no analogue for real numbers that you'll learn in Calculus II. However, in an upper level course (MA360, for instance) you learn that the factorial function can be generalized to the Gamma function in the case of the real numbers.

- Example: $\sum_{k=1}^{\infty} \frac{2\cdot 4\cdot 6\cdot \ldots \cdot 2k}{(2k)!}$ involves progressive product strings including the factorial term in the denominator. Hence apply **RaT,** since setting up such a ratio will create easy cancellation of most of the terms in the product string:

$$\rho = \lim_{k\to\infty} \frac{|a_{k+1}|}{|a_k|} = \lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty} \frac{2\cdot 4\cdot 6\cdot \ldots \cdot 2k\cdot 2(k+1)}{(2(k+1))!} \cdot \frac{(2k)!}{2\cdot 4\cdot 6\cdot \ldots \cdot 2k}$$

$$= \lim_{k\to\infty} \frac{2(k+1)(2k)!}{(2k+2)(2k+1)(2k)!} = \lim_{k\to\infty} \frac{1}{(2k+1)} = 0 < 1 \Rightarrow \therefore \sum_{k=1}^{\infty} a_k \_converges$$

- Example: $\sum_{k=1}^{\infty} k\left(\frac{2}{3}\right)^k$ involves a variable exponent term. Hence apply **RoT**, which will eliminate the k-th power term:

$$r = \lim_{k\to\infty} \sqrt[k]{|a_k|} = \lim_{k\to\infty} \left(|a_k|\right)^{1/k} = \lim_{k\to\infty} \sqrt[k]{k\left(\tfrac{2}{3}\right)^k} = \lim_{k\to\infty} k^{1/k}\left(\tfrac{2}{3}\right) = \tfrac{2}{3}\lim_{k\to\infty} k^{1/k}$$

The limit $\lim_{k\to\infty} k^{1/k}$ is an example of a ∞^0 indeterminate form. Before one can apply L'Hopital's Rule to it, however, one must look at the natural logarithm of this limit:[3]

$$\ln\left(\lim_{k\to\infty} k^{1/k}\right) = \lim_{k\to\infty} \ln\left(k^{1/k}\right) = \lim_{k\to\infty} \left(\frac{1}{k}\right)\cdot \ln k = \lim_{k\to\infty} \frac{\ln k}{k}$$

$$\xrightarrow{LHR} \lim_{k\to\infty} \frac{1/k}{1} = \lim_{k\to\infty} \frac{1}{k} = 0 \Rightarrow \lim_{k\to\infty} k^{1/k} = e^{\lim_{k\to\infty} \ln\left(k^{1/k}\right)} = e^0 = 1$$

Hence: $$r = \tfrac{2}{3}\lim_{k\to\infty} k^{1/k} = \tfrac{2}{3}e^0 = \tfrac{2}{3} < 1 \Rightarrow \sum_{k=1}^{\infty} a_k < \infty \_(i.e., converges)$$

Note 1 : The above example is simple enough that one could also apply the **RaT:**

$$\rho = \lim_{k\to\infty} \frac{|a_{k+1}|}{|a_k|} = \lim_{k\to\infty} \frac{(k+1)\left(\frac{2}{3}\right)^{k+1}}{k\left(\frac{2}{3}\right)^k} = \lim_{k\to\infty} \frac{(k+1)\left(\frac{2}{3}\right)}{k}$$

$$= \tfrac{2}{3}\lim_{k\to\infty} \frac{k+1}{k} = \tfrac{2}{3}\lim_{k\to\infty}\left(1+\tfrac{1}{k}\right) = \tfrac{2}{3} < 1 \Rightarrow \sum_{k=1}^{\infty} a_k < \infty$$

However, bear in mind that though we'd expect that $\rho < 1$ as well, the fact that $\rho = r$ in this case is purely coincidental!

---

[3] For more details, see: http://www.glue.umd.edu/%7Ewkallfel/MA261-2/Oct30notesb.pdf

Note 2 : Knowing which test to select in advance (**RaT** or **RoT**) which may prove the most efficient is not so straightforward to tell at the outset. As a general rule of thumb: Use **RaT** when the expression is some relatively complicated ratio that doesn't lend itself easily to SCT or LCT. **RaT** *is highly recommended* if the terms involve a product string (like a factorial) indexed according to k.[4] Bear in mind however that both tests can give inconclusive results! (In the $\rho = 1$ or $r = 1$ case).

