MA262 **Calculus II: Notes for Feb 14**
Spring 2008 Kallfelz

- ***REVIEW EXERCISES-EXAM I***

I, a.) Use integration by parts to evaluate:

$\int e^{-st}t^2dt$ (where: s is a constant)

u	dv
t^2 (+)	e^{-st}
$2t$ (-)	$-\frac{1}{s}e^{-st}$
2 (+)	$\frac{1}{s^2}e^{-st}$
0	$-\frac{1}{s^3}e^{-st}$

Hence: $$\int e^{-st}t^2dt = -\tfrac{1}{s}e^{-st}t^2 - \tfrac{2}{s^2}te^{-st} - \tfrac{2}{s^3}e^{-st} + C = -\tfrac{1}{s}e^{-st}\left[t^2 + \tfrac{2}{s}t + \tfrac{2}{s^2}\right] + C$$

b.) Use your answer in a.) to find:

$$L\{t^2\} = \int_0^\infty e^{-st}t^2dt = \lim_{b\to\infty}\int_0^b e^{-st}t^2dt = \lim_{b\to\infty} -\tfrac{1}{s}e^{-st}\left(t^2 + \tfrac{2}{s}t + \tfrac{2}{s^2}\right)\Big|_0^b$$

$$-\lim_{b\to\infty}\tfrac{1}{s}e^{-sb}\left(b^2 + \tfrac{2}{s}b + \tfrac{2}{s^2}\right) + \tfrac{1}{s}\left(\tfrac{2}{s^2}\right) = -\tfrac{1}{s}\lim_{b\to\infty}\frac{b^2 + \frac{2}{s}b + \frac{2}{s^2}}{e^{sb}} + \tfrac{2}{s^3}$$

$$= -\tfrac{1}{s}\lim_{b\to\infty}\frac{2b + \frac{2}{s}}{se^{sb}} + \tfrac{2}{s^3} = -\tfrac{1}{s}\lim_{b\to\infty}\frac{2}{s^2e^{sb}} + \tfrac{2}{s^3} = 0 + \tfrac{2}{s^3} = 2s^{-3}$$

(Where L'Hopital's Rule was repeatedly invoked in the first expression)

Note: It can be shown in that the 'power rule' for Laplace Transforms is:

$L\{t^n\} = \dfrac{\Gamma(n+1)}{s^{n+1}}$ **(where Γ is the Gamma Function $\Gamma(n+1) = n\,\Gamma(n)$ (details suppressed here, for more information, see: http://www.wam.umd.edu/~Ewkallfel/MA360/**

II.) Use a rationalizing substitution and integration by parts to evaluate:

$$\int \sin\sqrt{x}dx$$

$$u=\sqrt{x}\Rightarrow u^2=x\Rightarrow dx=2udu$$
$$\therefore \int \sin\sqrt{x}dx=2\int \sin uudu \Rightarrow dV=\sin udu, V=-\cos u, U=u, dU=du$$
$$\Rightarrow 2\int \sin udu=2\left\{-u\cos u+\int \cos udu\right\}=-2u\cos u+2\sin u+C=-2\sqrt{x}\cos\sqrt{x}+2\sin\sqrt{x}+C$$

III.a) Evaluate: $\int_{\pi/4}^{\pi/2}\csc^3 x\cot xdx$

$$\int_{\pi/4}^{\pi/2}\csc^3 x\cot xdx=\int_{\pi/4}^{\pi/2}\csc^2 x(\csc x\cot xdx)=-\int_{u_1=\csc(\pi/4)}^{u_2=\csc(\pi/2)}u^2du=\int_1^{\sqrt{2}}u^2du=\tfrac{1}{3}u^3\Big|_1^{\sqrt{2}}=\tfrac{1}{3}\left(2\sqrt{2}-1\right)$$

b.) $\int \sin^4 xdx$

$$\int \sin^4 xdx=\int \sin^2 x\sin^2 xdx=\tfrac{1}{4}\int(1-\cos 2x)(1-\cos 2x)dx=\tfrac{1}{4}\int\left(1-2\cos 2x+\cos^2 2x\right)dx$$
$$=\tfrac{1}{4}\int dx-\tfrac{1}{2}\int\cos 2x+\tfrac{1}{4}\int\cos^2 2xdx=\tfrac{1}{4}x-\tfrac{1}{4}\sin 2x+\tfrac{1}{8}\int(1+\cos 4x)dx=\tfrac{1}{4}x-\tfrac{1}{4}\sin 2x+\tfrac{1}{8}x+\tfrac{1}{8}\int\cos 4xdx$$
$$=\tfrac{1}{4}x-\tfrac{1}{4}\sin 2x+\tfrac{1}{8}x+\tfrac{1}{32}\sin 4x+C=\tfrac{3}{8}x-\tfrac{1}{4}\sin 2x+\tfrac{1}{32}\sin 4x+C$$

