MA 261 Fall 2007_ Kallfelz *ASSIGNMENT II KEY*

Directions: Please complete the following problems listed below, in legible and neat form. Illegible work will not be graded, and marked with a 0 grade.

I. (25) Consider the function: $f(x) = 2x^3 - 4x - 3$ on [1 , 3]. In this problem, you are to find the *fixed point* of *f*(*x*), i.e., the point *c* ∈ [1 , 3] such that *c* = *f* (*c*).

(a) (5) Prove that *f*(*x*) has a fixed point in [1 , 3]. To do this, construct a function: *g*(*x*) = *f*(*x*) – *x*, and use the *Intermediate value Theorem* on *g*(x).[1]

Let $g(x) = f(x) - x = (2x^3 - 4x - 3) - x = 2x^3 - 5x - 3$

Since *g* is a polynomial, *g* is continuous. $g(1) = 2 - 5 - 3 = -6 < 0$

$g(3) = 2 \cdot 27 - 5 \cdot 3 - 3 = 36 > 0$

Hence, by IVT, there exists at least one point *c* such that $1 < c < 3$ and $g(1) < g(c) < g(3)$, where $g(c) = 0$ (since $g(1) < 0 < g(3)$)

(b) (20) Use the Newton-Raphson method to approximate the fixed point of *f*(*x*) (i.e., the roots of *g*(*x*)), starting with $x_0 = 2$. Recall the Newton-Raphson algorithm (discussed Tuesday, October 2) : $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$.

You're welcome to use a software application like Excel here!

$$g(x) = 2x^3 - 5x - 3 \Rightarrow g'(x) = 6x^2 - 5 \Rightarrow x_{n+1} = x_n - \frac{2x_n^3 - 5x_n - 3}{6x_n^2 - 5}$$

n	x_n	$g(x_n)$	$g'(x_n)$	x_{n+1}
0	2	3	19	1.842105263
1	1.842105	0.29129611	15.3601108	1.82314081
2	1.823141	0.00396144	14.94305447	1.822875707
3	1.822876	7.6874E-07	14.93725506	1.822875656
4	1.822876	2.8422E-14	14.93725393	1.822875656
5	1.822876	0	14.93725393	1.822875656

[1] Recall from **August 30** notes: The **Intermediate Value Theorem (IVT) (Thm 2.11, page 74)** expresses a very attractive feature about continuous functions, namely that for any function *f* continuous on [*a* , *b*] we are *guaranteed* there exists *at least one point in the y-axis* *f*(c) such that: *f*(a) < *f*(c) < *f*(b) for *any c* ∈ (*a* , *b*). In other words, any point between the endpoints on the domain is going to produce *at least one* y-value between the points *f*(a) and *f*(b) in the range.

II. (25) a.) (8) p. 225, exercise 28

$$P = \left(500x - x^2\right) - \left(\tfrac{1}{2}x^2 - 77x + 3000\right)$$

$\Delta P \cong P'(x)\Delta x$ where: $P'(x) = 500 - 2x - x + 77 = -3x + 577$

$$\Delta x = x_1 - x_0 = 180 - 175 = 5$$
$$\Delta P \cong P'(x_0)\Delta x = [-3(175) + 577] \cdot 5 = 260$$

Percentage change:

$$\left(\frac{\Delta P}{P(x_0)}\right) \cdot 100 = \left(\frac{260}{-\frac{3}{2} \cdot 175^2 + 577 \cdot 175 - 3000}\right) = 0.499 \cong 0.5\%$$

b.) (7) p. 279, exercise 50

$$\int_{-2}^{6} x^2 \sqrt[3]{x+2}\,dx = \int_{-2}^{6} x^2 (x+2)^{1/3}\,dx$$

$$u = x + 2 \Rightarrow x = u - 2 \Rightarrow dx = du$$

$$\int_{-2}^{6} x^2 \sqrt[3]{x+2}\,dx = \int_{-2}^{6} x^2 (x+2)^{1/3}\,dx = \int_{u(-2)}^{u(6)} (u-2)^2 u^{1/3}\,du = \int_{0}^{8} \left(u^2 - 4u + 4\right) u^{1/3}\,du$$
$$= \int_{0}^{8} \left(u^{7/3} - 4u^{4/3} + 4u^{1/3}\right) du = \tfrac{3}{10} u^{10/3} - \tfrac{12}{7} u^{7/3} + 3u^{4/3} \Big|_0^8 = \tfrac{3}{10} \cdot 1024 - \tfrac{12}{7} \cdot 128 + 3 \cdot 16 - 0$$
$$= \tfrac{1536}{5} - \tfrac{1536}{7} + 48 = 1536\left(\tfrac{1}{5} - \tfrac{1}{7}\right) + 48 = \tfrac{3072}{35} + \tfrac{1680}{35} = \tfrac{4752}{35}$$