- Example: $\sum_{k=1}^{\infty} \frac{k!}{(ek)^k}$

Since a factorial term is involved, then adopt **RaT:**

$$\rho = \lim_{k\to\infty} \frac{|a_{k+1}|}{|a_k|} = \lim_{k\to\infty} \frac{\left(\frac{(k+1)!}{(e(k+1))^{k+1}}\right)}{\frac{k!}{k^k}} = \lim_{k\to\infty} \frac{\left(\frac{1}{e(k+1)}\right)\left(\frac{1}{e(k+1)}\right)^k (k+1)k!}{\left(\frac{1}{ek}\right)^k k!}$$

$$= \tfrac{1}{e}\lim_{k\to\infty} \left(\tfrac{1}{k+1}\right)\left(\tfrac{ek}{e(k+1)}\right)^k (k+1) = \tfrac{1}{e}\lim_{k\to\infty}\left(\tfrac{k}{k+1}\right)^k \Rightarrow \ln\left(\lim_{k\to\infty}\left(\tfrac{k}{k+1}\right)^k\right) = \lim_{k\to\infty} k\ln\left(\tfrac{k}{k+1}\right)$$

$$= \lim_{k\to\infty} k\ln\left(\tfrac{1}{1+1/k}\right) = \lim_{k\to\infty}\frac{\ln\left((1+\frac{1}{k})^{-1}\right)}{\frac{1}{k}} = -\lim_{k\to\infty}\frac{\ln\left(1+\frac{1}{k}\right)}{\frac{1}{k}} \xrightarrow{LHR} -\lim_{k\to\infty}\frac{-\left(1+\frac{1}{k}\right)^{-1}\left(-\frac{1}{k^2}\right)}{\left(-\frac{1}{k^2}\right)}$$

$$= \lim_{k\to\infty}\left(1+\tfrac{1}{k}\right)^{-1} = 1 \Rightarrow \lim_{k\to\infty}\left(\tfrac{k}{k+1}\right)^k = \exp\left(\lim_{k\to\infty}\left(1+\tfrac{1}{k}\right)^{-1}\right) = e^1$$

$$\therefore \rho = \tfrac{1}{e}\lim_{k\to\infty}\left(\tfrac{k}{k+1}\right)^k = \tfrac{1}{e}\exp\left(\lim_{k\to\infty}\left(1+\tfrac{1}{k}\right)^{-1}\right) = \tfrac{1}{e}\cdot e = 1$$

…So the test is inconclusive! As indicated by the footnote (n.4 below) trying to resolve this via **RoT** would prove itself to be prohibitively complicated.[5]

So, despite the factorial term, in principle, one could still try some of the comparison tests (SCT or LCT). A useful point to remember is:

$$a_k = \frac{k!}{(ek)^k} = \frac{k\cdot(k-1)\cdot\ldots\cdot 2\cdot 1}{(ek)\cdot(ek)\cdot\ldots\cdot(ek)\cdot(ek)} = \left(\frac{k}{ek}\right)\cdot\left(\frac{k-1}{ek}\right)\cdot\ldots\cdot\left(\frac{2}{ek}\right)\cdot\left(\frac{1}{ek}\right)$$

---

[4] As mentioned in class, it's conceivable to attempt **RoT** in this case, though the evaluation of such a limit can get very ugly, very fast. Consider the simple case: $\lim_{k\to\infty}(k!)^{1/k}$. One must take the natural log first:

$$\ln\left(\lim_{k\to\infty}(k!)^{1/k}\right) = \lim_{k\to\infty}\ln\left((k!)^{1/k}\right) = \lim_{k\to\infty}\left(\frac{\ln k!}{k}\right) = \lim_{k\to\infty}\left(\frac{\ln k + \ln(k-1) + \ldots + \ln 2 + \ln 1}{k}\right)$$

$$= \lim_{k\to\infty}\frac{\sum_{j=2}^{k}\ln j}{k} \xrightarrow{LHR} \lim_{k\to\infty}\frac{\sum_{j=2}^{k}\frac{1}{j}}{1} = \lim_{k\to\infty}\sum_{j=2}^{k}\frac{1}{j} = \sum_{j=2}^{\infty}\frac{1}{j} = \infty$$

…producing a divergent p-series (where $p = 1$). Hence it's not *impossible* to adopt **RoT**, but even this simple case indicates how laborious the procedure is when confronted with a factorial.

[5] As opposed to the above case (shown in n.4 above) here you'd have the following expression on your hands: $r = \lim_{k\to\infty}\sqrt[k]{\frac{k!}{(ek)^k}} = \frac{1}{e}\lim_{k\to\infty}\frac{1}{k}\sqrt[k]{k!}$, creating further headaches when you try to apply L'Hopital's Rule!