IV.) Evaluate using a trigonometric substitution

a.) $\int_0^5 x^2\sqrt{25-x^2}dx \qquad x=5\sin\theta\Rightarrow dx=5\cos\theta d\theta$

$$\therefore \int_0^5 x^2\sqrt{25-x^2}dx=\int_{\arcsin(0/5)}^{\arcsin(5/5)}5^2\sin^2\theta\cdot 5\cos\theta\cdot 5\cos\theta d\theta=5^4\int_0^{\pi/2}\sin^2\theta\cos^2\theta d\theta$$
$$=\tfrac{5^2}{4}\int_0^{\pi/2}(1-\cos 2\theta)(1+\cos 2\theta)d\theta=\tfrac{5^4}{4}\int\left(1-\cos^2 2\theta\right)d\theta=\tfrac{5^4}{4}\int\left(1-\tfrac{1}{2}(1+\cos 4\theta)\right)d\theta$$
$$=\tfrac{5^4}{8}\int(1-\cos 4\theta)d\theta=\tfrac{5^4}{4}\theta-\tfrac{5^4}{32}\sin 4\theta+C$$

b.) $\int \frac{dx}{\sqrt{x^2-2x-3}}$

$\therefore \int \frac{dx}{\sqrt{x^2-2x-3}} = \int \frac{dx}{\sqrt{x^2-2x+1-4}} = \int \frac{dx}{\sqrt{(x-1)^2-2^2}} = \int \frac{du}{\sqrt{u^2-2^2}}$

$(u = x-1) \Rightarrow u = 2\sec\theta \to du = 2\sec\theta\tan\theta d\theta$

$\int \frac{du}{\sqrt{u^2-2^2}} = \int \frac{2\sec\theta\tan\theta}{\sqrt{4\tan^2\theta}} d\theta = \int \sec\theta d\theta = \ln|\sec\theta + \tan\theta| + C$

$x-1 = 2\sec\theta \Rightarrow \sec\theta = \frac{x-1}{2} = \frac{HYP}{ADJ} \Rightarrow \tan\theta = \frac{OPP}{ADJ} = \frac{\sqrt{(x-1)^2-4}}{2}$

$= \frac{1}{2}\sqrt{x^2-2x-3}$

$\therefore \ln|\sec\theta+\tan\theta| + C = \ln\left|\frac{1}{2}\left(x-1+\sqrt{x^2-2x-3}\right)\right| + C = \ln\left|x-1+\sqrt{x^2-2x-3}\right| - \ln 2 + C$

$= \ln\left|x-1+\sqrt{x^2-2x-3}\right| + C$

V.) Evaluate Using partial fractions

$\int \frac{\sec^2\theta}{\tan^3\theta - \tan^2\theta} d\theta = \int \frac{\sec^2\theta d\theta}{\tan^2\theta(\tan\theta-1)} = \int \frac{du}{u^2(u-1)}$

$\frac{1}{u^2(u-1)} = \frac{A_1}{u} + \frac{A_2}{u^2} + \frac{A_3}{(u-1)} \Rightarrow 1 = A_1 u(u-1) + A_2(u-1) + A_3 u^2$

$u = 0 \Rightarrow -A_2 = 1 \to A_2 = -1$

$u = 1 \Rightarrow A_3 = 1$

$\therefore 1 = A_1(u^2-u) - (u-1) + u^2 = u^2(A_1+1) - u(A_1+1) + 1 \Rightarrow A_1 = -1$

$\therefore \int \frac{du}{u^2(u-1)} = \int \left[-\frac{1}{u} - \frac{1}{u^2} + \frac{1}{u-1}\right] du = -\ln|u| + \frac{1}{u} + \ln|u-1| + C = \ln\left|\frac{u-1}{u}\right| + \frac{1}{u}$

