MA262 **Calculus II: Notes for Feb 21**
Spring 2008 Kallfelz

- *SEQUENCES*

Recall from (Feb 14) notes:

1. **Thm10.1** (The functional equivalence thm). Given $a_n = f(n)$ and $\lim_{x\to\infty} f(x) = L$, then: $\lim_{n\to\infty} a_n = L$

2. **Tm 10.2** (Sum, product, difference, quotient). If $a_n \to L, b_n \to M$ then:
   i.) $(ca_n \pm db_n) \to cL \pm dM$ (for any constants c , d)
   ii.) $(a_n b_n) \to LM$
   iii.) $\frac{a_n}{b_n} \to \frac{L}{M}$ (provided $M \neq 0$)

3. **Thm10.3** (Sandwich Thm). Given sequences $\{b_n\}$, $\{c_n\}$ such that for all n: $c_n \le a_n \le b_n$, and $c_n \to L, b_n \to L$, then $a_n \to L$

4. **Thm10.4** (Abs value thm). If $|a_n| \to 0$ then $a_n \to 0$ (Note how this follows straight from Thm 10.3, by property of the absolute values: $-|a_n| \le a_n \le |a_n|$)

5. **Thm 10.5** (Monotone thm). If a sequence is monotone increasing (decreasing) and bounded above (below), then it converges.

…With the additional definitions

$\lim_{n\to\infty} a_n = L$, which one can abbreviate: $a_n \to L$. Precisely, this means (recall the above definition of limit):

$\lim_{n\to\infty} a_n = L$ means:

For any arbitrarily small ε such that $0 < |a_n - L| < \varepsilon$ there exists an arbitrarily large N such that $0 < |a_n - L| < \varepsilon$ for all $n > N$.

- **Defn. 1** A sequence is **bonded above (or below)** if there exists some number M such that: $a_n \le M$ (or $M \le a_n$) for all n. Note: A sequence can be bounded and still not converge! Consider: $a_n = (-1)^n$. Such a sequence is bounded above and below by +1, -1, but doesn't converge—it just oscillates between those values forever.

- **Defn.** 2: A sequence is **monotone increasing (or decreasing)** if for all *k*: $a_{k+1} \geq a_k$

  (or $a_{k+1} \leq a_k$), which obviously can be also expressed as: $\frac{a_{k+1}}{a_k} \geq 1$ or $\frac{a_{k+1}}{a_k} \leq 1$.

  Note further that if the inequalities are *strict*, then we say that the sequences are **strictly monotone (increasing or decreasing).**

Thm 10.1 will prove itself sufficient for the establishing of convergence/divergence for relatively simple cases:

- Example (# 46, §10.1)

$a_n = n\sin\left(\tfrac{1}{n}\right)$

At first blush it may look like the sequence would diverge: *n* acting as a 'coefficient' increasing in an unbounded manner, as well as the oscillating sinusoidal term. However:

$$a_n = n\sin\left(\tfrac{1}{n}\right) = \frac{\sin\left(\frac{1}{n}\right)}{\frac{1}{n}} \xrightarrow{Thm 10.1} \lim_{x\to\infty} f(x) = \lim_{x\to\infty}\frac{\sin\left(\frac{1}{x}\right)}{\frac{1}{x}} = \lim_{u\to 0}\frac{\sin u}{u} = 1 \Rightarrow a_n \to 1$$

The limit $\lim_{u\to 0}\frac{\sin u}{u} = 1$ can be established rigorously by L'Hopital's Rule, as Cory Heiges pointed out today in class. However it is also true (if you recall from your Calculus I when the subject of differentiating sinusoidal functions was first introduced) that it can be established by more basic means as well.[1]

- Example (# 50, §10.1)

$$a_n = ne^{-n/2} = \frac{n}{e^{n/2}} \xrightarrow{Thm 10.1} \lim_{x\to\infty}\frac{x}{e^{x/2}} \xrightarrow{LHR} \lim_{x\to\infty}\frac{1}{\frac{1}{2}e^{x/2}} = 0 \Rightarrow a_n \to 0$$

To examine its monotonicity:

$\frac{a_{n+1}}{a_n} = \frac{(n+1)e^{-(n+1)/2}}{ne^{-n/2}} = \left(1+\tfrac{1}{n}\right)e^{-1/2} = \frac{\left(1+\frac{1}{n}\right)}{\sqrt{e}} < 1$, for all $n \geq 1$. Hence it's monotone decreasing.