Hence using the above decomposition, it's certainly true that:

$$a_k = \left(\frac{k}{ek}\right)\cdot\left(\frac{k-1}{ek}\right)\cdot\ldots\cdot\left(\frac{2}{ek}\right)\cdot\left(\frac{1}{ek}\right) \le \left(\frac{k}{ek}\right)\cdot\left(\frac{k}{ek}\right)\cdot\ldots\cdot\left(\frac{k}{ek}\right)\left(\frac{k}{ek}\right) = \frac{1}{e^k}$$

In other words, for all k: $a_k = \dfrac{k!}{(ke)^k} \le \dfrac{1}{e^k} = e^{-k} = b_k$

However, examining this simpler series: $\sum_{k=1}^{\infty} \frac{1}{e^k} = \sum_{k=1}^{\infty} e^{-k}$, it is evident that it converges, by virtue of the integral test:

$$\sum_{k=1}^{\infty} e^{-k} \Leftrightarrow \int_1^{\infty} e^{-x}dx = \lim_{b\to\infty} \int_1^{b} e^{-x}dx = -\lim_{b\to\infty} e^{-x}\Big|_1^{b} = \lim_{b\to\infty} e^{-x}\Big|_b^{1} = e^{-1} - \lim_{b\to\infty} e^{-b} = \tfrac{1}{e} < \infty$$

$$\therefore \sum_{k=1}^{\infty} e^{-k} < \infty$$

Hence by SCT we can conclude that: $\sum_{k=1}^{\infty} \frac{k!}{(ek)^k} < \infty$

Note 4: Consider the subtle adjustment, if *e* in the denominator term were removed, i.e.: $\sum_{k=1}^{\infty} \frac{k!}{(ek)^k} \longrightarrow \sum_{k=1}^{\infty} \frac{k!}{k^k}$ …then (as the above derivation indicates) the convergence of this re-adjusted series could have been easily resolved using **RaT.**

- Example (§10.6, # 29):

$$\frac{1}{(\ln 3)^3} + \frac{1}{(\ln 4)^4} + \ldots + \frac{1}{(\ln n)^n} + \ldots = \sum_{k=3}^{\infty} (\ln k)^{-k}$$

Using **RoT:**

$$r = \lim_{k\to\infty} \sqrt[k]{(\ln k)^{-k}} = \lim_{k\to\infty} \frac{1}{\ln k} = 0 < 1 \Rightarrow \sum_{k=3}^{\infty} (\ln k)^{-k} < \infty\_(i.e., conv)$$

- Example (§10.6, # 50):

$\sum_{k=1}^{\infty} \frac{3\cdot 5\cdot 7\cdot \ldots \cdot(2k+1)}{18^k(2k-1)k!}$. Due to the presence of multiplication strings, adopt the **RaT:**

$$\rho = \lim_{k\to\infty} \frac{|a_{k+1}|}{|a_k|} = \lim_{k\to\infty} \frac{\frac{3\cdot5\cdot7\cdot\ldots\cdot(2k+1)\cdot(2k+3)}{18^{k+1}(2k+1)(k+1)!}}{\frac{3\cdot5\cdot7\cdot\ldots\cdot(2k+1)}{18^k(2k-1)k!}} = \lim_{k\to\infty} \frac{(2k+3)(2k-1)}{18(2k+1)(k+1)}$$

$$= \tfrac{1}{18}\lim_{k\to\infty} \frac{(2k+3)(2k-1)}{(2k+1)(k+1)} = \tfrac{1}{18}\lim_{k\to\infty} \frac{4k^2+4k-3}{2k^2+3k+1} = \tfrac{1}{18}\lim_{k\to\infty} \frac{4+\frac{4}{k}-\frac{3}{k^2}}{2+\frac{3}{k}+\frac{1}{k^2}}$$

$$= \tfrac{1}{18}\cdot 2 = \tfrac{1}{9} < 1 \Rightarrow \sum_{k=1}^{\infty} a_k < \infty \_(i.e., converges)$$

- *ALTERNATING SERIES AND CONDITIONAL CONVERGENCE*

Definition 1: An *alternating finite series* is one that can be represented in the form: $\sum_{k=1}^{n}(-1)^k a_k$ (alternating in odd terms) or $\sum_{k=1}^{n}(-1)^{k-1} a_k$ (alternating in even terms). If the series is infinite, then odd/even alternating series are represented by: $\sum_{k=1}^{\infty}(-1)^k a_k$, $\sum_{k=1}^{\infty}(-1)^{k-1} a_k$ (respectively).

- **Note 5** : Just because a series has negative and positive terms does not guarantee it alternates! (Note that the above definition shows that term-by-term, the neighboring terms differ in sign—i.e. if $a_k > 0$ then $a_{k+1} < 0$, or vice versa.) For example, the series: $\sum_{k=1}^{\infty} \frac{\sin\left(\frac{1}{n^2}\right)}{n}$ doesn't alternate, though some of its terms are negative (note that the default measure of sine is in radians.)

Thm (10.16) (Absolute Convergence Test-AbST) If $\sum_{k=1}^{n} |a_k| < \infty$, then $\sum_{k=1}^{\infty} a_k < \infty$.