$= \ln\left|\frac{\tan\theta-1}{\tan\theta}\right| + \frac{1}{\tan\theta} + C = \ln|1-\cot\theta| + \cot\theta + C$

BONUS

Given: $\int \frac{du}{a^2-u^2}$

a.) Use Trig substitution

$$u = a\sin\theta \Rightarrow \int \frac{du}{a^2-u^2} = \int \frac{a\cos\theta d\theta}{a^2\cos^2\theta} = \tfrac{1}{a}\int \sec\theta d\theta = \tfrac{1}{a}\ln|\sec\theta+\tan\theta| + C$$

$$\sin\theta = \frac{u}{a} = \frac{OPP}{HYP} \Rightarrow \sec\theta = \frac{HYP}{ADJ} = \frac{a}{\sqrt{a^2-u^2}}, \tan\theta = \frac{OPP}{ADJ} = \frac{u}{\sqrt{a^2-u^2}}$$

$$\therefore \tfrac{1}{a}\ln|\sec\theta+\tan\theta| + C = \tfrac{1}{a}\ln\left|\frac{u+a}{\sqrt{a^2-u^2}}\right| + C = \tfrac{1}{a}\ln\left|\frac{u+a}{\sqrt{(a+u)(a-u)}}\right| + C$$

$$= \tfrac{1}{a}\ln\sqrt{\frac{a+u}{a-u}} + C = \tfrac{1}{2a}\ln\left|\frac{a+u}{a-u}\right| + C = \tfrac{1}{2a}\left(\ln|a+u| - \ln|a-u|\right) + C$$

b.) Check your answer using partial fractions

$$\int \frac{du}{a^2-u^2} = \int \frac{1}{(a-u)(a+u)}du \Rightarrow \frac{1}{(a-u)(a+u)} = \frac{A_1}{(a-u)} + \frac{A_2}{(a+u)} \Rightarrow 1 = A_1(a+u) + A_2(a-u)$$

$$u = -a \rightarrow 1 = -2aA_2 \rightarrow A_2 = -\tfrac{1}{2a}$$

$$u = a \rightarrow 1 = 2aA_1 \rightarrow A_1 = \tfrac{1}{2a}$$

$$\therefore \int \frac{du}{a^2-u^2} = \tfrac{1}{2a}\int\left(\frac{1}{a-u} - \frac{1}{a+u}\right)du = -\tfrac{1}{2a}\ln|a-u| + \tfrac{1}{2a}\ln|a+u| + C$$

- *SEQUENCES*

A *sequence* is simply a function $f: N \rightarrow R$ (where N are the counting numbers, i.e. the positive integers). Abbreviate: $\{a_n\}$ = $a_1, a_2, a_3, \ldots$ where the n-th term of the sequence: $a_n = f(n)$. Note also that a sequence can be defined *recursively*: $a_k = g(a_{k-1})$ (where g is some function). Sometimes it takes some work to obtain the n-th term formula $a_n = f(n)$ from the recursive definition.

For example: Given the recursive definition: $a_1 = 2,\ a_k = 2a_{k-1}$

Find the n-th term formula: $a_1 = 2, a_2 = 2a_1, a_3 = 2a_2 = 2\cdot 2a_1 = 2^3 \Rightarrow a_n = 2^n$

- Recall the definition of the *limit* of a function[1]:

$\lim_{x\rightarrow c} f(x) = L$ means:

For any arbitrarily small ε such that $0 < |f(x) - L| < \varepsilon$ there exists an arbitrarily small δ such that $0 < |x - c| < \delta$ such that $\delta \rightarrow 0$ as $\varepsilon \rightarrow 0$

[1] For a review, see the last pages of August 30 notes, Calclus http://www.glue.umd.edu/%7Ewkallfel/MA261-2/MA261Aug30notes.pdf

When investigating sequences, of especial interest is to investigate their convergence or divergence properties. If a sequence *converges*, then there exists a finite L such that:

$\lim_{n\to\infty} a_n = L$, which one can abbreviate: $a_n \to L$. Precisely, this means (recall the above definition of limit):

$\lim_{n\to\infty} a_n = L$ means:

For any arbitrarily small ε such that $0 < |a_n - L| < \varepsilon$ there exists an arbitrarily large N such that $0 < |a_n - L| < \varepsilon$ for all $n > N$.

