MATH 261
EXAM I
September 27, 2007

Name: KEY

Directions: For maximum credit, please show all work in most reasonable detail. If you run out of room, you may pick up extra paper (supplied by instructor/proctor) and attach to this exam. No books or notes. Formula sheet provided. Calculator permitted, <u>with programming/memory mode shut off.</u> Please choose FIVE from the following SIX (worth 20 pts each.) If you do more, I will grade the best <u>five</u>. Bonus problem included. Good luck!

I. **Calculate the limits. Or if they do not exist, show or explain why, in terms of left and right hand limits**

a.) (5 pts) $\lim_{x\to -3}\dfrac{x^3+27}{x+3}$

$$\lim_{x\to -3}\frac{\cancel{(x+3)}(x^2-3x+9)}{\cancel{(x+3)}} = \lim_{x\to -3}(x^2-3x+9)$$

$$= 27$$

b.) (5 pts) $\lim_{x\to 5}\dfrac{1}{x-5}$

$$\lim_{x\to 5^+}\frac{1}{x-5} = \infty$$

$$\lim_{x\to 5^-}\frac{1}{x-5} = -\infty$$

$$\therefore \lim_{x\to 5}\frac{1}{x-5} \text{ DNE}$$

c.) (5 pts) $\lim_{x\to\infty}\left(\sqrt{x-1}-\sqrt{x+1}\right)$

$$= \lim_{x\to\infty}\left(\frac{\sqrt{x-1}-\sqrt{x+1}}{1}\cdot\frac{\sqrt{x-1}+\sqrt{x+1}}{\sqrt{x-1}+\sqrt{x+1}}\right)$$

$$= \lim_{x\to\infty}\left(\frac{x-1-\cancel{x}+1}{\sqrt{x-1}+\sqrt{x+1}}\right) = = -2\lim_{x\to\infty}\frac{1}{\sqrt{x-1}+\sqrt{x+1}}$$

$$\overset{\sim}{=} -2\lim_{x\to\infty}\frac{1}{2\sqrt{x}} = 0$$

d.) (5 pts) $\lim_{x\to 0}\left(\frac{1-\sqrt{x+1}}{x}\right)$ $\frac{0}{0}$ INDET. ∴ REDUCE

$$\lim_{x\to 0}\frac{1-\sqrt{x+1}}{x}\cdot\frac{1+\sqrt{x+1}}{1+\sqrt{x+1}}$$

$$= \lim_{x\to 0}\frac{1-x-1}{x(1+\sqrt{x+1})} = \lim_{x\to 0}\frac{-x}{x(1+\sqrt{x+1})}$$

$$= -\lim_{x\to 0}\frac{1}{(1+\sqrt{x+1})} = -\frac{1}{2}$$

II.) Given $f(x)=\left(\frac{x-1}{x+1}\right)^2$

a.) (8) Find $f'(x)$ **using the Chain Rule *first*, (and then the Quotient Rule)**

$$f'(x) = 2\left(\frac{x-1}{x+1}\right)^{1}\frac{d}{dx}\left(\frac{x-1}{x+1}\right) = 2\left(\frac{x-1}{x+1}\right)\frac{\{(x+1)-(x-1)\}}{(x+1)^2}$$

$$= 2\left(\frac{x-1}{x+1}\right)\frac{2}{(x+1)^2} = \frac{4(x-1)}{(x+1)^3}$$

b.) (12) Find $f'(x)$ **using the Quotient Rule *first* (I.e. re-write**

$f(x)=\left(\frac{x-1}{x+1}\right)^2=\frac{(x-1)^2}{(x+1)^2}$**, and use the Quotient Rule**

Show that your answer is the same as in a.)

$$f'(x) = \frac{d}{dx}\left\{\frac{(x-1)^2}{(x+1)^2}\right\} = \frac{(x+1)^2\,2(x-1)-(x-1)^2\,2(x+1)}{(x+1)^4}$$