So if the series $\sum_{k=1}^{\infty} a_k$ is an alternating series, and satisfies the above Thm (10.16), then it's guaranteed to converge. In fact, this kind of robust convergence (in terms of the absolute values of each term) one defines the associated alternating series to **converge absolutely.**

Interestingly, however, there are cases in which some alternating series $\sum_{k=1}^{n}(-1)^k a_k < \infty$, but $\sum_{k=1}^{n} |(-1)^k a_k| = \sum_{k=1}^{\infty} |a_k| = \infty$. I.e., the alternating series itself converges, but its associated series of absolute value terms diverges. In this case one defines the alternating series to **conditionally converge.**

Conditional convergence is a much weaker notion than absolute convergence. For example, an absolutely convergent alternating series will always converge to the same answer, regardless of the how one arranges its terms. On the other hand, this is generally not true for a conditionally convergent alternating series: the answer depends on the particular ordering of its terms.

To name a really elementary example:

The series: $\sum_{k=1}^{\infty}(-1)^n \frac{1}{n} = -1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - ....$ is conditionally convergent, for reasons to be discussed shortly. (It is immediately apparent that its absolute value terms diverge, since $\sum_{k=1}^{\infty} |(-1)^n \frac{1}{n}| = \sum_{k=1}^{\infty} \frac{1}{n}$ is a divergent p – series (since $p = 1$).

But $\sum_{k=1}^{\infty}(-1)^n \frac{1}{n} = -1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - ....$ converges to a definite answer. However, one is not free to arbitrarily rearrange terms in the series, and expect the same answer. For instance, if one grouped all the positive (even-indexed) terms first, and the negative (odd-indexed) terms second, one would have:

$$\sum_{k=1}^{\infty}(-1)^n \frac{1}{n} = -1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - ... = \frac{1}{2} + \frac{1}{4} + ... - \left(1 + \frac{1}{3} + \frac{1}{5} + ...\right) = \sum_{k=1}^{\infty} \frac{1}{2k} - \sum_{k=1}^{\infty} \frac{1}{2k+1}$$
$$= \frac{1}{2}\sum_{k=1}^{\infty} \frac{1}{k} - \frac{1}{2}\sum_{k=1}^{\infty} \frac{1}{k + \frac{1}{2}}$$

Of course, the first series is a divergent p-series, and the second series can be shown to diverge as well, by using the SCT: $k + \frac{1}{2} < k + 1 \Rightarrow \frac{1}{k+1} < \frac{1}{k + \frac{1}{2}}$, for all $k \geq 1$.

But the series: $\sum_{k=1}^{\infty} \frac{1}{k+1}$ clearly diverges, which can be easily shown by way of the integral test: $\int_1^{\infty} \frac{dx}{x+1} = \lim_{b \to \infty} \int_1^{b} \frac{dx}{x+1} = \lim_{b \to \infty} \ln(x+1)\Big|_1^b = \lim_{b \to \infty} \ln(b+1) - \ln 2 = \infty$

So the rearranged series: $\frac{1}{2}\sum_{k=1}^{\infty} \frac{1}{k} - \frac{1}{2}\sum_{k=1}^{\infty} \frac{1}{k + \frac{1}{2}}$ produces a $(\infty - \infty)$ indeterminacy, therefore from all the above information one cannot conclude that it converges, let alone to the same value as the original series $\sum_{k=1}^{\infty}(-1)^n \frac{1}{n} = -1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - ....$, if it could even be definitely established that this rearranged series still converges at all!

To account for cases of conditional convergence, adopt the **AST (Alternating Series Test, expressed in Thm 10.14, text)**

Thm 10.14 (AST)

Given: $\sum_{k=1}^{n}(-1)^{k-1} a_k$ or $\sum_{k=1}^{n}(-1)^k a_k$ (where $a_k > 0$ for all k). These series converge if the following conditions are met:
(a.) (Monotone decreasing) $a_{k+1} \leq a_k$
(b.) $\lim_{k\to\infty} a_k = 0$

- **Note 6:** Recall Thm 10.5 (§10.1): One immediately recognizes that sufficiency conditions (b.) follows from (a.), since the sequence $\{a_k\}$ is monotone decreasing and bounded below by 0 (as all of $a_k > 0$).

- Example (§10.5, # 11): $\sum_{k=0}^{\infty}(-1)^k e^{-k}$

To determine convergence, first apply **AbST:**

$\sum_{k=0}^{\infty}|(-1)^k e^{-k}| = \sum_{k=0}^{\infty} e^{-k}$, which is a convergent series as is readily apparent by the integral test: $\int_0^{\infty} e^{-x}dx = \lim_{b\to\infty}\int_0^{b} e^{-x}dx = -\lim_{b\to\infty} e^{-x}\Big|_0^b = e^0 - \lim_{b\to\infty} e^{-b} = 1 < \infty$

Hence $\sum_{k=0}^{\infty}(-1)^k e^{-k}$ is absolutely convergent.