- Example

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[1] See pp. 5-6 of http://www.glue.umd.edu/%7Ewkallfel/MA261-2/trig.pdf

$$a_n = 2\ln 3n - \ln\left(n^2+1\right) = \ln\left(9n^2\right) - \ln\left(n^2+1\right) = \ln\left(\frac{9n^2}{n^2+1}\right)$$

$$\xrightarrow{Thm10.1} \lim_{x\to\infty} \ln\left(\frac{9x^2}{x^2+1}\right) = \ln\left(\lim_{x\to\infty} \frac{9}{1+\frac{1}{x^2}}\right) = \ln 9 \Rightarrow a_n \to \ln 9$$

- <u>Example (# 40, §10.1)</u>

$$a_n = \frac{n^2}{2n+1} - \frac{n^2}{2n-1} = n^2\left(\frac{2n-1-2n-1}{4n^2-1}\right) = \frac{-2n^2}{4n^2-1} \xrightarrow{Thm10.1} \lim_{x\to\infty} \frac{-2x^2}{4x^2-1}$$

$$= \lim_{x\to\infty} \frac{1}{\frac{1}{2x^2}+2} = \tfrac{1}{2} \Rightarrow a_n = \tfrac{1}{2}$$

- *SERIES*

Recall some of special series formulae:

Arithmetic (finite): $$\sum_{k=0}^{n-1}(a+kd) = \sum_{j=1}^{n}\left(a+(j-1)d\right) = \frac{n\left(a_1+a_n\right)}{2}$$

…where a_1 and a_n are the first and last terms of the arithmetic series:

$$a_1 = a \qquad\qquad a_n = a + (n-1)d$$

Geometric (finite): $$\sum_{k=0}^{n-1} ar^k = \sum_{j=1}^{n} ar^{j-1} = a\frac{1-r^n}{1-r}$$

Geometric (infinite):

$$\sum_{k=0}^{\infty} ar^k = \lim_{n\to\infty} \sum_{k=0}^{n-1} ar^k = a\lim_{n\to\infty} \frac{1-r^n}{1-r} = \frac{a}{1-r},\ for\left|r\right| < 1$$

Telescoping: $$\sum_{k=1}^{n}\left(a_k - a_{k-1}\right) = \left(a_n - a_{n-1}\right) + \left(a_{n-1} - a_{n-2}\right) + \ldots + \left(a_2 - a_1\right) = a_n - a_1$$

In addition, §10.2 offered the following analogue to Thm10.2:

$$\sum_{k=1}^{n}\left(\alpha a_k + \beta b_k\right) = \alpha\sum_{k=1}^{n} a_k + \beta\sum_{k=1}^{n} b_k \qquad \text{(for any real numbers } \alpha, \beta\text{)}$$

…a property that *always* holds, due to commutativity and distributivity:

$$\begin{aligned}\sum_{k=1}^{n}(\alpha a_k+\beta b_k)&=(\alpha a_1+\beta b_1)+(\alpha a_2+\beta b_2)+\ldots+(\alpha a_n+\beta b_n)\\&=(\alpha a_1+\alpha a_2+\ldots+\alpha a_n)+(\beta b_1+\beta b_2+\ldots+\beta b_n)\\&=\alpha(a_1+a_2+\ldots+a_n)+\beta(b_1+b_2+\ldots+b_n)\\&=\alpha\sum_{k=1}^{n}a_k+\beta\sum_{k=1}^{n}b_k\end{aligned}$$

From which it immediately follows that in the convergent infinite case:

$$\sum_{k=1}^{\infty}a_k=L,\sum_{k=1}^{\infty}b_k=M\Rightarrow\sum_{k=1}^{\infty}(\alpha a_1+\beta b_k)=\alpha\sum_{k=1}^{\infty}a_k+\beta\sum_{k=1}^{\infty}b_k=\alpha L+\beta M$$