- **Defn. 1** A sequence is **bonded above (or below**) if there exists some number M such that: $a_n \le M$ (or $M \le a_n$) for all n. Note: A sequence can be bounded and still not converge! Consider: $a_n = (-1)^n$. Such a sequence is bounded above and below by +1, -1, but doesn't converge—it just oscillates between those values forever.
- **Defn.** 2: A sequence is **monotone increasing (or decreasing**) if for all k: $a_{k+1} \ge a_k$

  (or $a_{k+1} \le a_k$), which obviously can be also expressed as: $\frac{a_{k+1}}{a_k} \ge 1$ or $\frac{a_{k+1}}{a_k} \le 1$.

  Note further that if the inequalities are *strict*, then we say that the sequences are **strictly monotone (increasing or decreasing).**

Section 10.1 lists useful limit/convergence theorems for sequences (most of those track the limits theorems for functions, but some uniquely pertain to sequences).

1. **Thm10.1** (The functional equivalence thm). Given $a_n = f(n)$ and $\lim_{x\to\infty} f(x) = L$, then: $\lim_{n\to\infty} a_n = L$

2. **Tm 10.2** (Sum, product, difference, quotient). If $a_n \to L, b_n \to M$ then:
   i.) $(ca_n \pm db_n) \to cL \pm dM$ (for any constants c , d)
   ii.) $(a_n b_n) \to LM$
   iii.) $\frac{a_n}{b_n} \to \frac{L}{M}$ (provided $M \ne 0$)

3. **Thm10.3** (Sandwich Them). Given sequences $\{b_n\}$, $\{c_n\}$ such that for all n: $c_n \le a_n \le b_n$, and $c_n \to L, b_n \to L$, then $a_n \to L$

4. **Thm10.4** (Abs value thm). If $|a_n| \to 0$ then $a_n \to 0$ (Note how this follows straight from Thm 10.3, by property of the absolute values: $-|a_n| \le a_n \le |a_n|$)

5. **Thm 10.5** (Monotone thm). If a sequence is monotone increasing (decreasing) and bounded above (below), then it converges.

- Example (#29, §10.1): Given $a_n = \dfrac{3n^2 - n + 4}{2n^2 + 1}$ determine if is converges and its limit.

Using Thm10.1:

$$a_n = \frac{3n^2 - n + 4}{2n^2 + 1} \Leftrightarrow f(x) = \frac{3x^2 - x + 4}{2x^2 + 1} \Leftrightarrow \lim_{x\to\infty} \frac{3x^2 - x + 4}{2x^2 + 1}$$
$$\Leftrightarrow \lim_{x\to\infty} \frac{3 - \frac{1}{x} + \frac{4}{x^2}}{2 + \frac{1}{x^2}} = \frac{3}{2} \Rightarrow \lim_{n\to\infty} a_n = \frac{3}{2}$$

- Example (#38, §10.1): $a_n = \dfrac{(n-2)!}{n!}$

Thm 10.1 is unavailable (why?) Use Thm 10.5. First, note that that a_n is bounded below by 0. Determine if the sequence is monotone dcreasing:

$$\frac{a_{k+1}}{a_k} = \frac{(k-1)!}{(k+1)!} \cdot \frac{k!}{(k-2)!} = \frac{k!}{(k+1)!} \cdot \frac{(k-1)!}{(k-2)!} = \frac{k-1}{k+1} < 1, \text{ for all } k > 1.$$

Hence the sequence is monotone decreasing and bounded below so it converges.