- Example (§10.5, # 21): $\sum_{k=1}^{\infty}(-1)^{k+1} \operatorname{csc} hk$

To determine convergence, first apply **AbST:**

$$\sum_{k=1}^{\infty}|a_k| = \sum_{k=1}^{\infty}\frac{2}{e^k - e^{-k}}$$

It turns out that this above series converges. This can be established in (at least) two ways:

Method 1 (Integral Test)

$$\int_1^{\infty} \frac{2}{e^x - e^{-x}}\,dx = 2\lim_{b\to\infty}\int_1^b \frac{dx}{e^x - e^{-x}} = 2\lim_{b\to\infty}\int_1^b \frac{dx}{e^x\left(1-e^{-2x}\right)} = 2\lim_{b\to\infty}\int_1^b \frac{e^{-x}dx}{\left(1-e^{-2x}\right)}$$

Hence performing the *u*-substitution: $u = e^{-x} \Rightarrow du = -e^{-x}dx$:

$$2\lim_{b\to\infty}\int_{u(1)}^{u(b)} \frac{-du}{\left(1-u^2\right)} = 2\lim_{b\to\infty}\int_{u(b)=e^{-b}}^{u(1)=e^{-1}} \frac{du}{1-u^2} = 2\lim_{c\to 0}\int_c^{e^{-1}} \frac{du}{1-u^2}$$

This above integral can either be evaluated by partial fractions or via a trig-substitution. Opting for the latter:

$$u = \sin\theta \Rightarrow du = \cos\theta d\theta \Rightarrow \theta = \arcsin(u)$$

$$2\lim_{c\to 0}\int_c^{e^{-1}} \frac{du}{1-u^2} = 2\lim_{\beta\to\arcsin 0=0}\int_{\beta}^{\arcsin(e^{-2})} \frac{\cos\theta d\theta}{1-\sin^2\theta} = 2\lim_{\beta\to 0}\int_{\beta}^{\arcsin e^{-1}} \sec\theta d\theta$$

$$= 2\lim_{\beta\to 0}\ln\left|\sec\theta + \tan\theta\right|\Big|_{\beta}^{\arcsin e^{-1}} = 2\left\{\ln\left|\sec\left(\arcsin e^{-1}\right) + \tan\left(\arcsin e^{-1}\right)\right| - \lim_{\beta\to 0}\ln\left|\sec\beta + \tan\beta\right|\right\}$$

$$= 2\left\{\ln\left|\frac{e}{\sqrt{e^2-1}} + \frac{1}{\sqrt{e^2-1}}\right| - \ln\left|\sec 0 + \tan 0\right|\right\} = 2\ln\left|\frac{e-1}{\sqrt{e^2-1}}\right| = 2\ln\sqrt{\frac{e-1}{e+1`}} = \ln\left(\frac{e-1}{e+1}\right) < \infty$$

(Note how the expressions: $\sec\left(\arcsin e^{-1}\right), \tan\left(\arcsin e^{-1}\right)$ were arrived at. Since

$$\theta = \arcsin u \Rightarrow \sin\left(\arcsin e^{-1}\right) = e^{-1} = \frac{1}{e} = \frac{OPP}{HYP} \Rightarrow ADJ = \sqrt{HYP^2 - OPP^2} = \sqrt{e^2-1}$$

$$\therefore \sec\left(\arcsin e^{-1}\right) = \frac{HYP}{ADJ} = \frac{e}{\sqrt{e^2-1}}, \tan\left(\arcsin e^{-1}\right) = \frac{OPP}{ADJ} = \frac{1}{\sqrt{e^2-1}}$$

This method is also quite cumbersome! One could adopt the SCT as well:

Method 2 (SCT)

$e^k - e^{-k} > e^k - 1 > e^{k-1} \Rightarrow \dfrac{1}{e^k - e^{-k}} < \dfrac{1}{e^k - 1} < e^{1-k} = \dfrac{e}{e^k}$, for all $k \geq 1$.

The series: $\sum_{k=1}^{\infty} \frac{e}{e^k} = e\sum_{k=1}^{\infty} e^{-k}$ converges, as is easily demonstrated via the integral test (see Example (§10.5, # 11) above).

Hence, as established by both methods, $\sum_{k=1}^{\infty} (-1)^{k+1} \operatorname{csc} hk$ is absolutel convergent.