c.) (10) p. 288, exercise 32

$$y_0 = f(x_0) = 0,\ y_1 = f(x_1) = 50,\ y_2 = f(x_2) = 54,\ y_3 = f(x_3) = 82,$$
$$y_4 = f(x_4) = 82,\ y_5 = f(x_5) = 73,\ y_6 = f(x_6) = 75,\ y_7 = f(x_7) = 80,\ y_8 = f(x_8) = 0$$

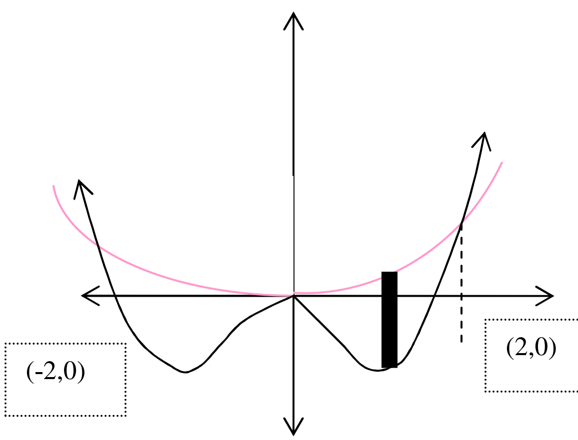
$$\Delta x = 20,\ x_0 = a = 0,\ x_8 = b = 160$$

n	x_n	f(x)	f(b) - f(a)/2n
1	20	50	0
2	40	54	
3	60	82	

4	80	82
5	100	73
6	120	75
7	140	80
8	160	0
Sum		496
Sum*(b-a)/n		19840
RESULT		**19840**

III. (25) a.) (5) p. 300, exercise 22

$$y_1 = x^4 - 2x^2,\ y_2 = 2x^2 \Rightarrow y_1 = y_2 \Rightarrow x^4 - 4x^2 = 0 \Rightarrow x^2\left(x^2 - 4\right) = 0$$
$$x_1 = -2, x_2 = 0, x_3 = 2$$



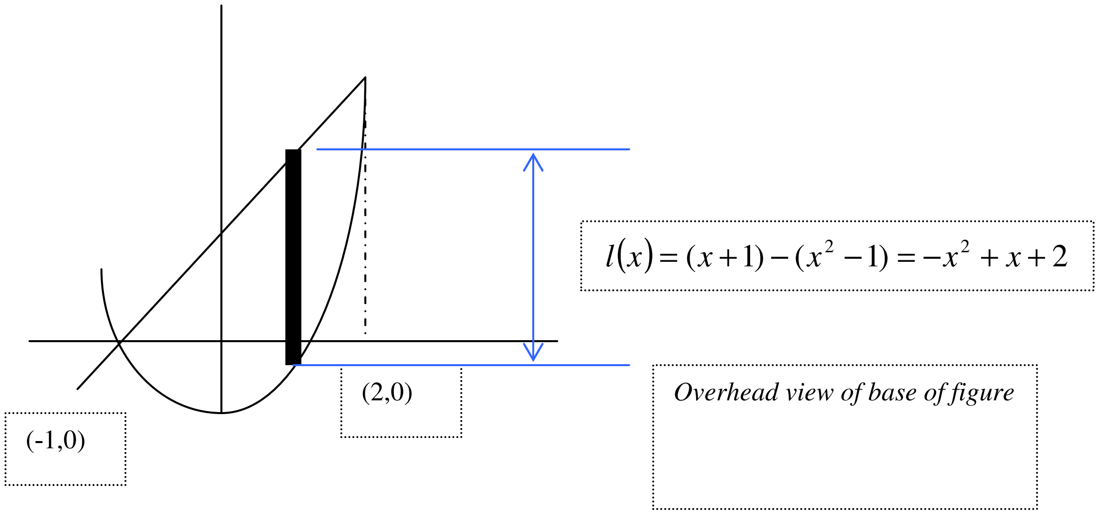
Because of symmetry:

$$A = \int_{-2}^{2}[y_2 - y_1]dx = 2\int_{0}^{2}\left[2x^2 - \left(x^4 - 2x^2\right)\right]dx = 2\int_{0}^{2}\left(4x^2 - x^4\right)dx = 2\left(\tfrac{4}{3}x^3 - \tfrac{1}{5}x^5\right)\Big|_0^2$$
$$= 2x^3\left(\tfrac{4}{3} - \tfrac{1}{5}x^2\right)\Big|_0^2 = 16\left(\tfrac{4}{3} - \tfrac{4}{5}\right) = 64\left(\tfrac{1}{3} - \tfrac{1}{5}\right) = 64\left(\tfrac{2}{15}\right) = \tfrac{128}{15}$$

b.) (10) p. 312, exercise 46

Base: $y_1 = x + 1,\ y_2 = x^2 - 1$

$$y_1 = y_2 \Rightarrow x^2 - x - 2 = 0 = (x-2)(x+1) \Rightarrow x_1 = -1, x_2 = 2$$



a.)

$$V = \int_{-1}^{2} (l(x))^2 dx = \int_{-1}^{2} \left(-x^2 + x + 2\right)^2 dx = \int_{-1}^{2} \left(x^4 - x^3 - 2x^2 - x^3 + x^2 + 2x - 2x^2 + 2x + 4\right) dx$$

$$= \int_{-1}^{2} \left(x^4 - 2x^3 - 3x^2 + 4x + 4\right) dx = \left(\tfrac{1}{5}x^5 - \tfrac{1}{2}x^4 - x^3 + 2x^2 + 4x\right)\Big|_{-1}^{2}$$

$$= x\left(\tfrac{1}{5}x^4 - \tfrac{1}{2}x^3 - x^2 + 2x + 2\right)\Big|_{-1}^{2} = 2\left(\tfrac{16}{5} - 4 - 4 + 4 + 2\right) + \left(\tfrac{1}{5} + \tfrac{1}{2} - 1 - 2 + 2\right)$$

$$= 2\left(\tfrac{16}{5} - 2\right) + \left(\tfrac{1}{5} - \tfrac{1}{2}\right) = \tfrac{32}{5} - 4 - \tfrac{3}{10} = \tfrac{64-40-3}{10} = \tfrac{21}{10}$$

b.)

$$V = \int_{-1}^{2} (l(x)\cdot 1)\ dx = \int_{-1}^{2} \left(-x^2 + x + 2\right) dx = \left(-\tfrac{1}{3}x^3 + \tfrac{1}{2}x^2 + 2x\right)\Big|_{-1}^{2} = x\left(-\tfrac{1}{3}x^2 + \tfrac{1}{2}x + 2\right)\Big|_{-1}^{2}$$

$$= 2\left(-\tfrac{4}{3} + 1 + 2\right) + \left(-\tfrac{1}{3} - \tfrac{1}{2} + 2\right) = 2\left(\tfrac{5}{3}\right) + \tfrac{7}{6} = \tfrac{27}{6}$$

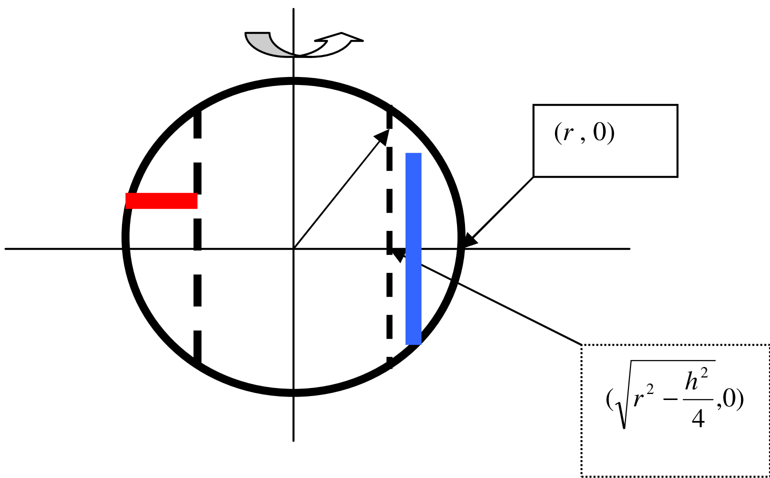
c.) $A(x) = \tfrac{1}{2}\pi ab = \tfrac{1}{2}\pi\left(\dfrac{l(x)}{2}\right)\cdot 2 = \tfrac{\pi}{2}\left(l(x)\right) = \tfrac{\pi}{2}\left(-x^2 + x + 2\right)$