Also a rather weak convergence theorem:

$$\textit{If}\ \sum_{k=1}^{\infty}a_n<\infty,\ \textit{then}\ \lim_{n\to\infty}a_n=0$$

The reason why the above is rather weak as a test for convergence is simply because convergence is presupposed! (It's a sufficiency condition). The converse is false! (I.e just because $\lim_{n\to\infty}a_n=0$ does *not* guarantee $\sum_{k=1}^{\infty}a_n<\infty$.[2] However, when expressed in *contrapositive*[3] form: " *If* $\lim_{n\to\infty}a_n\neq 0$, *then* $\sum_{k=1}^{\infty}a_n\to\infty$", the theorem becomes a useful test for *divergence*. (Since divergence is a necessary condition, or a consequence, not a presupposition).

- <u>Example (# 34, §10.2)</u>

$$\sum_{n=5}^{\infty}2\left(-\tfrac{3}{4}\right)^n=2\sum_{k=1}^{\infty}\left(-\tfrac{3}{4}\right)^{k+4}=2\sum_{k=1}^{\infty}\left(-\tfrac{3}{4}\right)^{k-1}\left(-\tfrac{3}{4}\right)^5=-2\left(\tfrac{3}{4}\right)^5\sum_{k=1}^{\infty}\left(-\tfrac{3}{4}\right)^{k-1}$$

$$=-2\left(\tfrac{3}{4}\right)^5\left(\frac{1}{1-\left(-\tfrac{3}{4}\right)}\right)=-2\left(\tfrac{3}{4}\right)^5\left(\frac{1}{\tfrac{7}{4}}\right)=-2\left(\tfrac{3}{4}\right)^5\left(\tfrac{4}{7}\right)=\frac{-2\cdot 3^5}{4^4\cdot 7}\approx -0.271$$

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[2] For example: $\sum_{k=1}^{\infty}\frac{1}{k}$ is a divergent Harmonic series. However, $\lim_{k\to\infty}\frac{1}{k}=0$

[3] Any conditional statement: "If *p* then *q*" can be equivalently expressed as "If not-*q*, then not-*p*."

Aside from the trick of index-shifting, you could derive the geometric series expression by brute force: writing out successive terms:

$$2\sum_{n=5}^{\infty}\left(-\tfrac{3}{4}\right)^n = 2\left[\left(-\tfrac{3}{4}\right)^5 + \left(-\tfrac{3}{4}\right)^6 + \left(-\tfrac{3}{4}\right)^7 + \ldots\right] = 2\left(-\tfrac{3}{4}\right)^5\left[1 + \left(-\tfrac{3}{4}\right) + \left(-\tfrac{3}{4}\right)^2 + \ldots\right] = 2\left(-\tfrac{3}{4}\right)^5\sum_{k=0}^{\infty}\left(-\tfrac{3}{4}\right)^k$$

$$= 2\left(-\tfrac{3}{4}\right)\left[\frac{1}{1+\tfrac{3}{4}}\right] = etc\ldots$$

- Example (# 38, §10.2)

$$\sum_{k=1}^{\infty}\frac{1}{(2k+1)(2k+3)}$$

Using partial fractions:

$$\frac{1}{(2k+1)(2k+3)} = \frac{A_1}{(2k+1)} + \frac{A_2}{(2k+3)} \Rightarrow 1 = A_1(2k+3) + A_2(2k+1)$$

$$k = -\tfrac{3}{2} \rightarrow 1 = -2A_2 \rightarrow A_2 = -\tfrac{1}{2}$$
$$k = -\tfrac{1}{2} \rightarrow 1 = 2A_1 \rightarrow A_1 = \tfrac{1}{2}$$

$$\therefore \sum_{k=1}^{\infty}\frac{1}{(2k+1)(2k+3)} = \tfrac{1}{2}\left(\sum_{k=1}^{\infty}\frac{1}{2k+1} - \frac{1}{2k+3}\right)$$
$$= \tfrac{1}{2}\left\{\left(\tfrac{1}{3}-\tfrac{1}{5}\right) + \left(\tfrac{1}{5}-\tfrac{1}{7}\right) + \left(\tfrac{1}{7}-\tfrac{1}{9}\right) + \ldots\right\} = \tfrac{1}{2}\cdot\tfrac{1}{3} = \tfrac{1}{6}$$