- Example (§10.5, # 29):

$\sum_{k=1}^{\infty} \frac{(-1)^k}{\sqrt{k}}$. Examining for absolute convergence: $\sum_{k=1}^{\infty} \left| \frac{(-1)^k}{\sqrt{k}} \right| = \sum_{k=1}^{\infty} \frac{1}{k^{1/2}}$

We recognize this to be a divergent p-series (since $p = {}^1/_2 < 1$). However, this doesn't necessarily mean that $\sum_{k=1}^{\infty} \frac{(-1)^k}{\sqrt{k}}$ diverges! To test for the possibility of conditional convergence, adopt the **AST:**

a.) Monotone decreasing? $a_{k+1} \le a_k \Rightarrow \frac{a_{k+1}}{a_k} \le 1$

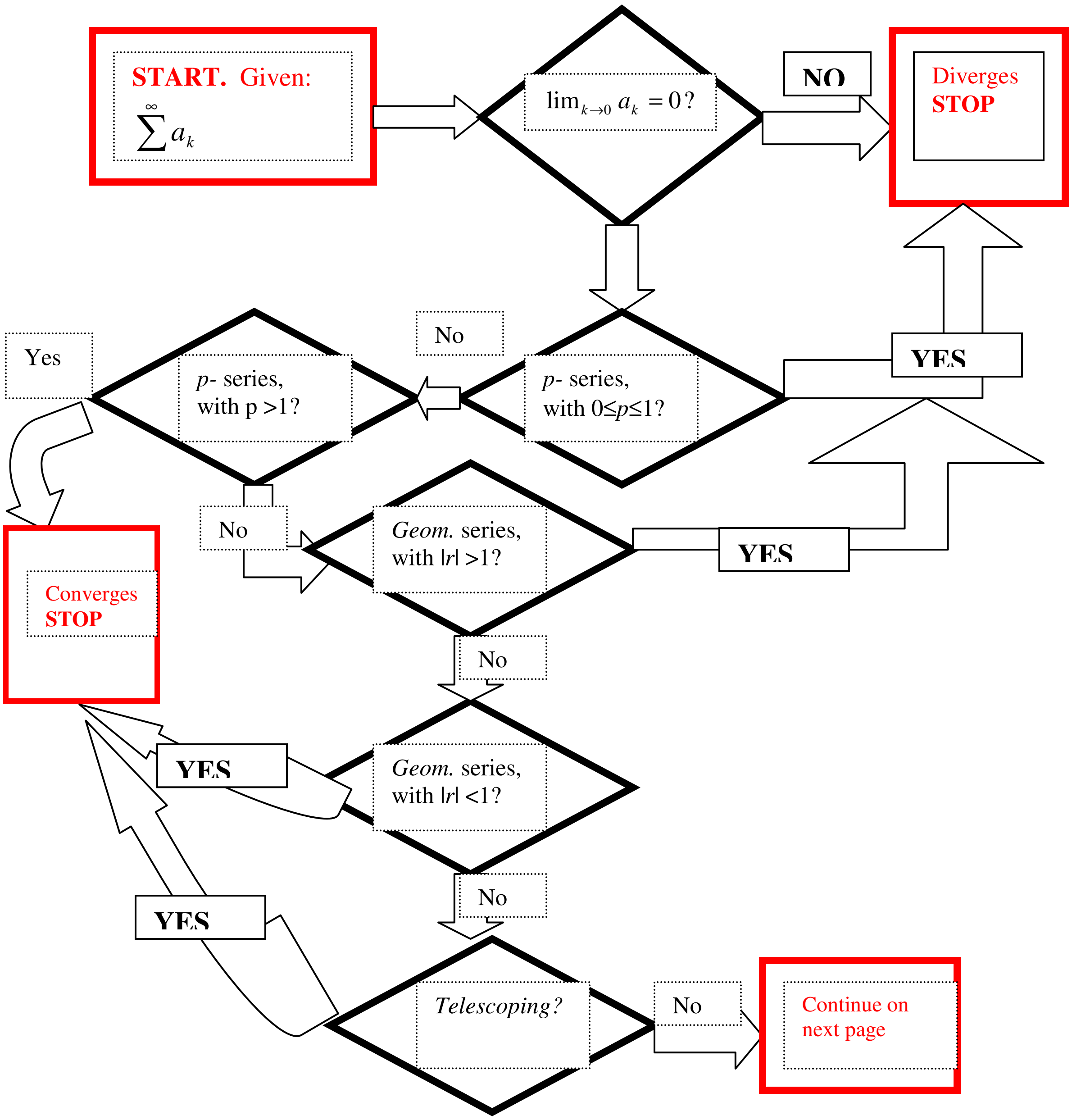
Observe: $\frac{a_{k+1}}{a_k} = \frac{(k+1)^{-1/2}}{k^{-1/2}} = \sqrt{\frac{k}{k+1}} = \sqrt{\frac{1}{1+\frac{1}{k}}} < 1$, for all $k \ge 1$.