$V = \int_{-1}^{2} A(x)dx = \tfrac{\pi}{2}\int_{-1}^{2}\left(-x^2 + x + 2\right)dx$, which differs from result in b.) above by the multiplicative constant $\tfrac{\pi}{2}$. Hence: $V = \tfrac{\pi}{2}\cdot\tfrac{27}{6} = \tfrac{27\pi}{12} = \tfrac{9}{4}\pi$

d.) $A(x) = \tfrac{1}{2}BH = \tfrac{1}{2}\left(\dfrac{l(x)}{2}\right)\left(\dfrac{\sqrt{3}}{2}l(x)\right) = \tfrac{\sqrt{3}}{8}\left(l(x)\right)^2$

$V = \frac{\sqrt{3}}{8}\int_{-1}^{2}(l(x))^2\,dx$, which differs from the result in a.) by the multiplicative constant $\frac{\sqrt{3}}{8}$, hence: $V = \frac{21\sqrt{3}}{80}$

d.) (10) p. 320, exercise 34



Method 1: Shells

$$V = 2\cdot 2\pi \int_{\sqrt{r^2-\frac{h^2}{4}}}^{r} x\sqrt{r^2 - x^2}\,dx$$

$$u(x) = r^2 - x^2 \Rightarrow du = -2x\,dx \Rightarrow x\,dx = -\tfrac{1}{2}du$$

$$u\left(\sqrt{r^2 - \tfrac{h^2}{4}}\right) = \tfrac{h^2}{4},\ u(r) = 0$$

$$\therefore V = 4\pi \int_{h^2/4}^{0} u^{1/2}\left(-\tfrac{1}{2}du\right) = -2\pi \int_{h^2/4}^{0} u^{1/2}du = 2\pi \int_{0}^{h^2/4} u^{1/2}du = 2\pi \tfrac{2}{3}u^{3/2}\Big|_{0}^{h^2/4}$$

$$= \tfrac{4}{3}\pi\left(\tfrac{h^2}{4}\right)\left(\tfrac{h}{2}\right) = \tfrac{\pi}{6}h^3$$

Method 2: Washers

$$V = 2\pi \int_{0}^{h/2}\left[\left(\sqrt{r^2 - y^2}\right)^2 - \left(\sqrt{r^2 - \frac{h^2}{4}}\right)^2\right]dy = 2\pi \int_{0}^{h/2}\left(r^2 - y^2 - r^2 + \tfrac{1}{4}h^2\right)dy$$

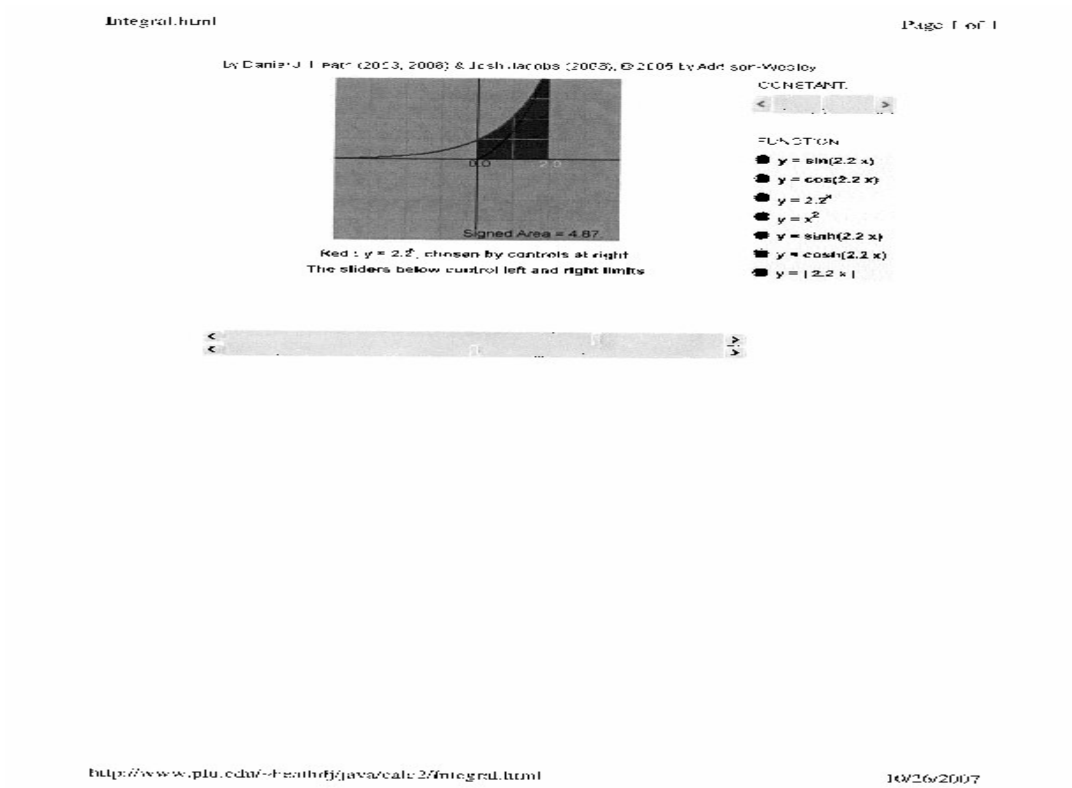
$$= 2\pi \int_0^{h/2} \left(\tfrac{1}{4}h^2 - y^2\right) dy = 2\pi\left(\tfrac{1}{4}h^2 y - \tfrac{1}{3}y^3\right)\Big|_0^{h/2} = 2\pi y\left(\tfrac{1}{4}h^2 - \tfrac{1}{3}y^2\right)\Big|_0^{h/2}$$