To find its limit: $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \dfrac{(n-2)!}{n!} = \lim_{n\to\infty} \dfrac{1}{n(n-1)} = 0$

- Example: $a_n = \dfrac{\sin n}{n}$

Observe that: $-1 \le \sin n \le 1 \Rightarrow -\dfrac{1}{n} \le \dfrac{\sin n}{n} \le \dfrac{1}{n}$

However: $\lim_{n\to\infty} -\dfrac{1}{n} = 0 = \lim_{n\to\infty} \dfrac{1}{n} \Rightarrow \lim_{n\to\infty} \dfrac{\sin n}{n} = 0$, by **Thm10.3**

Defn. A Sequence A_n is a Cauchy Sequence if for all $\varepsilon > 0$ there exists an integer $N(\varepsilon)$ such that
Ffor all n,m >N:

$$/A_n - A_m/ < \varepsilon$$

Theorem: A sequence converges $\Leftrightarrow$ It's a Cauchy Sequence:

Proof ($\Rightarrow$) We need to show that <u>if</u> $A_n \to L$, <u>then</u> A_n is Cauchy

$$|A_n - A_m| = |A_n - L + L - A_m| \leq |A_n - L| + |L - A_m| < 2\varepsilon \text{ for all } m,n > Max\{K,L\}$$

(where K,L are the integer indices such that: $|A_n - L| < \varepsilon$ for all $n > K$ and $|A_m - L| < \varepsilon$ for all $m > L$)

$\Leftarrow$ ***We use: 1) A Cauchy Sequence is Bounded (we'll prove this) 2) Every bounded sequence contains a convergent subsequence (Boltzano-Weierstrauss) (We'll accept this)***

1) A Cauchy Sequence is Bounded:

From Triangle Inequalty[2], $|A_n| - |A_m| \leq |A_n - A_m| < \varepsilon$, *or for* **all** m,n: $|A_n| < |A_m| + \varepsilon$

Define: $K = lub\{|A_1| + \varepsilon, |A_2| + \varepsilon, \ldots, |A_{m-1}| + \varepsilon, |A_m| + \varepsilon\}$, *therefore:* $|A_n| < K < \infty$

- ***We'll accept the B-W-T, and assume there's a subsequence A_K in A_n (precisely stated: For some indices K: $A_K \subseteq A_n$) such that: $A_K \to L$. We need to show that this implies that the <u>entire</u> Cauchy sequence converges to L, then (not just some of its terms)***

$$|A_n - L| = |A_n - A_K + A_K - L| \leq |A_n - A_K| + |L - A_K| < 2\varepsilon$$

$\downarrow$ $\qquad\qquad\qquad\qquad$ $\downarrow$

0 $\qquad$ 0 *as n,K get very large since the sequence is Cauchy,*

and by premise

Example: $A_1 = 1 \quad A_2 = 2 \qquad A_n = {}^1/_2\,[A_{n-1} + A_{n-2}]$

$A_3 = {}^1/_2\,[A_2 + A_1] = {}^3/_2 \qquad |A_3 - A_2| \leq {}^1/_2$

$A_4 = {}^1/_2\,[A_3 + A_2] = {}^5/_4 \qquad |A_3 - A_2| \leq {}^1/_4 = {}^1/_{2^2}$

$A_5 = {}^1/_2\,[A_4 + A_3] = {}^{11}/_8 \qquad |A_4 - A_3| \leq {}^1/_4 = {}^1/_{2^3}$

…By cranking out terms, we see it's Certainly Cauchy, therefore (as perhaps our intuition would confirm) its limit lies between 1 and 2. Stay tuned on how to calculate the limit of such a sequnce

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[2] *(WLOG assume $|A_n| \geq |A_m|$)*

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Infinite Series And Sequences

If $\Sigma a_k \rightarrow L < \infty$ then $a_k \rightarrow 0$.

Proof: $S_k = \sum_{j=1}^{k} a_j \qquad S_{k+1} = \sum_{j=1}^{k+1} a_j$

$$S_{k+1} - S_k = \sum_{j=1}^{k+1} a_j - \sum_{j=1}^{k} a_j = \{a_1+a_2+\ldots+a_k+a_{k+1}\} - \{a_1+a_2+\ldots+a_k\} = a_{k+1}$$

But: $\lim_{k\to\infty} S_{k+1} = \lim_{k\to\infty} S_k = L$, *so: :* $\lim_{k\to\infty} (S_{k+1} - S_k) = \mathbf{0} = \mathbf{\lim_{k\to\infty} a_{k+1}} = \mathbf{\lim_{k\to\infty} a_k}$