$$= \frac{2(x+1)(x-1)[\cancel{x}+1-(\cancel{x}-1)]}{(x+1)^4}$$

$$= \frac{2(x-1)[2]}{(x+1)^3} = \frac{4(x-1)}{(x+1)^3}$$

III.) Given $f(x)=\left|x^2-16\right|$

a.) (10) Find $f'(x)$

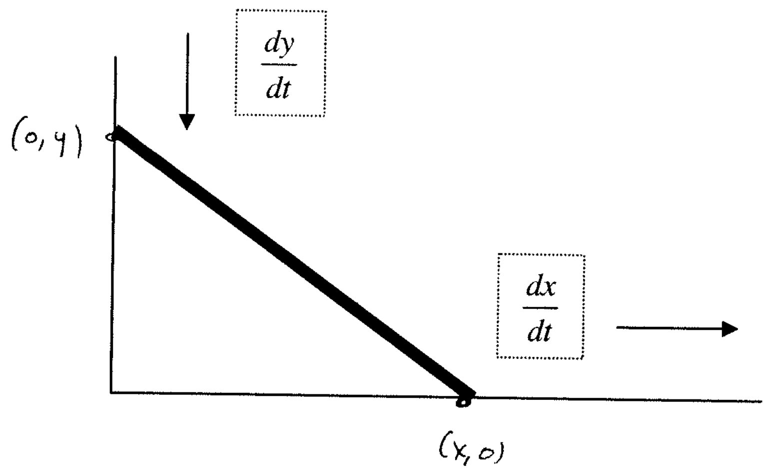
$$f(x)=\begin{cases} x^2-16 & x^2-16\geq 0 \Rightarrow x\geq 4 \text{ or } x\leq -4 \\ 16-x^2 & x^2-16<0 \Rightarrow -4<x<4 \end{cases}$$

$$f'(x)=\begin{cases} 2x & \text{for } x\geq 4 \text{ or } x\leq -4 \\ -2x & \text{for } -4<x<4 \end{cases}$$

b.) (10) Find $f''(x)$

$$f''(x)=\begin{cases} 2 & \text{for } x\geq 4 \text{ or } x\leq 4 \\ -2 & \text{for } -4<x<4 \end{cases}$$

IV.) A ladder of length 15 feet leans up against a wall (see figure below.) Suppose its tip is sliding down at a rate of $\frac{dy}{dt} = -10 \, ^{ft}/_{sec}$, when $y = 9$ feet. How fast is its bottom right tip moving?



$$L^2 = x^2 + y^2 \qquad (L = 15 \, ft)$$

$$\frac{d}{dt} L^2 = 0 = 2x\frac{dx}{dt} + 2y\frac{dy}{dt} \Rightarrow x\frac{dx}{dt} = -y\frac{dy}{dt}$$

$$\frac{dx}{dt} = -\frac{y}{x}\frac{dy}{dt}$$

$$y = 9 \quad x = \sqrt{15^2 - 9^2} = \sqrt{144} = 12$$

$$\frac{dx}{dt} = -\frac{3}{4}\left(\frac{dy}{dt}\right) = \frac{3}{4}(10 \, ft/sec) = \frac{15}{2} \, ft/sec$$

V.) (20) Use the MVT to show that for any quadratic polynomial,
$p(x) = Ax^2 + Bx + C$

defined on $[a, b]$ the point c satisfying the condition of the MVT lies at the midpoint of the interval. (The midpoint of $[a, b]$ is: $c = \dfrac{a+b}{2} = \frac{1}{2}(a+b)$)

$$p(a) = Aa^2 + Ba + C$$
$$p(b) = Ab^2 + Bb + C$$
$$\frac{p(b) - p(a)}{(b-a)} = \frac{(Ab^2 + Bb + C) - (Aa^2 + Ba + C)}{(b-a)}$$
$$= \frac{A(b^2 - a^2) + B(b-a)}{(b-a)} = \frac{A(b+a)(b-a) + B(b-a)}{(b-a)}$$
$$= \frac{\cancel{(b-a)}[A(b+a) + B]}{\cancel{(b-a)}} = A(b+a) + B$$
$$f'(x) = 2Ax + B$$
$$f'(c) = 2Ac + B$$
$$f'(c) = 2\cancel{A}c + \cancel{B} = \frac{f(b) - f(a)}{b-a} = A(b+a) + \cancel{B}$$
$$\therefore 2\cancel{A}c = \cancel{A}(b+a)$$
$$\boxed{c = \frac{1}{2}(b+a)}$$

VI.) **Given:** $f(x) = \dfrac{-2x}{x^2+2}$

Observe: $-f(x) = f(-x)$ $(\therefore f \text{ is odd})$

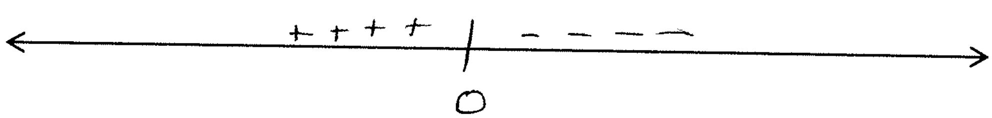
a.) **(2)** Find f's asymptotes (horizontal)

$$\lim_{x \to \pm\infty} f(x) = \lim_{x \to \pm\infty} \frac{-2x}{x^2+2} = -2 \lim_{x \to \pm\infty} \frac{\frac{1}{x}}{1 + \frac{2}{x^2}} = 0$$