$$= 2\pi\left(\tfrac{h}{2}\right)\left[\tfrac{1}{4}h^2 - \tfrac{1}{12}h^2\right] = \frac{\pi h}{4}\left[h^2 - \tfrac{1}{3}h^2\right] = \pi\left(\tfrac{h}{4}\right)\left(\tfrac{2}{3}h^2\right) = \tfrac{1}{6}\pi h^3$$

IV.) (25) Integration of exponential functions.

a.) (5) Log on to http://www.plu.edu/~heathdj/java/calc2/Integral.html
Use the applet to determine the area under the curve $y = 2.2^x$ for the domain: [0, 2]

From Applet : Area = 4.87



b.) (8) Verify your result in a.) by computing the integral by hand.

$$A = \int_0^2 (2.2)^x dx = \tfrac{1}{\ln 2.2}(2.2)^x\Big|_0^2 = \tfrac{1}{\ln 2.2}\left((2.2)^2 - (2.2)^0\right) = 4.87$$

c.) (7) p. 369, exercise 24

$$y = x^2 e^x - 2xe^x + 2e^x$$

Method 1 (easier!) Factor out e^{-x} and then use product rule

$$y = x^2 e^x - 2xe^x + 2e^x = \left(x^2 - 2x + 2\right)e^x$$
$$\frac{dy}{dx} = \frac{d}{dx}\left[\left(x^2 - 2x + 2\right)e^x\right] = \left[\frac{d}{dx}\left(x^2 - 2x + 2\right)\right]e^x + \left(x^2 - 2x + 2\right)\left[\frac{d}{dx}e^x\right]$$
$$= (2x-2)e^x + \left(x^2 - 2x + 2\right)e^x = \left[(2x-2) + \left(x^2 - 2x + 2\right)\right]e^x = x^2 e^x$$

Method 2 (more tedious!) Apply product rule separately on all three terms

$$y = x^2 e^x - 2xe^x + 2e^x$$
$$\frac{dy}{dx} = \frac{d}{dx}\left(x^2 e^x\right) + \frac{d}{dx}\left(-2xe^x\right) + \frac{d}{dx}\left(2e^x\right) = \frac{d}{dx}\left(x^2 e^x\right) - 2\frac{d}{dx}\left(xe^x\right) + 2\frac{d}{dx}e^x$$
$$= \left(2xe^x + x^2 e^x\right) - 2\left(e^x + xe^x\right) + 2e^x = x^2 e^x$$

d.) (5) p. 370, exercise 57

$$\int \frac{e^x + e^{-x}}{\sqrt{e^x - e^{-x}}}dx$$

$$u = e^x - e^{-x} \Rightarrow du = \left(e^x + e^{-x}\right)dx$$
$$\therefore \int \frac{e^{-x} + e^x}{\sqrt{e^x - e^{-x}}}dx = \int \frac{du}{\sqrt{u}} = \int u^{-1/2}du = 2u^{1/2} + C = 2\sqrt{e^x - e^{-x}} + C$$

Check: $\frac{d}{dx}(2\left(e^x - e^{-x}\right)^{1/2} + C) = \left(e^x - e^{-x}\right)^{-1/2}\left(e^x + e^{-x}\right) = \frac{e^x + e^{-x}}{\sqrt{e^x - e^{-x}}}$