- Example (# 47, §10.2)

$$\sum_{n=1}^{\infty}\left(\tfrac{1}{n} - \tfrac{1}{n+2}\right) = \left(1-\tfrac{1}{3}\right) + \left(\tfrac{1}{2}-\tfrac{1}{4}\right) + \left(\tfrac{1}{3}-\tfrac{1}{5}\right) + \left(\tfrac{1}{4}-\tfrac{1}{6}\right) + \ldots = 1 + \tfrac{1}{2} = \tfrac{3}{2}$$

- *THE INTEGRAL AND COMPARISON TESTS*

Analogous to Thm10.1 is the **Integral Test:**

For $\sum_{k=1}^{\infty} a_n$, if $a_n = f(n)$, then $\sum_{k=1}^{\infty} a_n$ converges/diverges depending on whether $\int_1^{\infty} f(x)dx = \lim_{b\to\infty}\int_1^{b} f(x)dx$ converges/diverges.

Recall the improper integral: $\int_1^\infty \frac{dx}{x^p} = \begin{cases} \frac{1}{p-1} & p > 1 \\ \infty & 0 \le p \le 1 \end{cases}$

Hence based on the Integral Test one may infer:

The ***p* –Series:** $\sum_{k=1}^\infty \frac{1}{k^p} = \begin{cases} converges & (p > 1) \\ diverges & 0 \le p \le 1 \end{cases}$

- <u>Example (# 47, §10.3)</u>

$$\sum_{n=2}^\infty \frac{1}{n\sqrt{n^2-1}} \xrightarrow{I.T.} \int_2^\infty \frac{dx}{x\sqrt{x^2-1}} \Rightarrow x = \sec\theta, dx = \sec\theta\tan\theta d\theta$$

$$\therefore \lim_{b\to\infty} \int_2^b \frac{dx}{x\sqrt{x^2-1}} = \lim_{\beta\to arc\sec(\infty)=\pi/2} \int_{arc\sec(2)=\pi/3}^{\beta} \frac{\sec\theta\tan\theta}{\sec\theta\sqrt{\sec^2\theta-1}} d\theta$$

$$= \lim_{\beta\to\pi/2} \int_{\pi/3}^{\beta} \frac{\sec\theta\tan\theta}{\sec\theta\tan\theta} d\theta = \lim_{\beta\to\pi/2} \theta\Big|_{\pi/2}^{\beta} = \tfrac{\pi}{2} - \tfrac{\pi}{3} = \tfrac{\pi}{6} \Rightarrow \sum_{n=2}^\infty \frac{1}{n\sqrt{n^2-1}} < \infty$$

- <u>Example (# 28, §10.3)</u>

$\sum_{n=1}^\infty \left(\frac{1}{n^2} - \frac{1}{n^3}\right) = \sum_{n=1}^\infty \frac{1}{n^2} - \sum_{n=1}^\infty \frac{1}{n^3} < \infty$, since this is the difference between two *p*-series (with $p = 2$, $p = 3$), hence the difference of two finite values (since both these series converge).[4]

- <u>Example (# 29, §10.3)</u>

$\sum_{n=1}^\infty \left(1 + \frac{1}{n}\right)^n$. Note that:

$$a_n = \left(1 + \tfrac{1}{n}\right)^n \Rightarrow \lim_{n\to\infty} a_n = \lim_{n\to\infty} \left(1 + \tfrac{1}{n}\right)^n = e \ne 0 \Rightarrow \sum_{n=1}^\infty \left(1 + \tfrac{1}{n}\right)^n \to \infty$$

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[4] More formally, use is made of the Theorem in §10.2:

$$\sum_{k=1}^\infty a_k = L, \sum_{k=1}^\infty b_k = M \Rightarrow \sum_{k=1}^\infty (\alpha a_1 + \beta b_k) = \alpha \sum_{k=1}^\infty a_k + \beta \sum_{k=1}^\infty b_k = \alpha L + \beta M$$

(I.e. using the *n*-th term test for divergence mentioned above: *If* $\lim_{n\to\infty} a_n \neq 0$, *then* $\sum_{k=1}^{\infty} a_n \to \infty$.)