- ***The converse is false! (i.e. If $a_k \rightarrow 0$ <u>this does not imply that</u> $\Sigma a_k \rightarrow L < \infty$)***
  Counterexample:
  Consider: $S_n = \sum_{k=1}^{n} \frac{1}{k}$. *Certainly,* $a_k \rightarrow 0$, *but* $\lim_{n\to\infty} S_n = \infty$

- ***Logically, since ($P \Rightarrow Q$) $\Leftrightarrow$ ($\sim Q \Rightarrow \sim P$) (A conditional statement is logically equivalent to its contrapositive.) The contrapositive (If $a_k \rightarrow K$, (where $K \neq 0$), then $\Sigma a_k \rightarrow \infty$) form serves obviously as a useful test for necessary conditions leading to <u>divergence.</u>***

- ***Defn.: Suppose $\{a_n\}$ is a sequence ($n \geq 1$). Define the <u>sequence of partial sums of</u> $\{a_n\}$ as:***

$S_1 = \Sigma^1_{k=1} a_k = a_1$, $S_2 = \Sigma^2_{k=1} a_k = a_1 + a_2, \ldots,$ $S_n = \Sigma^n_{k=1} a_k = a_1 + a_2 + \ldots + a_n$

- **Remark: Note that these partial sums define a <u>sequence</u> { S_n } while each sum is defined as a <u>series</u> with respect to $\{a_n\}$. In fact, such a process of forming sequences of partial sum of other sequences of partial sums of other sequences of… can be indefinitely iterated. This process, for example, is how a digital computer "upscales" from binary {∅,1} artihmetic on an assembly level to the**

sophisticated computations it performs on Third, Fourth, and higher generational languages (levels)

- **For a an infinite series to converge it's equivalent to say the sequence of partial sums converges (we've investigated convergent/divergent sequences in Ch10), we may write:**

$$\boldsymbol{lim}_{n\to\infty} S_n = \boldsymbol{lim}_{n\to\infty} \sum_{k=1}^{n} a_k < \infty,$$

or in abbreviated form: $\sum a_k \to L < \infty,$

Three Important Series:

I) "Telescoping" $\sum_{k=1}^{n} A_k$ ***where:*** $A_k = a_k - a_{k-1}$ ***(each term is the difference between two consecutive terms of a sequence)***

Observe: $\sum_{k=1}^{n} A_k = \sum_{k=1}^{n} a_k - a_{k-1} = (a_1 - a_0) + (a_2 - a_1) + (a_3 - a_2) + \ldots + (a_n - a_{n-1}) = \boldsymbol{a_n - a_0}$ ***(Why?)***

II) "Arithmetic" $\sum_{k=1}^{n} A_k$ ***where:*** $A_k = A_1 + (k-1)d \quad (k \geq 1)$

Observe: $\sum_{k=1}^{n} A_k = \dfrac{n(A_1 + A_n)}{2}$

Proof:

$$\begin{aligned} &\sum_{k=1}^{n} A_k = [A_1 + (1-1)d] + [A_1 + (2-1)d] + [A_1 + (3-1)d] + \ldots + [A_1 + (n-1)d] \\ + \;&\sum_{k=1}^{n} A_k = [A_1 + (n-1)d] + [A_1 + (n-2)d] + [A_1 + (n-3)d] + \ldots + [A_1 + (1-1)d] \\ \hline &2\sum_{k=1}^{n} A_k = [2A_1 + (n-1)d] + [2A_1 + (n-1)d] + [2A_1 + (n-1)d] + \ldots + [2A_1 + (n-1)d] \\ &= n\,[2A_1 + (n-1)d] \end{aligned}$$

Hence: $\sum_{k=1}^{n} A_k = \dfrac{n\,[2A_1 + (n-1)d]}{2} = \dfrac{n\,[A_1 + (A_1 + (n-1)d)]}{2} = \boldsymbol{\dfrac{n(A_1 + A_n)}{2}}$

By the same token: $\sum_{k=0}^{n} A_k = \dfrac{(n+1)(A_0 + A_n)}{2}$ ***where: where:*** $A_k = A_0 + kd \quad (k \geq 0)$

II) "Geometric" $\sum^{n}_{k=1} A_k$ ***where:*** $A_k = A_1 r^{(k-1)}$ $(k \geq 1)$