b.) **(3)** Find partition points of f and sketch a sign chart for f

$f(x) = 0 \Rightarrow x_0 = 0$

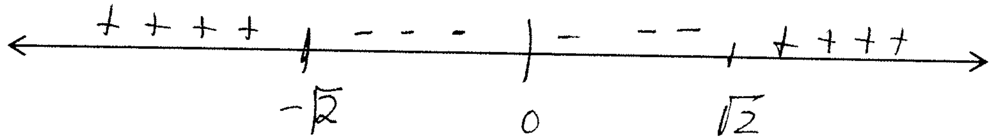
No partition points setting denominator $= 0$ (irreducible)



c.) **(5)** Find all critical points for f and sketch a sign chart for f'

$$f'(x) = -2\left\{\frac{(x^2+2) - x(2x)}{(x^2+2)^2}\right\} = -2\left\{\frac{-x^2+2}{(x^2+2)^2}\right\}$$

$$= \frac{-2(2-x^2)}{(x^2+2)^2} \qquad f'(x) = 0 \Rightarrow x_{1/2} = \pm\sqrt{2}$$



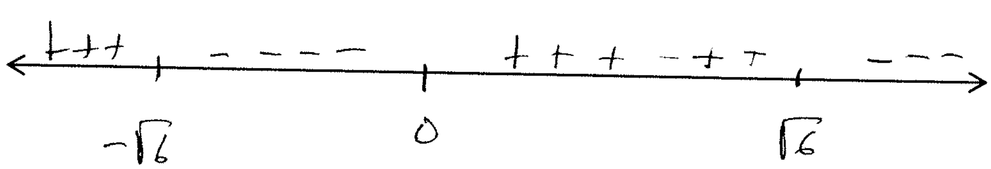
d.) (~~5~~) Find all inflection points for f and sketch a sign chart for f''
(10)

$$f''(x) = \frac{d}{dx} f'(x) = \frac{d}{dx}\frac{-2(2-x^2)}{(x^2+2)^2} = -2\left\{\frac{(x^2+2)^2(-2x) - (2-x^2)2(x^2+2)(}{(x^2+2)^4}\right.$$

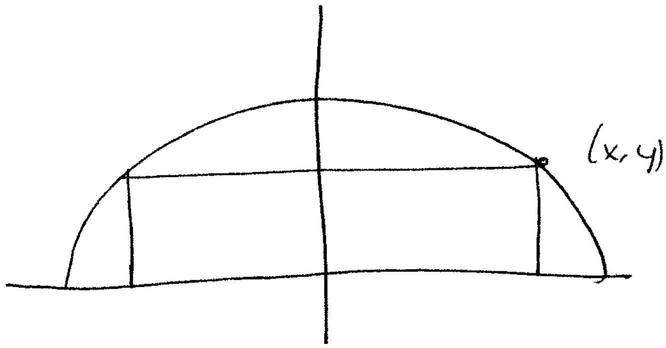
$$= \frac{-2(x^2+2)\left[(x^2+2)(-2x) - (2-x^2)4\cancel{x}\right]}{(x^2+2)^4}$$

$$= \frac{4\left[(x^2+2)x + 2(2-x^2)\boxed{x}\right]}{(x^2+2)^3}$$

$$= \frac{4x[6-x^2]}{(x^2+2)^3} \qquad x_1 = 0 \qquad x_{5,6} = \pm\sqrt{6}$$



BONUS: Find the maximum area of a rectangle bounded below by the x axis and inscribed inside the top half of a circle: $x^2 + y^2 = 64$



$A = 2xy$

$y = \sqrt{64 - x^2}$

$A(x) = 2x(64 - x^2)^{1/2}$

$A'(x) = 2(64-x^2)^{1/2} + 2x \cdot \frac{1}{2}(64-x^2)^{-1/2}(-2x)$

$= 2(64-x^2)^{1/2} - \cancel{2}x^2(64-x^2)^{-1/2}$

$= 2(64-x^2)^{1/2}\left[1 - \frac{x^2}{(64-x^2)}\right]$

$= 2(64-x^2)^{-1/2}[(64-x^2) - x^2]$

$= \frac{2(64-2x^2)}{\sqrt{64-x^2}}$

$A_{max} = 2xy$

$= 2 \cdot 4\sqrt{2}\sqrt{64-32}$

$= 2 \cdot 32 = 64$

$A'(x) = 0 \Rightarrow 64 - 2x^2 = 0 \Rightarrow x = \sqrt{32} = 4\sqrt{2}$

$A''(x) = 2\frac{d}{dx}\left\{\frac{64-2x^2}{\sqrt{64-x^2}}\right\} = 4\frac{d}{dx}\left\{\frac{32-x^2}{\sqrt{64-x^2}}\right\}$

$= 4\cancel{\frac{d}{dx}}\left\{\frac{\sqrt{64-x^2}(-2x) - (32-x^2)\frac{1}{2}(64-x^2)^{-1/2}(-2x)}{(32-x^2)}\right\}$

$A''(4\sqrt{2}) < 0 \Rightarrow 4\sqrt{2}$ is a max.