- Example (# 34, §10.3)

Determine the domain for the Riemann Zeta-function: $\zeta(x) = \sum_{n=1}^{\infty} n^{-x} = \sum_{n=1}^{\infty} \frac{1}{n^x}$

Basically, one should immediately consider the *p*-series. The above function is well-defined only if the sum converges. One recognizes immediately its form equivalent to the *p*-series.

Hence: $$Dom\,\zeta(x) = \left\{ x \mid \zeta(x) = \sum_{n=1}^{\infty} \frac{1}{n^x} < \infty \right\} = \{x \mid x > 1\} = (1, \infty)$$

Recall the Sandwich Thm for Sequences: **Thm10.3** (Sandwich Thm). Given sequences $\{b_n\}$, $\{c_n\}$ such that for all *n*: $c_n \le a_n \le b_n$, and $c_n \to L, b_n \to L$, then $a_n \to L$. The analog for Thm10.3 for the case of series is found in the following **comparison tests:**

- **Simple Comparison Test (SCT)**

Given $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$. (i) If $a_n \le b_n$ for *all n* in the sums, and if $\sum_{n=1}^{\infty} b_n < \infty$, then: $\sum_{n=1}^{\infty} a_n < \infty$. (ii) Conversely, if $b_n \le a_n$ for *all n* in the sums, and if $\sum_{n=1}^{\infty} b_n \to \infty$, then: $\sum_{n=1}^{\infty} a_n \to \infty$

- **Limit Comparison Test (LCT)**

Given $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$, with $a_n > 0, b_n > 0$ for *all* n in the sum. Then if $0 < L < \infty$, where[5]: $L = \lim_{n\to\infty}\left(\frac{b_n}{a_n}\right)$, then the convergence or divergence of $\sum_{n=1}^{\infty} a_n$ depends on the convergence/divergence of $\sum_{n=1}^{\infty} b_n$ (and obviously vice versa as well).[6]

In many ways, the **SCT** may seem more attractive than the **LCT**, in terms of its relative simplicity. However, note that the **LCT** imposes a rather strong condition: one must show that $a_n \le b_n$ (or $b_n \le a_n$) for *all* n in the sums. For complicated cases, this can prove difficult (if not impossible to show). While the **LCT** one needs only to construct the above limit, a rather weak (and therefore applicable in many cases) condition. When adopting the **LCT** or the **SCT** keep the following strategy in mind:

> To construct the 'simpler' series Σb_n from Σa_n (where the series Σb_n has convergence/divergence which is a straightforward matter to achieve, whether by p-series, geometric, or n-th term, or integral test), it is most easily accomplished by preserving *only* the leading terms (terms of highest power) in the rational expression of a_n in Σa_n .

- Example (# 12, §10.4)

$\sum_{n=1}^{\infty} \frac{4^n}{3^n - 1}$ Obviously trying to use the Integral test is not such a good idea, (to say the least) since the integral $\int \frac{3^x}{4^x - 1} dx$ is very messy to evaluate. So use a comparison test:

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[5] Note that the authors define: $L = \lim_{n\to\infty}\left(\frac{a_n}{b_n}\right)$, but this distinction is arbitrary (it's a 'difference that makes no difference', so to speak. As is evident, if $0 < L < \infty$, then certainly its reciprocal is also positive and finite.

[6] More formally, this is often written in terms of the "if and only if" connector. I.e., $\sum_{n=1}^{\infty} a_n$ converges/diverges *if and only if* $\sum_{n=1}^{\infty} b_n$ converges/diverges.

Adopting the above strategy, consider $\sum_{n=1}^{\infty} b_n$ where: $b_n = \frac{4^n}{3^n} = \left(\frac{4}{3}\right)^n$, which was constructed by truncating all but the leading term(s) in the denominator and in the numerator. (The latter case is obviously trivial, since there's one term in the numerator!)