Observe: $\sum^{n}_{k=1} A_k = \dfrac{A_1(1 - r^n)}{1 - r}$

Proof: $\sum^{n}_{k=1} A_k = [A_1 r^{(1-1)}] + [A_1 r^{(2-1)}] + [A_1 r^{(3-1)}] + \ldots + [A_1 r^{(n-1)}]$

$-\quad r\sum^{n}_{k=1} A_k = [A_1 r^{(1)}] + [A_1 r^{(2)}] + [A_1 r^{(3)}] + \ldots + [A_1 r^{(n)}]$

$(1-r)\sum^{n}_{k=1} A_k = A_1 - A_1 r^{(n)}$

Hence: $\sum^{n}_{k=1} A_k = \dfrac{A_1(1 - r^n)}{1 - r}$

or: $\sum^{n}_{k=0} A_k = \dfrac{A_0(1 - r^{n+1})}{1 - r}$ ***where:*** $A_k = A_0 r^k$ $(k \geq 0)$ ***(easy to verify)***

Obviously the infinite series converges to $\dfrac{A_0}{1 - r}$ ***provided*** $0 \leq |r| < 1$

$$\sum_{k=0}^{\infty} \frac{1}{(k+1)(k+3)}$$

Answer: *Using partial fractions,* $\dfrac{1}{(k+1)(k+3)} = \dfrac{A}{(k+1)} + \dfrac{B}{(k+3)} \Rightarrow A = {}^1/_2.\ \ B = -{}^1/_2.$

Hence: $\sum[(k+1)(k+3)]^{-1} = \lim_{n\to\infty} {}^1/_2 \sum^{n}_{k=0} [(k+1)^{-1} - (k+3)^{-1}]$

$= \lim_{n\to\infty} {}^1/_2 \ \{ [(0+1)^{-1} - (0+3)^{-1}] + [(1+1)^{-1} - (1+3)^{-1}] + [(2+1)^{-1} - (2+3)^{-1}] + \ldots + [(n+1)^{-1} - (n+3)^{-1}] \}$

$= \lim_{n\to\infty} {}^1/_2 \{[1 + {}^1/_2] - [(n+1)^{-1} + (n+3)^{-1}]\}$ ***(After cancelling all telescoping terms)***

$\mathbf{= {}^1/_2 \{[1 + {}^1/_2]\} = {}^3/_4}$

$\sum_{k=1}^{2}(3k+1)$ $= [2(4 + 7)]/2 = 11$ ***(Using Arithmetic Series Formula))***

$\sum_{k=0}^{3}(-1)^k 2^{k+1}$ == 2 $\lim_{n\to\infty} \frac{1}{2} \sum^{n}_{k=0} (-2)^k$ ***Geometric series, diverges (/r / = 2 >1)***

$\sum_{k=0}^{\infty}(\frac{25}{10^k} - \frac{6}{100^k})$ = *25* $\lim_{n\to\infty} \sum^{n}_{k=0} (0.1)^k$ - *6* $\lim_{n\to\infty} \sum^{n}_{k=0} (0.01)^k$

=25[1/0.9] - 6[1/.99] = 250/9 - 600/99 = 2750/99 (Using infinite geometric series convergence formula)

Other Useful Formulae

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(n+2)}{6} \qquad \sum_{k=1}^{n} k^3 = \left[\sum_{k=1}^{n} k\right]^2$$

Proof By Induction:

(Inductive Step Only) Assume: $\sum^{k}_{(j=1)} j^2 = \frac{k(k+1)(2k+1)}{6}$

Prove: $\sum^{(k+1)}_{(j=1)} j^2 = \frac{(k+1)(k+2)(2k+3)}{6}$

(Proof):

$$\sum^{k+1}_{(j=1)} j^2 = \sum^{k}_{(j=1)} j^2 + (k+1)^2 = \frac{k(k+1)(2k+1)}{6} + 6\frac{(k+1)^2}{6}$$

$$= \frac{(k+1)[(2k+1)k + (6k+6)]}{6}$$

$$= \frac{(k+1)[(2k^2 + 7k+6]}{6}$$

$$= \frac{(k+1)(k+2)(2k+3)}{6}$$

$$\sum_{k=1}^{n} k(k^2 - 5)$$

Answer: $\sum(k^3 - 5k) = \sum k^3 - 5\sum k = \dfrac{n^2(n+1)^2}{4} - 5\dfrac{n(n+1)}{2} = \dfrac{n(n+1)[\,n(n+1) - 10]}{4}$