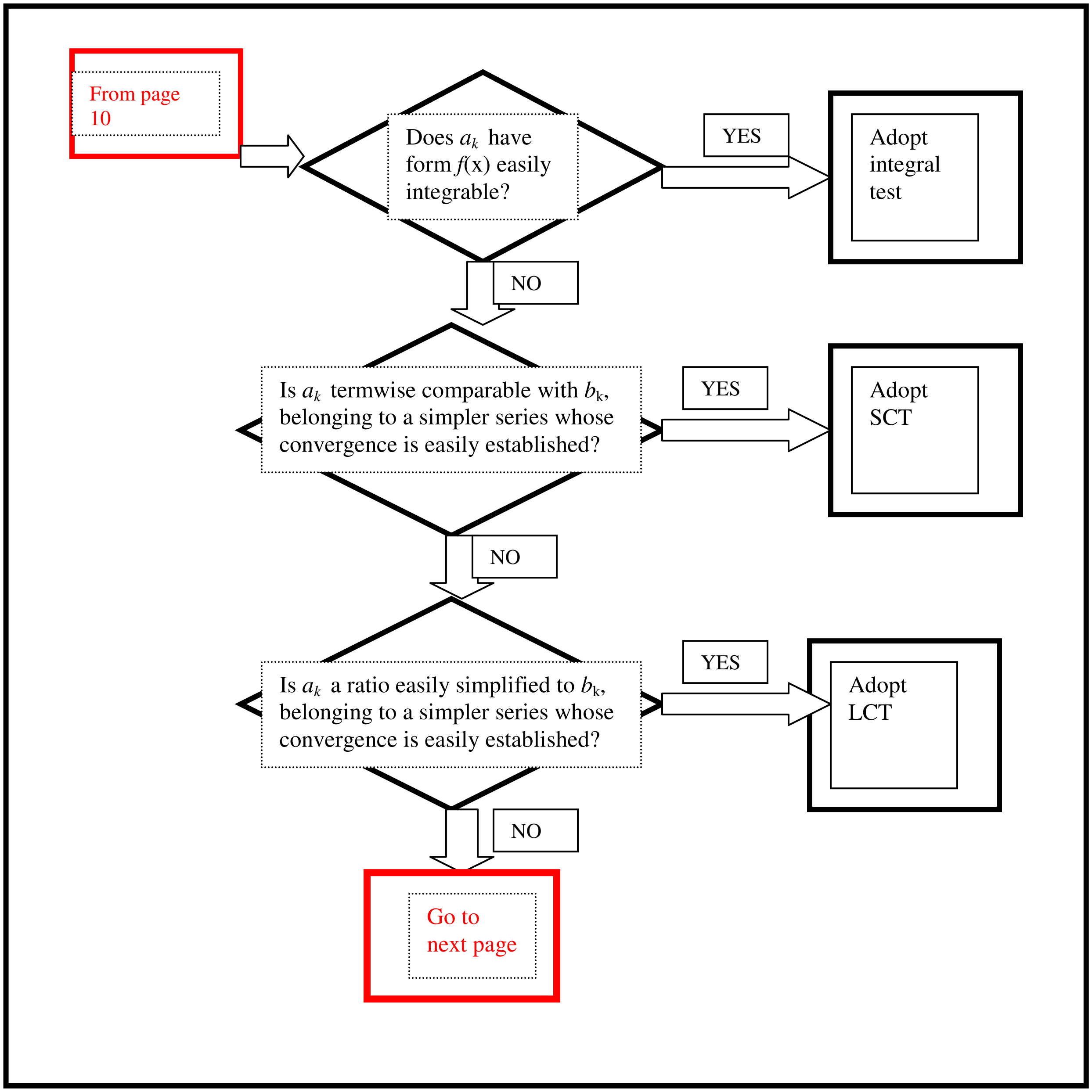
- **Note 7:** One could adopt a more rigorous procedure for testing monotonicty, which in certain cases is actually labor-saving. Adopt the functional form comparison and take its derivative: $a_k = \frac{1}{\sqrt{k}} \Leftrightarrow f(x) = x^{-1/2} \Rightarrow f'(x) = -\frac{1}{2} x^{-3/2} < 0$ for all $x \ge 1$.