Of course one recognizes that $\sum_{n=1}^{\infty} b_n$ is a diverging geometric series (why?). Moreover, note that obviously for all $n \geq 1$: $\frac{4^n}{3^n} < \frac{4^n}{3^n - 1}$. Hence by **SCT,** $\sum_{n=1}^{\infty} \frac{4^n}{3^n - 1}$ diverges.

- Example (# 11, §10.4)

$\sum_{n=0}^{\infty} e^{-n^2}$. As in the previous, the integral test is off limits since we *never* introduced techniques (thus far) to evaluate an integral of the form: $\int e^{-x^2} dx$. So use a comparison test.

Observe that $b_n = e^{-n} > e^{-n^2} = a_n$ for all $n \geq 1$. Consider: $\sum_{n=0}^{\infty} e^{-n} = \sum_{n=0}^{\infty} b_n$. This series is an obvious candidate for the integral test:

$$\sum_{n=0}^{\infty} e^{-n} \xrightarrow{I.T.} \int_0^{\infty} e^{-x} dx = \lim_{b \to \infty} \int_0^{b} e^{-x} dx = -\lim_{b \to \infty} e^{-x} \Big|_0^b$$

$$= \lim_{b \to \infty} e^{-x} \Big|_b^0 = e^0 - \lim_{b \to \infty} e^{-b} = 1 \Rightarrow \sum_{n=0}^{\infty} b_n < \infty$$

Thus $\sum_{n=0}^{\infty} e^{-n^2}$ converges by **SCT**

- Example (# 18, §10.4)

$\sum_{n=0}^{\infty} a_n = \sum_{n=0}^{\infty} \frac{5n - 3}{n^2 - 2n + 5}$. Following the above suggestion we can construct $\sum_{n=1}^{\infty} b_n$ just by keeping the leading terms of the numerator and denominator in the above, and truncating the rest: $\sum_{n=1}^{\infty} b_n = \sum_{n=0}^{\infty} \frac{5n}{n^2} = 5\sum_{n=0}^{\infty} \frac{1}{n}$, which is easily recognized as a divergent $p = 1$ series. However, to compare the a_n and b_n terms in such a manner as to guarantee $a_n \le b_n$ (or $b_n \le a_n$) for *all* n in the sums might be difficult if not impossible to show. The **LCT** should then come to mind, and note that $a_n > 0$, and $b_n > 0$ for all n, hence satisfying the constraint of the **LCT.**

Form:

$$L = \lim_{n\to\infty}\left(\frac{b_n}{a_n}\right) = \lim_{n\to\infty} \frac{n^2 - 2n + 5}{n(5n-3)} = \lim_{n\to\infty} \frac{n^2 - 2n + 5}{5n^2 - 3n} = \lim_{n\to\infty} \frac{1 - \frac{2}{n} + \frac{5}{n^2}}{5 - \frac{3}{n}} = \frac{1}{5}$$

Obviously $0 < L < \infty$. Hence since $\sum_{n=1}^{\infty} b_n$ diverges, then: $\sum_{n=0}^{\infty} a_n = \sum_{n=0}^{\infty} \frac{5n-3}{n^2 - 2n + 5}$ diverges.

- <u>Example (# 24, §10.4)</u>

$\sum_{n=1}^{\infty} \frac{1}{n + \sqrt{n^2 + 1}}$ Consider: $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n + \sqrt{n^2}} = \sum_{n=1}^{\infty} \frac{1}{2n} = \frac{1}{2}\sum_{n=1}^{\infty} \frac{1}{n}$, a divergent ($p = 1$) p-series. Form:

$$L = \lim_{n\to\infty}\left(\frac{b_n}{a_n}\right) = \lim_{n\to\infty} \frac{2n}{n + \sqrt{n^2 + 1}} = \lim_{n\to\infty} \frac{2}{1 + \sqrt{1 + \frac{1}{n^2}}} = 1$$

Obviously $0 < L < \infty$. Hence since $\sum_{n=1}^{\infty} b_n$ diverges, then: $\sum_{n=0}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{1}{n + \sqrt{n^2 + 1}}$ diverges.