$= \dfrac{n(n+1)(n^2 + n - 9)}{4}$

$$\sum_{k=2}^{\infty} \frac{3^{k-1}}{4^{3k+1}}$$

Using convergent geom. Series formula

↓

Answer: $= {}^{1}/_{12}\sum_{k=2}^{\infty}({}^{3}/_{4^3})^{k} = {}^{1}/_{12}\sum_{j=0}^{\infty}({}^{3}/_{64})^{j+2} = {}^{9}/_{12*(64)^2}\sum_{j=0}^{\infty}({}^{3}/_{64})^{j} = [{}^{9}/_{12*(64)^2}]\ [1/(1-({}^{3}/_{64}))]$

← Though it may seem less efficient, we can shift index starting off at 0 to compare geometric serie forms

↑

Factoring out 3^{-1} from numerator and 4 from the denominator

$= [{}^{9}/_{12*64*61}] = [{}^{3}/_{4*64*61}]$

) Find a formula for the partial sums of $\sum_{k=1}^{\infty} \ln\left[\dfrac{k}{k+1}\right]$

Series Telescopes

↓

Answer: $S_n = \sum_{k=1}^{n} [\, \ln k - \ln(k+1)] = \mathbf{[\ln 1 - \ln(n+1)]} = \mathbf{\ln[1/(n+1)]}$

Determine the smallest integer N such that $\left|L - s_N\right| < 10^{-4}$ *where L is the infinite sum and S_N is the Nth partial sum for the series:* $\sum_{k=0}^{\infty}\left(\frac{2}{3}\right)^k$

Convergent Infinite geometric series
↓
Answer: $L = 1/(1 - {}^2/_3) = 3$

$S_N = [1 - ({}^2/_3)^{N+1}]/(1 - {}^2/_3) = 3[1 - ({}^2/_3)^{N+1}]$ ←***Using finite geometric series formula***

$|L - S_N| = |3[1 - ({}^2/_3)^{N+1}] - 3| = ({}^2/_3)^{N+1} < 10^{-4}$

or: $|(N+1)\ln(2/3)| > 4\ln(10) \Rightarrow N \geq \lceil -4\ln 10/\ln(2/3) + \ln(2/3) \rceil = 23$

Tests for Convergence

- ***The Integral Test:*** $\sum_{k=1}^{\infty} f(k) < \infty \Leftrightarrow \int_1^{\infty} f(x)dx < \infty$

  Where f is continuous, decreasing, and positive

Use the integral test when it's fairly obvious that summand f(k) becomes an integral function when replacing k with x. ***Important****:* ***f(k) must be continuous, decreasing, positive (for k*** $\geq$***1) !***

Example: $\sum_{k=1}^{\infty} [k\ln k]^{-1}$

Answer: *1/(klnk) is positive, decreasing. So it's a candidate:* $\lim_{b\to\infty}\int_1^b [x\ln x]^{-1}dx = \lim_{b\to\infty}\ln((\ln b)) - \ln 1$

Diverges

- ***Comparison Test:*** $\sum a_k \,(a_k \geq 0)$ **:** $\sum a_k$ ***converges provided there exists a convergent series :*** $\sum c_k$ ***where*** $a_k \leq c_k$ ***for all k.*** $\sum a_k$ ***diverges provided there exists a divergent series (with nonegative terms) series :*** $\sum c_k$ ***where*** $c_k \leq a_k$ ***for all k.***

Use this simple comparison test whenever the summand can be compared termwise with a series you know converges/diverges. (Example: the geometric series, $\sum_{k=1}^{\infty} r^k$ *, the p-series:* $\sum_{k=1}^{\infty} k^{-p}$*)*

Example: $\sum_{k=1}^{\infty} \frac{2 + \sin k}{k^2}$

Answer: $\sin k \leq 1$ so $(2+\sin k)k^{-2} \leq 3/k^2$ so: $\sum_{k=1}^{\infty}(2+\sin k)k^{-2} \leq 3\sum_{k=1}^{\infty} k^{-2}$ $\leftarrow$***p-series***

(converges, since p=2 >1)

Hence series converges by Simple Comparison Test