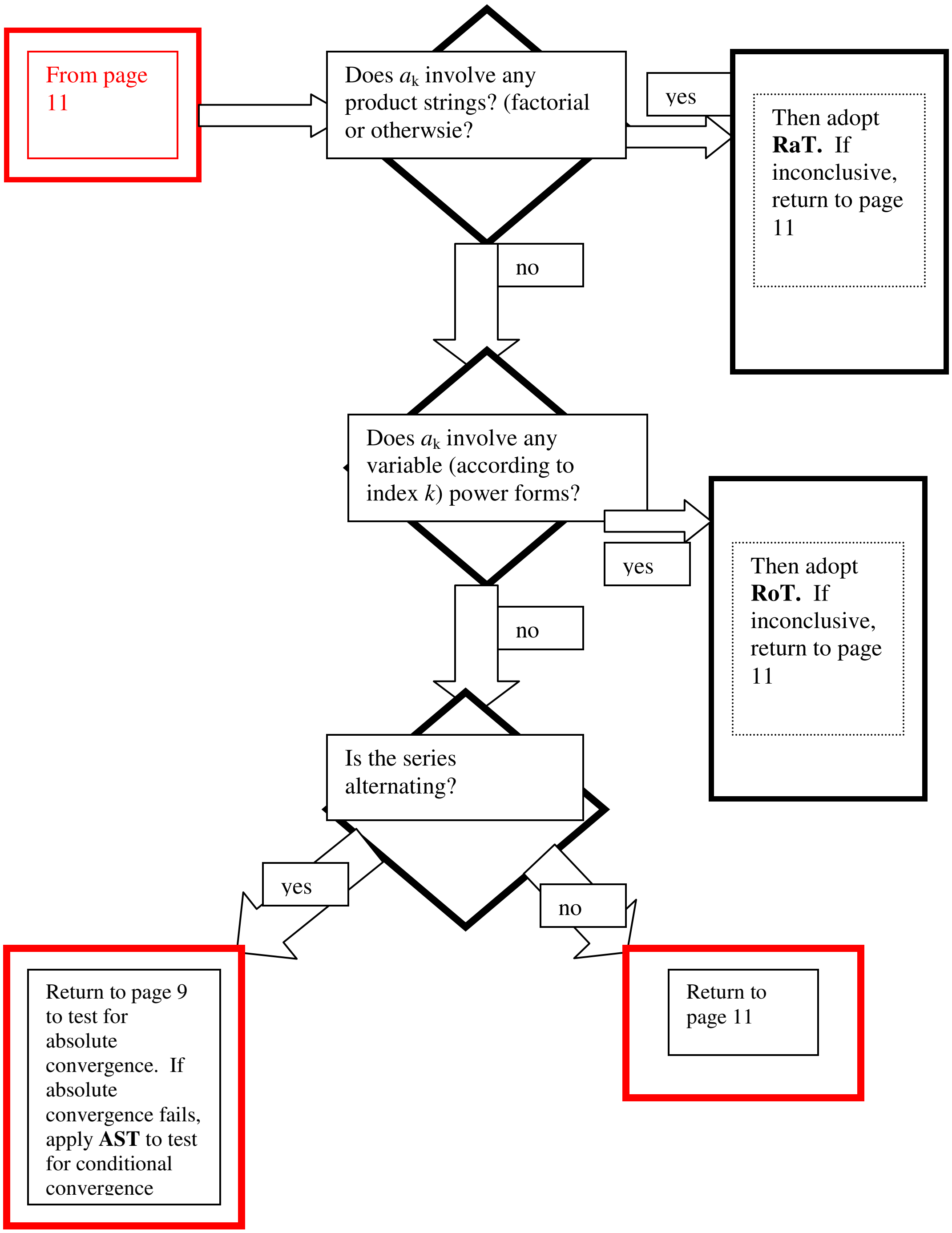
b.) $\lim_{k\to\infty} a_k = \lim_{k\to\infty} \frac{1}{\sqrt{k}} = 0$

Hence $\sum_{k=1}^{\infty} \frac{(-1)^k}{\sqrt{k}}$ converges by the AST but diverges by AbST. So the alternating series conditionally converges.

Everything we've covered so far suggests the following flow chart (Recall Table 10.2, p. 604) in terms of what to do when:


START. Given: $\sum^{\infty} a_k$
$\lim_{k\to 0} a_k = 0$?
NO
Diverges STOP
YES
p- series, with 0≤p≤1?
No
p- series, with p >1?
Yes
Converges STOP
No
Geom. series, with |r| >1?
YES
No
Geom. series, with |r| <1?
YES
No
Telescoping?
YES
No
Continue on next page


From page 10
Does a_k have form $f(x)$ easily integrable?
YES
Adopt integral test
NO
Is a_k termwise comparable with b_k, belonging to a simpler series whose convergence is easily established?
YES
Adopt SCT
NO
Is a_k a ratio easily simplified to b_k, belonging to a simpler series whose convergence is easily established?
YES
Adopt LCT
NO
Go to next page



From page 11
Does a_k involve any product strings? (factorial or otherwsie?
yes
Then adopt **RaT.** If inconclusive, return to page 11
no
Does a_k involve any variable (according to index k) power forms?
yes
Then adopt **RoT.** If inconclusive, return to page 11
no
Is the series alternating?
yes
no
Return to page 9 to test for absolute convergence. If absolute convergence fails, apply **AST** to test for conditional convergence
Return to page 11